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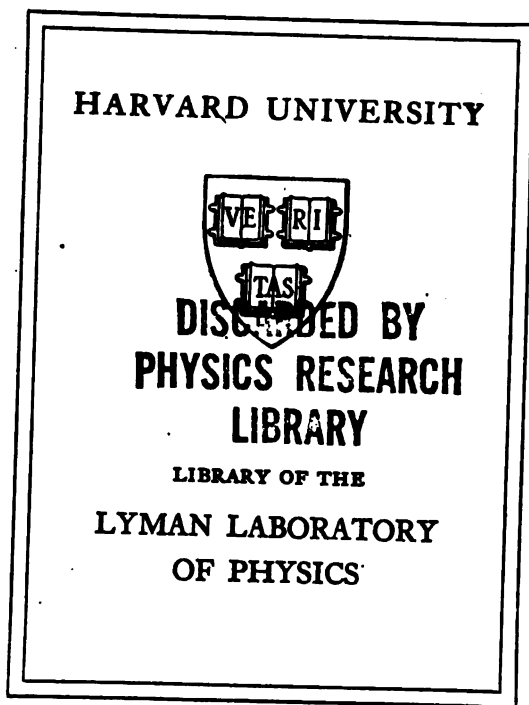
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DEPARTMENT OF COMMERCE AND LABOR

BULLETIN
OF THE
BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

VOLUME 6
1909-10



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1910

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CONTENTS OF VOLUME 6.

NO. 1.

[Issued October, 1909.]

	Page.
116. DETERMINATION OF THE RATIO OF TRANSFORMATION AND OF THE PHASE RELATIONS IN TRANSFORMERS..... <i>E. B. Rosa and M. G. Lloyd</i>	1
117. DETERMINATION OF THE MAGNETIC INDUCTION IN STRAIGHT BARS..... <i>Charles W. Burrows</i>	31
118. METHOD FOR CONSTRUCTING THE NATURAL SCALE OF PURE COLOR..... <i>P. G. Nutting</i>	89
119. APPROXIMATE EXPERIMENTAL METHOD FOR THE ANALYSIS OF EMF WAVES..... <i>P. G. Agnew</i>	95
120. NOTE ON THE THERMOELECTRIC PROPERTIES OF TANTALUM AND TUNGSTEN..... <i>W. W. Coblentz</i>	107
121. ESTIMATION OF THE TEMPERATURE OF COPPER BY MEANS OF OPTICAL PYROMETERS..... <i>George K. Burgess</i>	111
122. RESOLVING POWERS OF OBJECTIVES..... <i>P. G. Nutting</i>	121
123. THEORY OF THE HAMPSON LIQUEFIER..... <i>Edgar Buckingham</i>	125

NO. 2.

[Issued November, 1909.]

124. PLATINUM RESISTANCE THERMOMETRY AT HIGH TEMPERATURES..... <i>C. W. Waidner and G. K. Burgess</i>	149
125. DAYLIGHT EFFICIENCY OF ARTIFICIAL ILLUMINANTS..... <i>Herbert E. Ives</i>	231
126. COUPLED CIRCUITS IN WHICH THE SECONDARY HAS DISTRIBUTED INDUCTANCE AND CAPACITY..... <i>Louis Cohen</i>	247
127. EFFECT OF PHASE HARMONICS UPON ACOUSTIC QUALITY..... <i>M. G. Lloyd and P. G. Agnew</i>	255
128. WHITE LIGHT FROM THE MERCURY ARC AND ITS COMPLEMENTARY..... <i>Herbert E. Ives</i>	265
129. REGULATION OF POTENTIAL TRANSFORMERS AND MAGNETIZING CURRENT.. <i>M. G. Lloyd and P. G. Agnew</i>	273
130. DETERMINATION OF THE CONSTANTS OF INSTRUMENT TRANSFORMERS..... <i>P. G. Agnew and T. T. Fitch</i>	281

NO. 3.

[Issued February, 1910.]

131. SELECTIVE RADIATION FROM VARIOUS SOLIDS, II..... <i>W. W. Coblentz</i>	301
132. LUMINOUS EFFICIENCY OF THE FIREFLY..... <i>Herbert E. Ives and W. W. Coblentz</i>	321
133. LUMINOSITY AND TEMPERATURE..... <i>P. G. Nutting</i>	337
134. THEORETICAL AND EXPERIMENTAL STUDY OF THE VIBRATION GALVANOMETER..... <i>Frank Wenner</i>	347
135. SPECIFIC HEAT OF SOME CALCIUM CHLORIDE SOLUTIONS BETWEEN -35° C. AND $+20^{\circ}$ C..... <i>H. C. Dickinson, E. F. Mueller, and E. B. George</i>	379
136. DEFINITION OF THE IDEAL GAS..... <i>Edgar Buckingham</i>	409

NO. 4.

[Issued November, 1910.]

	Page.
137. MICA CONDENSERS AS STANDARDS OF CAPACITY..... <i>Harvey L. Curtis</i>	431
138. MUTUAL INDUCTANCE OF TWO PARALLEL COAXIAL CIRCLES IN TERMS OF HYPERGEOMETRICAL SERIES..... <i>Frederick W. Grover</i>	489
139. METHOD FOR THE ABSOLUTE MEASUREMENT OF ELECTRIC QUANTITY..... <i>Burton McCollum</i>	503
140. COMPARATIVE SENSITIVENESS OF SOME COMMON DETECTORS OF ELECTRIC OSCILLATIONS..... <i>Louis W. Austin</i>	527
141. PHOTOMETRIC UNITS AND NOMENCLATURE..... <i>E. B. Rosa</i>	543
142. MODIFIED METHOD FOR THE DETERMINATION OF RELATIVE WAVE-LENGTHS <i>Irwin G. Priest</i>	573

DEPARTMENT OF COMMERCE AND LABOR

Vol. 6

BULLETIN

No. 1

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

Issued October, 1909

CONTENTS

	Page
The Determination of the Ratio of Transformation and of the Phase Relations in Transformers <i>E. B. Rosa and M. G. Lloyd</i>	1
The Determination of the Magnetic Induction in Straight Bars <i>Charles W. Burrows</i>	31
A Method for Constructing the Natural Scale of Pure Color . . . <i>P. G. Nutting</i>	89
An Approximate Experimental Method for the Analysis of EMF Waves <i>P. G. Agnew</i>	95
Note on the Thermoelectric Properties of Tantalum and Tungsten <i>W. W. Coblentz</i>	107
The Estimation of the Temperature of Copper by Means of Optical Pyrometers <i>George K. Burgess</i>	111
The Resolving Power of Objectives <i>P. G. Nutting</i>	121
The Theory of the Hampson Liquefier <i>Edgar Buckingham</i>	125



WASHINGTON
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1909

THE DETERMINATION OF THE RATIO OF TRANSFORMATION AND OF THE PHASE RELATIONS IN TRANSFORMERS.

By E. B. Rosa and M. G. Lloyd.

Alternating-current transformers are so useful in the measurement of current and potential, by reducing the current or potential that must be applied directly to the instruments, that they have been extensively used in engineering work for the measurement of current, voltage, and power, not only for the heaviest currents and highest potentials, but also for currents and potentials of moderate values. Such transformers, when properly constructed, can safely be employed in connection with precision voltmeters, ammeters, and wattmeters for measurements of considerable accuracy. Indeed, if the constants of the transformers have been accurately measured, the precision of the results will depend chiefly on the indicating instruments, for the transformers themselves are more permanent and less liable to injury than the more delicate instruments used with them.

Transformer losses have been an object of much study, and their determination has become a familiar test; the measurement of ratios is one which may be carried out without complicated apparatus and is easily accomplished; but the question of phase relations seems to have remained a subject of theoretical study principally, and to have received scant experimental attention.¹ It is of importance, however, not merely as a matter of general interest or in the design of transformers, but also in the measurement of power. For measurements of voltage or of current, it is necessary to know only the ratio of transformation involved, but

¹ Since the above was written, a comprehensive article on the subject has appeared by L. T. Robinson, *Proc. A. I. E. E.*, 28, p. 981; 1909.

for measurements of power with a wattmeter the phase relations are also involved, and accuracy can not be assured unless these are known. Usually the phase of the secondary is so nearly the reverse of the primary phase that the error with high power-factors would be insignificant, but with low power-factors a large error might be introduced, as may be seen by reference to the numerical examples given below. In view of these facts, it is thought desirable to publish the methods of measurement of these quantities in use at the Bureau of Standards.²

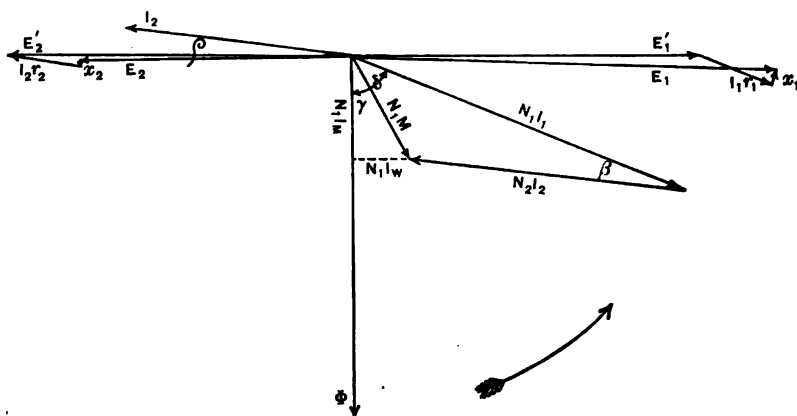


Fig. 1.

In the vector diagram, Fig. 1, are shown the various quantities which go to determine the ratio of the transformer. The length of the vector is proportional to the maximum value of the quantity, and it is considered to be rotating uniformly in a counter clockwise direction. Its projection on a fixed diameter will then represent the instantaneous value of the quantity, assuming it to be sinusoidal. The angles between the vectors represent the phase-angles of the corresponding quantities.

Φ represents the magnetic flux linking both primary and secondary windings. It induces in the secondary winding an electromotive force E_2 , and in the primary winding an electromotive force in the same direction, but of different magnitude, fixed by the number of turns. If we call the exciting current M , the cur-

²Since this was written, other methods have been developed; they are described by Agnew and Fitch, Phys. Rev. 28, p. 473, 1909, and will appear in a later number of this Bulletin.

rent-turns necessary to produce the flux Φ is represented by N_1M , and it is made up of two components N_1I_1 and N_2I_2 . The electromotive force applied to the primary terminals may be separated into three components; the first balances the induced emf. due to the flux Φ , and is represented by E_1' ; the second balances the emf. (of self-induction) x_1 due to any leakage flux which links with the primary, but not the secondary; the remainder sends current through the primary and is represented by I_1r_1 . The vector sum E_1 of these three components must be the terminal emf. of the primary winding.

The secondary terminal electromotive force, E_2 , represents what is left after deducting the ohmic drop I_2r_2 and the induced emf. x_2 (due to leakage flux linking the secondary alone) from the total emf. E_2' .

The ratio of E_1 to E_2 is known as the ratio of a potential transformer, and in general it differs appreciably from the ratio of E_1' to E_2' , which is the ratio of primary to secondary turns. The phase-angle between E_1 and E_2 reversed is the angle which it is proposed to measure. This may be either positive or negative.

If no current be taken from the secondary, E_2 becomes identical with E_2' ; N_1I_1 becomes identical with N_1M ; I_1r_1 becomes smaller and farther from E_1' in phase; the ratio is more nearly the ratio of turns; the secondary emf. is in advance of the primary. As the secondary current is increased the ohmic drop in the secondary increases and E_2 decreases; I_1 must increase to maintain M ; I_1r_1 increases and comes more nearly into phase with E_1' ; hence E_1 must be increased to maintain the same magnetic conditions, or if E_1 be maintained constant the flux decreases and E_2 suffers further contraction, so that the ratio is increased; x_1 will in general increase and the phase angle will approach zero and finally become negative.

If the applied voltage E_1 be altered, the same diagram will still represent the quantities to a different scale, provided the external secondary impedance be unchanged, so that I_2 retains its proportion to E_2 . The ratio and phase relations thus remain unchanged. This is strictly true of a transformer with air core, and with an iron core the deviation of the ratio from constancy becomes appreciable only when saturation is approached, so that x_1 and x_2 no longer remain proportional, and the permeability of the iron falls off to

such an extent that N_1M must be increased out of all proportion to E_1 . But within the working range of the transformer we may say that the ratio and phase angle are independent of the voltage.

A change of frequency involves a change of flux, which, in turn, requires a change in the magnetizing current. If the frequency be increased, the magnetizing current is decreased, but at the same time is thrown more nearly in phase with the emf., so that the change in ratio is slight. If a very large increase be made in the frequency, so that the flux is very low, x_1 and x_2 will be increased also, the ratio will be appreciably raised, and the phase angle decreased.

If, however, the voltage be changed in proportion to the frequency, so that the same magnetic flux is maintained, the conditions are little altered, and for the same impedance in secondary circuit the ratio is little affected. For the same secondary *current*, the ohmic drop is proportionally decreased for an increased frequency and the ratio also decreased. The effect here will depend upon the load and increase with it.

This diagram and the discussion have been based upon the supposition that all the quantities concerned follow a sinusoidal variation. In a transformer containing iron this is never realized, for if the applied electromotive force is sinusoidal, the current will not be, owing to the varying permeability of the iron. The discussion remains substantially valid, however, if we let the current vectors in the diagram represent the equivalent sine waves.

If the applied electromotive force is not sinusoidal, then neither the magnetic flux nor the other electromotive forces will follow the sine law of variation. Since the two induced electromotive forces have the same wave form, the terminal electromotive forces can differ from them, and the ratio can be affected, only in so far as the leakage and the resistance drop in the windings influence them. The qualitative effect may be determined from theoretical considerations.

The equation connecting the instantaneous electromotive forces in the primary is

$$e = N \frac{d\Phi}{dt} + ir + L \frac{di}{dt}$$

where

$$e = A \sin pt + Ah_3 \sin (3 pt + \theta_3) + Ah_5 \sin (5 pt + \theta_5) + \dots$$

is the applied electromotive force.

$$i = B \sin (pt + a) + Bk_3 \sin (3 pt + a_3) + Bk_5 \sin (5 pt + a_5) + \dots$$

is the primary current.

r = primary resistance

N = number of turns in the primary winding.

$L \frac{di}{dt}$ represents leakage reactance

For the secondary circuit we have a similar equation

$$e_2 = N_2 \frac{d\Phi}{dt} - i_2 r_2 - L_2 \frac{di_2}{dt}$$

Since the effect of resistance and leakage reactance is the same in both circuits, it is sufficient to consider the primary alone.

The ratio is expressed in terms of the effective terminal voltage. It is consequently necessary to express the effective voltage in terms of the above quantities.

$$\begin{aligned} E^2 &= \frac{2}{T} \int_0^{\frac{T}{2}} e^2 dt = \frac{2}{T} \int_0^{\frac{T}{2}} \left(eN \frac{d\Phi}{dt} + eir + eL \frac{di}{dt} \right) dt \\ \frac{di}{dt} &= Bp \cos (pt + a) + 3 Bk_3 p \cos (3 pt + a_3) \\ &\quad + 5 Bk_5 p \cos (5 pt + a_5) + \dots \\ \int_0^{\frac{T}{2}} eL \frac{di}{dt} dt &= pLAB \int_0^{\frac{T}{2}} [\sin pt \cos (pt + a) \\ &\quad + 3 h_3 k_3 \sin (3 pt + \theta_3) \cos (3 pt + a_3) + \dots] dt \\ &= -LAB \left[\frac{\pi}{2} \sin a + 3 h_3 k_3 \frac{\pi}{2} \sin (a_3 - \theta_3) \right. \\ &\quad \left. + 5 h_5 k_5 \frac{\pi}{2} \sin (a_5 - \theta_5) + \dots \right] \\ \frac{2}{T} \int_0^{\frac{T}{2}} eL \frac{di}{dt} dt &= -\frac{p}{2} LAB [\sin a + 3 h_3 k_3 \sin (a_3 - \theta_3) \\ &\quad + 5 h_5 k_5 \sin (a_5 - \theta_5) + \dots] \end{aligned}$$

$$\begin{aligned}
 \int_0^{\frac{T}{2}} e i r dt &= r A B \int_0^{\frac{T}{2}} [\sin pt \sin (pt + a) \\
 &\quad + h_3 k_3 \sin (3 pt + \theta_3) \sin (3 pt + a_3) + \dots] dt \\
 &= r A B \frac{\pi}{2p} [\cos a + h_3 k_3 \cos (a_3 - \theta_3) \\
 &\quad + h_5 k_5 \cos (a_5 - \theta_5) + \dots] \\
 \frac{2}{T} \int_0^{\frac{T}{2}} e i r dt &= \frac{1}{2} r A B [\cos a + h_3 k_3 \cos (a_3 - \theta_3) \\
 &\quad + h_5 k_5 \cos (a_5 - \theta_5) + \dots]
 \end{aligned}$$

The two terms which have been integrated determine the difference between the terminal emf. and the induced emf., and since the induced emf. varies in the same way in both primary and secondary, any change in the ratio due to wave form will be indicated by the change in the above terms.

For sinusoidal emf., $h_3 = 0 = h_5 = h_7 = \text{etc.}$

$$E^2 = \frac{2}{T} \int_0^{\frac{T}{2}} e N \frac{d\Phi}{dt} dt + \frac{1}{2} A_o B_o (r \cos a - pL \sin a)$$

where A_o and B_o denote the values of A and B for this particular case. For other wave forms with the same effective voltage, A and B will have different values.

For other wave forms,

$$\begin{aligned}
 E^2 &= \frac{2}{T} \int_0^{\frac{T}{2}} e N \frac{d\Phi}{dt} dt + \frac{1}{2} A B [\{r \cos a - pL \sin a\} + h_3 k_3 \{r \cos (a_3 - \theta_3) \\
 &\quad - 3 pL \sin (a_3 - \theta_3)\} + h_5 k_5 \{r \cos (a_5 - \theta_5) - 5 pL \sin (a_5 - \theta_5)\} + \dots]
 \end{aligned}$$

Let us first consider the effect upon the ratio at no load. For a constant effective voltage, distortion of any kind will reduce the value of A , since³

$$A_o^2 = A^2 (1 + h_3^2 + h_5^2 + \dots)$$

B may be either increased or diminished, since

$$\frac{\Phi}{\Phi_o} = \frac{f_o}{f}$$

³ See M. G. Lloyd, this Bulletin, 4, page 480, ff; 1908 (Reprint No. 88), for some of the relations here made use of.

where f is the form factor and B may be assumed to vary approximately as Φ . a will be negative and for a sine wave approximately $\frac{\pi}{3}$, making the terms in resistance and reactance both positive. It will be only slightly changed by wave distortion, and since any change will affect the sine and cosine oppositely the factor $r \cos a - pL \sin a$ may be regarded as not varying.

It has been shown by Bedell and Tuttle⁴ that $a_3 - a$ must be positive and lie between 30° and 180° . For a peaked wave θ_3 is in the neighborhood of 180° and hence $a_3 - \theta_3$ will probably lie in the third quadrant, making sine and cosine both negative. The term involving the third harmonic may then be either positive or negative, according as reactance or resistance predominates. We shall not attempt to follow the terms involving higher harmonics. We can see already that ordinarily a peaked wave will decrease the ratio; for the form factor is greater than for a sine wave, and hence Φ and B are less, while A is in all cases less. Hence the applied voltage departs further from the induced voltage.

For a flat potential wave on no load, the form factor is low, Φ and B are decreased, and A as before is decreased. θ_3 is now in the neighborhood of zero, and $a_3 - \theta_3$ will be positive and small. The resistance term for the third harmonic is now positive and the reactance term negative; the ratio will ordinarily be increased.

For a full load upon the transformer, the conditions are somewhat altered. If the load be noninductive, the current will have approximately the same wave form as the applied voltage and be almost in phase with it. Consequently, A and B will both be decreased by distortion and a will approach more nearly to zero. Since a_3 lies between 30° and 180° , $a_3 - \theta_3$ will be negative for a peaked wave and will usually be positive for a flat wave. Opposing the decrease in AB is the resistance component of the harmonic term, and for a peaked wave the reactance component also. If we neglect the magnetizing current, it can be shown that the resistance components of the harmonic terms will exactly neutralize the decrease in AB . Consequently, when the leakage in a trans-

⁴F Bedell and E. B. Tuttle, Trans. Am. Inst. Elect. Engrs., 25, p. 601, 1906.

former is small, the effect of wave form when loaded will be qualitatively the same as when unloaded, but probably less in magnitude. When the leakage is large, however, the reactance components may become important, since these have numerical coefficients equal to the order of the harmonic. With a peaked wave especially the ratio will be increased, while with a flat wave the increase will be less.

With a lagging secondary load the harmonics will be less prominent in the current than in the emf. wave. α will be negative and all the phase-angles about the same as for no load. The effect of leakage reactance will be more prominent than for non-inductive load and the effect of the resistance terms less important. Consequently, if the leakage be large, the ratio may be increased with a peaked wave and decreased with a flat wave; otherwise it will surely be decreased for a peaked wave and probably increased for a flat wave.

We see from this discussion and from the experimental results given below that the ratio of a potential transformer is quite definitely determined by given conditions, and, moreover, with a definite secondary circuit, is little affected by variation of voltage or moderate variation of frequency and wave form. Consequently, if a potential transformer be calibrated for the value of its ratio with different secondary impedances, it may be used as an instrument of precision. The phase-angle under normal conditions is so nearly zero that for most purposes the discrepancy is negligible. When used with wattmeters on low power-factors, however, this angle should be determined.

In the series transformer we are concerned, not with the electromotive forces, but with the ratio of the primary and secondary currents. This ratio depends upon the exciting current and the power factor of the load, as well as upon the ratio of turns.

Let ρ be the angle by which the secondary current lags behind the induced emf., and resolve each side of the triangle of current-turns into components parallel and normal to the direction of Φ . Let δ be the angle between the primary current and this direction and let I_W and I_M be the two components of M . Then, referring to the vector diagram, Fig. 1,

$$N_1 I_1 \cos \delta = N_2 I_2 \sin \rho + N_1 I_M$$

$$N_1 I_1 \sin \delta = N_2 I_2 \cos \rho + N_1 I_W$$

Squaring and adding the two equations, we have

$$N_1^2 I_1^2 = N_2^2 I_2^2 + N_1^2 M^2 + 2 N_2 I_2 N_1 (I_M \sin \rho + I_W \cos \rho)$$

or

$$\begin{aligned} \left(\frac{I_1}{I_2}\right)^2 &= \left(\frac{N_2}{N_1}\right)^2 + \left(\frac{M}{I_2}\right)^2 + 2 \frac{N_2}{N_1 I_2} (I_M \sin \rho + I_W \cos \rho) \\ \frac{I_1}{I_2} &= \frac{N_2}{N_1} \sqrt{1 + \left(\frac{N_1 M}{N_2 I_2}\right)^2 + 2 \frac{N_1}{N_2 I_2} (I_M \sin \rho + I_W \cos \rho)} \\ &= \frac{N_2}{N_1} \left[1 + \frac{1}{2} \left(\frac{N_1 M}{N_2 I_2}\right)^2 + \frac{N_1}{N_2 I_2} (I_M \sin \rho + I_W \cos \rho) \right] \end{aligned}$$

approximately.

The smaller M is with respect to I_2 , the more nearly the ratio becomes simply the inverse ratio of the numbers of turns. $\frac{M}{I_2}$ may be small from three causes. The iron of the core may be of high permeability, so that only a low magnetomotive force is needed. Secondly, the impedance of the secondary circuit may be low, permitting the necessary current to flow with a low magnetic flux. Finally, the load on the transformer may be large.

The deviation from ratio of turns increases with the angle ρ ; that is, with the reactance of the secondary circuit. The ratio of currents can only equal the ratio of turns by having a large negative value of ρ ; that is, a leading current in the secondary. The value of ρ necessary is determined by the relation

$$\cos (\delta - \gamma) = \frac{N_1 M}{2 I_2 N_2}$$

where

$$-\rho = 90^\circ + 2\gamma - \delta$$

for in this case

$$I_M \sin \rho + I_W \cos \rho = -\frac{1}{2} \frac{N_1 M^2}{N_2 I_2}$$

and

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

The angle β , denoting the phase difference between primary and secondary currents, is decreased by reactance in the secondary circuit and is increased for a leading current in secondary. The latter condition is scarcely one which would be attained in practice.

With a definite secondary circuit, and increasing current, Φ increases at the same rate, but at low flux densities M does not increase so fast, and the ratio is diminished. When the maximum permeability is passed, however, M increases faster than the secondary current and the ratio will begin to rise again. The phase-angle is also diminished rapidly at first, then becomes nearly constant, and finally increases at high flux densities. The larger the secondary resistance, the sooner this turning point is reached, but it will ordinarily lie beyond the full load of the transformer.

We see from the above that the ratio will depart least from the ratio of turns when the secondary circuit has low impedance and the core has a generous section of high permeability. For constancy of ratio, it is necessary to have constant permeability in the core with different inductions. This condition has been more nearly realized since the advent of silicon-alloy-steel, which has a high and slowly changing permeability at low inductions.

The effect of wave form upon the ratio of currents is in altering the necessary induction in the core and thereby the exciting current. The emf. induced in the secondary circuit must be proportional to the current in it, and its effective value is also proportional to the product of its form factor and the maximum flux in the core. If the form factor be increased, the maximum flux will be diminished and the exciting current likewise diminished. The numerical relations will depend upon the form of permeability curve of the core, but the direction of the effect will be that stated, the assurance of this being greater since high flux densities are never used in series transformers. Since a lower exciting current means a lower ratio, we may say in general that a peaked wave of current will give a lower ratio, and a flat wave will give a higher ratio. As the exciting current enters merely as a correction to the ratio, and as the wave form only slightly alters the exciting current, the effect of wave form will necessarily be slight.

POTENTIAL TRANSFORMERS.

The ratio of a potential transformer is determined by means of a differential dynamometer voltmeter. In this instrument the torque due to one set of coils is balanced against the torque due to the other set. Each set of coils consists of a pair of fixed coils and one moving coil between them. The two moving coils are rigidly connected, one above the other, but have separate leading-in wires. In determining the ratio of the primary and secondary voltages of a transformer the coils of each set are connected in series with each other and with a large noninductive resistance. One pair is supplied with current from the primary terminals, the other from the secondary terminals. Then, if each pair has the

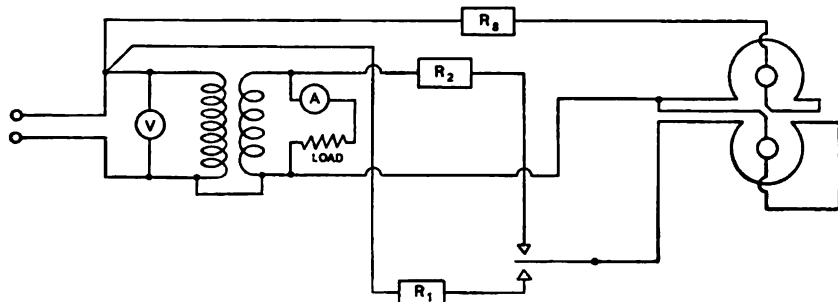


Fig. 2.

same constant (that is, if the torques are equal and opposite for equal currents in the coils), a balance is obtained when the resistances of the two circuits are proportional to the respective electromotive forces. To prevent interaction between the two sets of coils, the moving coil of one set is in the plane of the fixed coils of the other set. In other words, the two moving coils are mounted on the suspended system at right angles to each other. A current of about 0.025 ampere in each system gives sufficient sensibility so that a change of one part in five thousand may be detected.

In using the instrument, the primary emf. is applied through a suitable resistance to one set of coils; to the other set first the primary and then the secondary emf. is applied; the resistance

being adjusted each time for a balance (see Fig. 2). The ratio of the two latter resistances is the ratio of the electromotive forces at the terminals of the transformer.

For, let

k_1, k_2 be the constants of the two sets of coils;

R_1, R_2, R_3 the total resistances of the circuits in the respective cases;

E_1, E_2 the terminal electromotive forces acting simultaneously upon the two sets of coils;

E the emf. for the auxiliary measurement, which may or may not be the same as E_1 .

Then

$$\left(\frac{E}{R_3}\right)^2 k_1 = \left(\frac{E}{R_1}\right)^2 k_2 \text{ when the same emf. is applied to both.}$$

$$\left(\frac{E_1}{R_3}\right)^2 k_1 = \left(\frac{E_2}{R_2}\right)^2 k_2 \text{ when primary emf. is applied to one and secondary emf. to the other.}$$

From which

$$\left(\frac{k_1}{k_2}\right) = \left(\frac{R_3}{R_1}\right)^2 = \left(\frac{E_2}{E_1}\right)^2 \left(\frac{R_2}{R_3}\right)^2$$

and

$$\frac{E_1}{E_2} = \frac{R_1}{R_2}.$$

It is to be noticed that E need not be the same as E_1 ; that is, any change in the voltage produced by altering the load on the transformer or by fluctuation of the supply does not affect the result. For convenience in computation the resistance R_3 should be a round number, such as 1000, 5000, or 10000. R_3 is adjusted for balance with the switch in upper position, and R_1 is adjusted for balance with the switch down.

To determine the phase-angle between primary and secondary, the fixed coils are supplied as before, while the two moving coils are successively connected in series with a condenser and supplied with current from the primary terminals. This current is nearly in quadrature with the current in the fixed coils, and will produce very little deflection. Its value is afterwards determined by sending it through one pair of coils and noting the deflection.

It is to be remembered that the phase-angle is usually very small, and only one or two significant figures are necessary in the result.

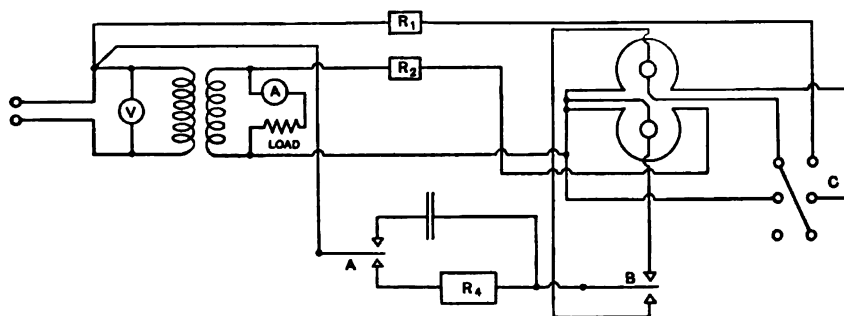


Fig. 3.

The connections are shown in Fig. 3. With switches A and B thrown down and switch C thrown up, the deflection is noted. Switch B is then thrown up and the resistance R_4 adjusted to give the same deflection as before. This makes the two fields of equal deflecting strength. Switch A is then thrown up, connecting the condenser in series with the moving coils, and the deflections D_1 and D_2 are read for the two positions of switch B. Each deflection

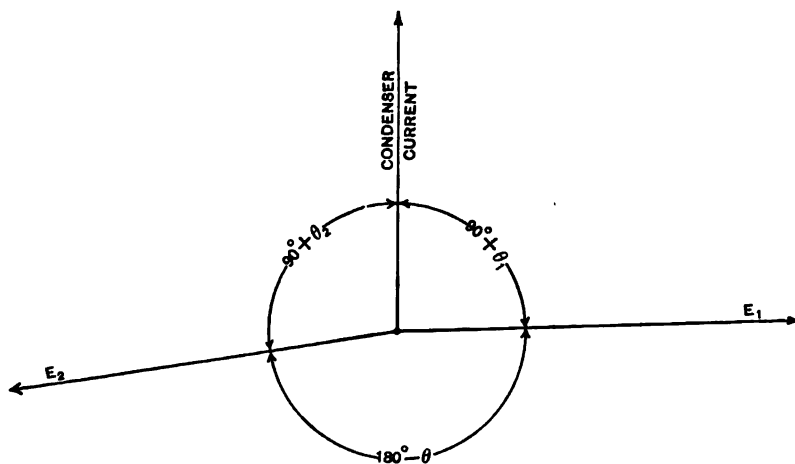


Fig. 4.

is a measure of the phase-angle between the condenser current and one of the terminal electromotive forces. Switches C and B are then thrown down, and the deflection D_3 noted with switch A thrown up. Finally the instrumental constant is determined by throwing

switch *A* down and *C* up, and observing the deflection *D*₄. This deflection should be made about the same as *D*₃, by adjusting the resistance *R*₄, whose value must be known.

Let

*k*₁ *k*₂ = constants of instrument as before.

90° + *θ*₁, 90° + *θ*₂ = angles between condenser current and terminal electromotive forces.

180° - *θ* = lag of secondary behind primary emf., so that

$$\theta = \theta_1 + \theta_2$$

*D*₁ *D*₂ *D*₃ *D*₄ = deflections.

*R*₁ *R*₂ *R*₃ = resistances as before.

*R*₄ = resistance whose admittance is approximately same as that of condenser.

*I*₃ = condenser current.

$$I_4 = \frac{E_1}{R_4} = \text{current for calibrating.}$$

With the condenser current, *I*₃, in each moving coil in turn we have

$$D_1 = k_1 I_3 \frac{E_1}{R_1} \cos (90^\circ + \theta_1) = -k_1 I_3 \frac{E_1}{R_1} \sin \theta_1$$

$$D_2 = -k_2 I_3 \frac{E_2}{R_2} \sin \theta_2 = -k_1 I_3 \frac{E_1}{R_1} \sin \theta_2$$

if *k*₂ $\frac{E_2}{R_2}$ be made equal to *k*₁ $\frac{E_1}{R_1}$ as mentioned above.

Then

$$-\sin \theta = -\sin \theta_1 - \sin \theta_2 = + \frac{R_1}{k_1 I_3 E_1} (D_1 + D_2)$$

$$D_3 = k_1 I_3^2$$

$$D_4 = k_1 I_4 \frac{E_1}{R_1} = k_1 \frac{E_1^2}{R_1 R_4}$$

Hence

$$\sqrt{D_3 D_4} = k_1 \frac{E_1 I_3}{\sqrt{R_1 R_4}}$$

and

$$\sin \theta = - \sqrt{\frac{R_1}{R_4} \frac{(D_1 + D_2)}{\sqrt{D_3 D_4}}}$$

If $D = D_1 + D_2$ and $D_4 = D_3$ then

$$\sin \theta = - \sqrt{\frac{R_1 D}{R_4 D_3}}$$

For the highest accuracy k_1 should be determined separately for the deflections D_1 , D_2 and the larger deflections D_3 , D_4 . Ordinarily it may be taken as the same in both cases.

We see then that the phase-angle may be determined by four observed deflections if the two resistances be known and a steady voltage is available. If the voltage is not maintained at a constant value throughout the observations, it should be observed at the time of each reading and corrections made for it. Small fluctuations, however, would make no appreciable error.

TABLE I.

Transformer D.—120/120 volts, 60 cycles, 500 watts. Primary resistance 0.39 ohm. Secondary resistance 0.68 ohm.

[Tested July 21, 27, 1905, at 110 volts, 60 cycles; exciting current=0.52 ampere.]

Secondary Current*	Ratio	R_1	R_2
0	1.001	5117	5111
1	1.010	5163	5111
2	1.019	5209	5111
3	1.028	5139	5000
4	1.037	5186	5000
5	1.047	5234	5000
6	1.057	5285	5000
7	1.067	5337	5000
8	1.078	5388	5000

* Current taken by instrument neglected.

Great care must be used in giving the proper algebraic sign to the deflections D_1 and D_2 . Ordinarily θ_1 will be negative and θ_2

positive. If the connections to the instrument be such as to make the deflection D_1 in the same direction for both coils, this means that D_1 and D_2 will be in opposite directions. D_1 will have the direction of D_4 , and should be considered positive, while D_2 is considered negative. This makes θ a positive angle when D_2 is numerically greater.

If this precaution be taken when connecting the instrument, the deflections D_1 and D_2 may be mechanically combined in the instrument by sending the condenser current through both moving coils in series. A single reading then gives $D_1 + D_2$.

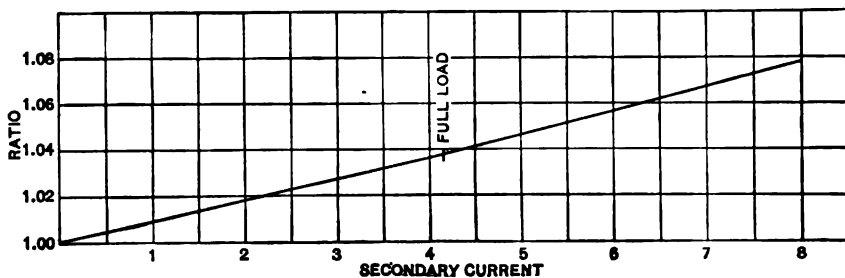


Fig. 5.

It has been assumed in the above measurement that a sine wave of electromotive force was used. If a distorted wave be used, the condenser current will have the harmonics magnified, and will not have the same wave form as the other currents in the apparatus. Since the secondary electromotive force has approximately the same wave form as the primary, the phase angle has still a very definite meaning, but it would be better to replace the condenser current by another whose phase is displaced in some other way. In the experiments given below a sine wave was used.

It may also be mentioned here that the noninductive resistances used in series with the dynamometer coils should be large enough to make the inductance of these coils negligible at the frequency used. Since different multipliers are used with the two sets of coils (except for ratio 1:1) the lag would, otherwise, be different in the two field coils and would introduce an equal error in the measurement of phase-angle. In getting the ratio it would be sufficient to use the impedance in place of the resistance of the instrument coils.

TABLE II.

Transformer G.—1100/110 volts, 60–125 cycles, 50 watts.

[Tested April 11, 1905. Secondary resistance constant. Slight overload.]

Cycles	E_s	Ratio
59.7	60	10.17
59.0	70	10.17
59.0	80	10.17
59.0	90	10.17
59.0	100	10.17
59.0	110	10.17
59.0	120	10.17
59.0	130	10.175
44.5	90	10.18
55.	90	10.18
60.	90	10.17
No load on secondary		
40	93	9.925
45	100	9.917
55	100	9.904
60	100	9.904

Table I gives the readings and results of a set of observations upon a 1:1 transformer to determine the variation of ratio with load. It is to be noted that the ratio changes almost 4 per cent between no load and full load. These values are plotted in Fig. 5.

Table II shows the effect of changes in voltage and frequency with constant secondary resistance. The ratio decreases slightly as frequency rises, but the change with voltage is less than 0.1 per cent.

Table III shows the changes with voltage and with secondary resistance in another transformer.

TABLE III.

Transformer H.—3000/120 volts, 60 cycles.

[Tested April 29, 1905. Stepping up, 60 cycles. Similar transformer connected to secondary and resistance varied in its secondary.]

E	Resistance of Secondary Circuit of Aux. Transformer	Inverse Ratio
80	9090	21.54
98	9090	21.58
120	9090	21.58 ₅
126	9090	21.58 ₅
120	1000	21.11
120	1500	21.26 ₅
120	2000	21.35
120	3000	21.44
120	5000	21.50 ₅
120	∞	21.61

In Tables IV and V results are given for two step-down transformers with secondary capacity of 500 and 400 amperes, respectively. The ratios rise rapidly with the load and at 200 amperes have changed 2.6 per cent and 3.6 per cent, respectively. Transformer E was tested also at 180 cycles, and the ratio in this case changes even more rapidly. A great difference is noticed also in the phase-angle. The ratios were also determined at 30 cycles and 55 volts. The ratios are plotted in the curves of Fig. 6 and the phase-angles in Fig. 7. The phase-angle under normal conditions is at first positive and decreases with the load. The ratios are also plotted in terms of secondary resistance in Fig. 8 for transformer E.

In Tables VI and VII are given the results of varying the wave form. The form of wave was varied by connecting two generators in series, the two being mounted on a single shaft and giving frequencies of 60 and 180 cycles, respectively. Each generator alone gives an approximate sine wave. One connection of the generators gives a peaked wave; by reversing the terminals of one machine this is changed into a flat or dimpled wave. The wave forms were determined on the oscillograph.

It will be seen that on no load the variation of ratio is less than 0.1 per cent. With the transformer loaded the effect is less, the ratio being less for a peaked wave, thus indicating that the ohmic drop is the determining factor and that the leakage effect is only apparent in decreasing the effect at full load.

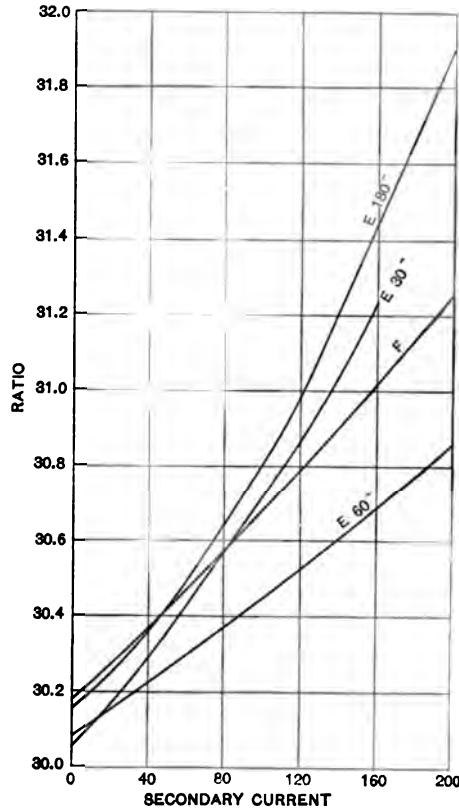


Fig. 6.

To make the effect of leakage apparent, a transformer was improvised by winding two coils upon opposite sides of a core of laminated iron. The results are given in Table VIII and show a very manifest increase in ratio with peaked wave, when the secondary was loaded.

Roessler⁵ found the ratio larger with a peaked wave, indicating large leakage in his transformer. This is explained by the fact that his coils were wound side by side and not one over the other. His results (as regards ratio) do not apply to good transformers, where the effect will usually be in the opposite direction and negligible in amount.

TABLE IV.

Transformer E.—120/4 volts, 60 cycles, 2000 watts. Primary resistance = 0.076 ohm. Secondary resistance = 0.00027 ohm.

[Tested July 17, 1905, at 110 volts; exciting current at 60 cycles = 1.3 amperes; exciting current at 180 cycles = 0.45 ampere.]

Secondary Current	Ratio		Phase Angle		Ratio at 30 cycles 55 volts
	60 cycles	180 cycles	60 cycles	180 cycles	
0	30.08	30.15	+0°39'	—3°35'	30.05
40	30.22	30.35	+0°29'	—4°18'	30.28
80	30.37	30.65	+0°18'	—4°50'	30.57
120	30.52	30.98	+0° 7'	—5°20'	30.87
160	30.68	31.44	—0° 4'	—5°30'	31.24
200	30.86	31.90	—0°15'	—5°36'	

TABLE V.

Transformer F.—120/4 volts, 50 cycles, 1600 watts. Primary resistance = 0.17 ohm. Secondary resistance = 0.00027 ohm.

[Tested July 7, 1905, at 110 volts, 60 cycles; exciting current = 0.65 ampere.]

Secondary Current	Ratio	Phase Angle
0	30.18	+0° 54'
40	30.37	+0° 24'
80.5	30.57	0'
121	30.79	—0° 26'
160	31.02	—0° 52'
200	31.26	—1° 22'

⁵G. Roessler, *Electrician*, 36, p. 151; 1895.

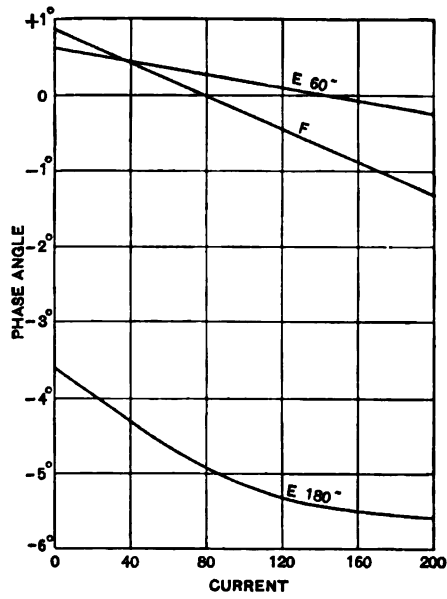


Fig. 7.

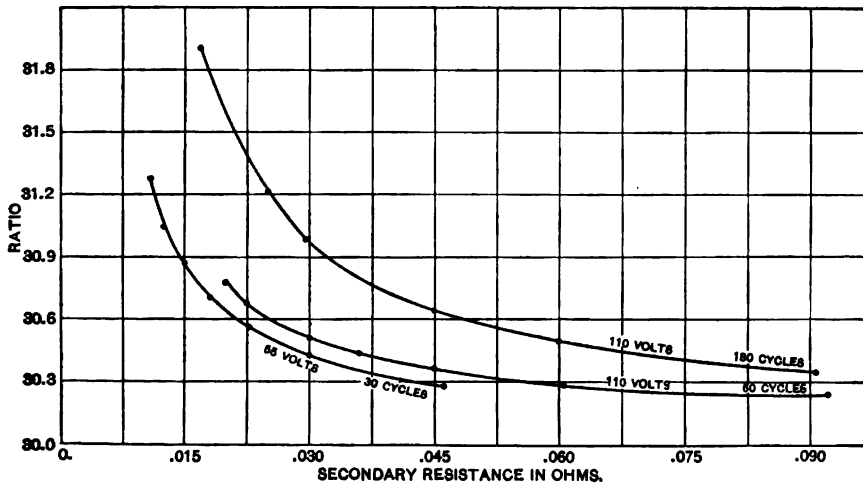


Fig. 8.

TABLE VI.

Transformer G.—1100/110 volts, 60–125 cycles, 50 watts.

[Tested February 17, 1908, at 60 cycles, 1100 volts, 30 per cent of third harmonic.]

Wave Form	Load	R ₁	R ₂	Ratio
Sine.....	None.....	40103	4058	9.880
Peak.....	None.....	40103	4061	9.873
Sine.....	Full.....	40103	3983	10.070
Dimple.....	Full.....	40103	3980	10.078
Peak.....	Full.....	40103	3984	10.068

TABLE VII.

Transformer D.—480/120 volts, 60 cycles, 500 watts.

[Tested February 17, 1908, at 60 cycles, 480 volts, 17 per cent of third harmonic.]

Wave Form	Load	R ₁	R ₂	Ratio
Sine.....	None.....	20116	5016	4.010
Peak.....	None.....	20116	5020	4.007
Flat.....	None.....	20116	5014	4.012
Flat.....	Full.....	20116	4833	4.161
Peak.....	Full.....	20116	4836	4.159
Sine.....	Full.....	20116	4834	4.160

TABLE VIII.

Special transformer.—60/60 volts.

[Tested February 19, 1908, at 60 cycles, 60 volts, 24 per cent of third harmonic.]

Wave Form	Load	R ₁	R ₂	Ratio
Sine.....	None.....	1943.5	1612	1.205
Peak.....	None.....	1935	1612	1.200
Flat.....	None.....	1984	1612	1.230
Flat.....	1.5 ampere.....	4203	1612	2.605
Peak.....	1.5 ampere.....	4243	1612	2.630
Sine.....	1.5 ampere.....	4113	1612	2.548

CURRENT TRANSFORMERS.

The currents in the primary and secondary of a series transformer may be determined by a dynamometer in each circuit, of the type already described⁶ in this bulletin. They are astatic, wound on frames of mahogany, have field coils which are wound with stranded wire (for the higher ranges), air damping, and the deflections are read with telescope and circular scale. As shown in the article cited, after being calibrated on direct current these instruments are correct for alternating currents of a wide range of frequency and any wave form.

The current flows through the field coils of the dynamometer and through a standard resistance in series. The moving coil is connected through a noninductive resistance of suitable value to the terminals of the standard resistance. The deflection of the instrument is a measure of the power expended in the standard resistance, and consequently is determined by the square of the current.

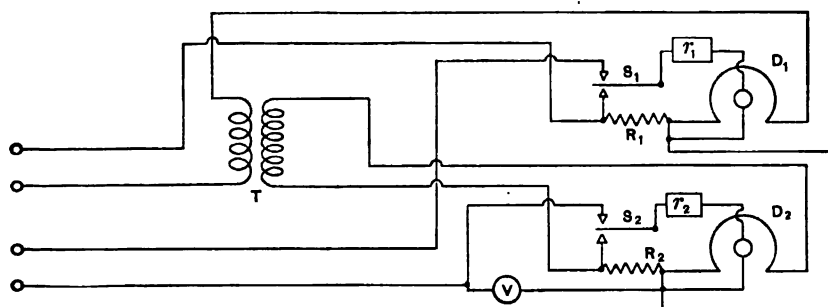


Fig. 9.

To determine the phase relation between the currents in primary and secondary, the two moving coils may be disconnected from the standard resistances and connected in series with each other. They are supplied with current exactly in quadrature with the primary current, so that there is no deflection of the dynamometer whose field coils are in the primary circuit. If the current in the secondary circuit is not exactly reversed in phase with

⁶ E. B. Rosa, this Bulletin, 3, p. 43; 1907. Reprint No. 48.

respect to the primary there will be a deflection in the second dynamometer, and this serves to measure the phase difference.

Fig. 9 is a diagram of connections suitable for making both measurements by simply throwing two switches S_1 and S_2 . T represents the transformer, D_1 and D_2 the dynamometers, R_1 and R_2 the standard resistances, r_1 and r_2 resistances in series with the moving coils, V a voltmeter. The switches are thrown up for phase measurement.

Let I_2 be the secondary current, i the current in moving coil. Let β be the angle by which the secondary current reversed leads the primary.

Let d_1 be the deflection of the dynamometer and k its constant. Then if the phase of the current in the moving coils has been adjusted for no deflection in the dynamometer D_1 , $90^\circ - \beta$ will be the phase angle between the two currents in D_2 and we have

$$d_1 = kI_2 i \cos(90^\circ - \beta) = kI_2 \frac{V}{r_1} \sin \beta$$

where r_1 includes the resistance of the moving coil.

After observing the deflection d_1 the switch S_2 is thrown over and r_2 is adjusted until the same deflection is again obtained. Let the new value be r_3 .

Then

$$d_2 = kI_2 \frac{R_2 I_2}{r_3 + R_2} \text{ and } \sin \beta = \frac{R_2 I_2}{r_3 + R_2} \frac{r_1}{V}$$

A current in quadrature with the primary may be obtained in various ways, but most conveniently from a two-phase circuit, the second phase being applied directly to the moving coils. To have adjustment, however, an arrangement of rheostats may be used as in Fig. 10, where a is common to the two phases, and connections are made at b and d .

If only single phase be available, an air-core transformer may be used in the primary circuit, and its secondary used as a source of current for the moving coils. Since the resistance of this circuit is large, the current would be in quadrature with the primary current. Or the potential of the source may be used in conjunc-

tion with a condenser and resistances to get the necessary phase. Another well-known method is shown in Fig. 11, where ABC are inductive coils and DE noninductive resistance coils. By adjustment of these the current in C may be brought into exact quadrature with the supply, or with the primary current. A phase transformer with adjustable secondary may also be used.

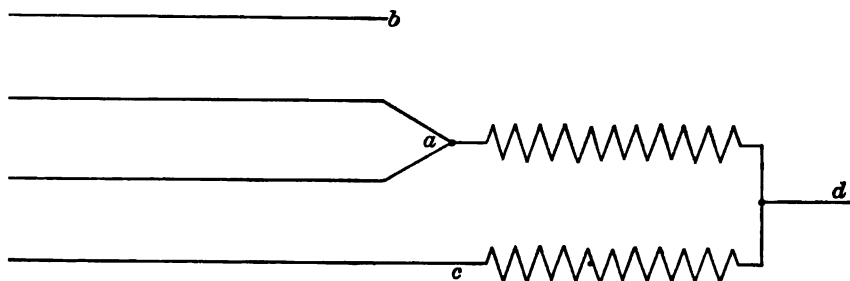


Fig. 10.

If it be not convenient to adjust for exact quadrature, the phase may be merely approximated and the deflection in D_1 observed, giving $\sin \nu$ where $90^\circ - \nu$ is the angle representing the phase relation between primary and moving coil circuits. Then the phase relation between primary and secondary is $\beta - \nu$. Care must be taken here to get the proper algebraic sign for ν .

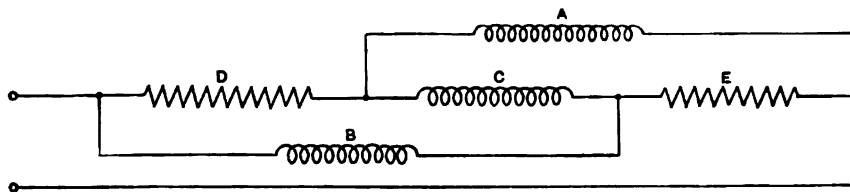


Fig. 11.

The method here indicated will give very accurate results for the conditions existing during the measurement, but as the accurate dynamometer used in the secondary circuit is of rather high resistance, the results would not be applicable to the same transformer when used with ordinary portable or switchboard instruments of low resistance. The method consequently is of little practical use. For ratio, readings taken on calibrated portable instruments of suitable type are usually sufficient, but more

accurate indirect methods (i. e., not requiring the measurement of each current independently) have been used and are still undergoing development in the bureau.

Another method for determining the phase-angle of a series transformer, which is simpler and requires no computation, has been applied by making use of a special generator set. This consists of a driving motor and two generators mounted on one shaft. The generators have revolving poles and fixed armatures, and the position of one armature can be shifted circumferentially with respect to the other by means of a worm gearing. Its position is read on a graduated scale.

The current from the movable armature is sent through the fixed coil of a dynamometer (see Fig. 12). The moving coil is connected through a suitable resistance to the terminals of a small noninductive resistance standard in the primary circuit, and the position of the armature is adjusted until there is no deflection. The moving coil is next connected to a similar resistance in the secondary circuit, and the armature again shifted until there is no deflection.

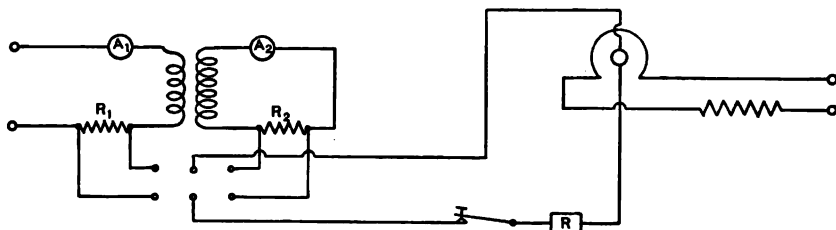


Fig. 12.

The number of electrical degrees between the two positions of the armature represents the angle β . In this case the generator has six poles, so that 120° of angular shift is equivalent to 360° electrical. The reading on the scale can be estimated to 0.1° or better, so that the uncertainty in the phase angle is not greater than 0.3° at most. When desired, this angle can be read with greater accuracy by means of finely graduated scale or vernier. The two non-inductive resistances for primary and secondary circuits are chosen to give about the same drop, so that the resistance of the moving coil circuit is not altered during the measurement; any lag in this circuit is not altered on shifting from primary to secondary.

TABLE IX.

Transformer J.—25–125 cycles, 125/5 amperes, 10 watts.

[Tested March 14, 1908, at 60 cycles. 0.001 ohm in primary; 0.025 ohm in secondary;
total resistance of secondary circuit, 0.054 ohm.]

Primary Current	Secondary Current	Ratio	Generator Settings		β
			Primary	Secondary	
140.3	5.20	27.0	32.05	28.75	9.9
125.5	4.65	27.0	31.70	28.25	10.4
100.5	3.68	27.3	31.10	27.15	11.8
75.6	2.73	27.7	30.65	26.0	14.0
50.4	1.78	28.3	60.1	54.6	16.5

An extra ammeter inserted in the secondary circuit, doubling the impedance.

135.3	4.72	28.6			
125.5	4.36	28.8	31.8	28.25	10.6
100.5	3.44	29.2			
75.6	2.50	30.2			
50.4	1.62	31.1	30.1	24.0	18.3

Same at 30 cycles. Secondary resistance=0.054 ohm.

140.3	4.91	28.6	32.0	27.7	12.9
125.5	4.36	28.8	31.6	27.0	13.8
100.5	3.41	29.5	31.0	25.95	15.2
75.6	2.50	30.2	30.5	24.3	18.6
50.4	1.57	32.1	29.9	22.3	22.8

This method has also the advantage that only a small resistance is required in the secondary circuit. In Tables IX, X, and XI the ratios were determined by means of calibrated portable ammeters, and the phase-angles by the method just described. Transformer J is of the type which can be slipped over a cable and the two parts of the core clamped together.

TABLE X.

Transformer K.—5/5 amperes.

[Tested March 21, 1908, at 60 cycles. 0.05 ohm in primary; 0.025 ohm in secondary; total resistance of secondary circuit, 0.054 ohm.]

Primary Current	Secondary Current	Ratio	Generator Settings		β
			Primary	Secondary	
5.00	4.99	1.002	12.95	12.90	0.1
4.04	4.03	1.0025	9.95	9.90	.1
3.02	3.01	1.003	6.8	6.75	.1
2.015	1.99	1.013	4.0	3.9	.3
1.02	0.995	1.025	1.6	1.55	.1
Same at 30 cycles.					
5.00	4.99	1.002	1.6	1.4	0.6
4.04	4.03	1.002	7.25	7.15	.3
3.03	3.01	1.007	5.0	4.8	.6
1.025	0.995	1.03	0.6	0.25	1.0
With 1 ohm additional in secondary circuit.					
5.00	4.955	1.009	1.7	1.4	0.9
Same with extra ammeter inserted in secondary circuit, doubling its impedance. 60 cycles.					
5.00	4.99	1.002	13.95	13.90	0.1
4.06	4.03	1.007	10.55	10.45	.3
3.03	3.01	1.007	7.3	7.25	.1
1.03	0.995	1.035	1.75	1.7	.1

TABLE XI.

Transformer K.—5/5 amperes.

[Tested March 21, 1908, at 60 cycles and full load.]

Extra Secondary Resistance	Ratio	β
0.0	1.003	
.2	1.003	
.4	1.003	
.6	1.005	
1.0	1.007	0.6

Transformer K has the secondary current (reversed) in almost exact phase with the primary, but the angle is not quite zero. By connecting a large impedance in the secondary circuit at full load the angle was made negative, but by an amount too small to measure though visible in the dynamometer deflection. By inserting 3 ohms in the secondary circuit with 2 amperes load β was increased to 1.8° , the maximum for this transformer under any of the conditions imposed.

The same method for phase-angles may be applied to potential transformers.

TABLE XII.

Transformer L.—5/5 amperes.

[Tested March 25, 1908, at 60 cycles. Secondary current, 4 amperes.]

Wave Form	Per cent Thrd Harmonic	Deflection of Primary Dynamometer		Per cent Increase in Ratio
			Mean.	
Peak.....	11	{ 22.33 .32 .33 }	22.33	0.0
Flat.....	11	{ 22.33 .33 }	22.33	0.0
Sine.....		{ 22.33 .34 }	22.33 _s	
Sine.....		{ 22.33 .32 }	22.32 _s	
Dimple.....	30	{ 22.36 .35 }	22.35 _s	0.05
Peak.....	30	{ 22.31 .82 }	22.31 _s	-0.03
Sine.....		{ 22.35 .32 }	22.33 _s	
Dimple.....	30	{ 22.36 .36 }	22.36	0.07
Peak.....	30	{ 22.30 .29 }	22.29 _s	-0.08
Peak.....	68	{ 22.27 .25 }	22.26	-0.15
Dimple.....	68	{ 22.41 .39 }	22.40	+0.17
Sine.....		{ 22.32 .33 }	22.32 _s	
Dimple.....	68	{ 22.38 .39 }	22.38 _s	+0.14
Peak.....	68	{ 22.25 .26 }	22.25 _s	-0.16

Table XII shows the effect of wave form upon the ratio of a current transformer. In this case a portable dynamometer ammeter was connected in the secondary circuit, and the current adjusted until its needle coincided with a division. The error here was not greater than 0.05 per cent. In the primary circuit a mirror dynamometer was included and its reading taken at the same time. The wave form was varied in the manner already described.

The effect is seen to be inappreciable except with a large component of harmonic, and would be negligible for all practical purposes unless the component of harmonic exceed half the value of the fundamental.

To exclude the possibility of the change being in the instruments and not in the transformer, the portable dynamometer was placed in series with the mirror dynamometer in the primary circuit and readings taken with the same current in each. These showed no discrepancies beyond the error in reading for the most distorted wave.

Transformer L is a duplicate of transformer K.

For other methods of measuring transformer ratios and phase-angles, we refer the reader to the works of Robinson,⁷ Drysdale,⁸ Makower,⁹ and Sumpner.¹⁰

We are indebted to Messrs. C. E. Reid, J. V. S. Fisher, and G. W. M. Vinal for assistance in taking the observations.

WASHINGTON, February 25, 1909.

⁷ L. T. Robinson, *Trans. A. I. E. E.*, **25**, p. 727; 1906.

⁸ C. V. Drysdale, *Electrician*, **58**, pp. 160, 199; 1906: *Phil. Mag.* **16**, p. 136; 1908.

⁹ A. J. Makower, *Electrician*, **58**, p. 695; 1907.

¹⁰ W. E. Sumpner, *Phil. Mag.*, **9**, p. 155; 1905.

THE DETERMINATION OF THE MAGNETIC INDUCTION IN STRAIGHT BARS.

By Charles W. Burrows.

CONTENTS.

	Page
I. STATEMENT OF THE PROBLEM.....	32
II. THEORY OF A DISTRIBUTED MAGNETOMOTIVE FORCE.....	33
III. PRELIMINARY INVESTIGATIONS.....	35
1. Yoke Reluctance.....	35
2. Flux Distribution in the Iron.....	39
3. Leakage Flux in Air.....	43
4. Reactive Force of the Yokes.....	45
5. Curved Yokes.....	48
6. Double Compensation.....	50
7. Various Forms of Yokes.....	52
8. Uniformity of Specimen.....	59
9. Distribution of Test Coils.....	62
IV. DIRECT READING METHOD OF MEASURING THE MAGNETOMOTIVE FORCE.....	63
V. THE MAGNETIZING COILS.....	69
VI. MEASUREMENT OF THE INDUCTION.....	70
1. The Galvanometer.....	70
2. Mutual Inductance.....	72
3. The Current.....	73
4. Test Coils.....	73
5. Cross Section of Specimen.....	73
6. Air Flux.....	74
7. Zero Method.....	75
8. Variable Mutual Inductance.....	78
VII. FULL SET-UP AND OPERATIONS.....	80
1. For Rings and Uncompensated Bars and Yoke.....	84
2. Compensated Bar and Yoke.....	85
VIII. GENERAL CONCLUSIONS.....	87

The present paper is a description of a method of making precision magnetic measurements in use at the Bureau of Standards, and also of some of the investigations which were made during the development of the method.

I. STATEMENT OF THE PROBLEM.

The problem of the measurement of the magnetic induction in a given specimen consists in the simultaneous determination of the magnetic flux density and the magnetizing force at a given point. The difficulties depend on the form of magnetic circuit. If the magnetic circuit is a toroid in which the thickness of metal is small compared to the diameter of the toroid, the magnetic measurements are easily made, because the magnetic flux density and force are approximately uniform over any cross section. If the test specimen is a long rod whose diameter is small compared with its length, the magnetic measurements for points distant from the ends are likewise readily made. In either case the magnetizing force is calculated from the current-turns of a uniformly wound magnetizing coil. The magnetic flux may be measured in terms of the quantity of electricity which flows through a circuit containing a few test turns wound over the specimen, when the magnetizing force is reversed. It may be measured also by comparing the electromotive forces developed in the test coil and the secondary of a mutual inductance when the magnetizing current and the primary current of the mutual inductance are reversed simultaneously. In practice, however, it is not convenient to have the test specimen in either of the forms just mentioned. Most convenient mechanically is a comparatively short rod of uniform cross section which may be easily machined. If we use a short rod, however, the material under test does not form the whole of the magnetic circuit, and we can no longer assume that the test material comprises the total reluctance of the magnetic circuit. In some cases the greater part of the total reluctance may be in the air path. In a particular case of a cylinder 25 diameters long whose permeability is 3770 ($= 300 \times 4\pi$) when the magnetizing force is unity, the reluctance of the air path is 19 times as large as that of the specimen.¹ In order to reduce the reluctance of the magnetic circuit, the ends of the bar may be connected by a soft iron yoke. With carefully designed and constructed yokes, this gives a very satisfactory magnetic circuit, and as a rough approximation we may assume

¹ Assuming Mann's values for demagnetizing factors.

that the whole applied magnetomotive force is used to magnetize the test specimens. The reluctance of the yokes must, however, be taken into consideration. This may be done in two ways. Using Ewing's double yoke method² with two different lengths between yokes of the test specimens and magnetizing coils, one can eliminate the reluctance of the yokes, assuming it to be the same in each case for corresponding values of the induction, and so obtain the true induction curve.

A second method of obtaining data free from errors due to the yokes is to apply to the yokes and joints a compensating magnetomotive force which shall overcome the reluctance of these parts of the magnetic circuit. When properly compensated, all parts of the magnetic circuit are at the same magnetic potential and consequently there is no magnetic leakage from one part of the circuit to another. This idea of a distributed and adjustable magnetomotive force was suggested to me by Prof. E. B. Rosa about two years ago. I am indebted to him not only for the original suggestion but also for his hearty cooperation throughout the course of the investigation. The theory of this compensating magnetomotive force is made clear from a consideration of the magnetic condition of a straight bar surrounded by a magnetizing solenoid.

II. THEORY OF A DISTRIBUTED MAGNETOMOTIVE FORCE.

In order that there shall be no magnetic leaking anywhere in the rod, it is necessary that the magnetic potential of the rod be everywhere the same. That is, that in the infinite rod the rise of magnetic potential in each element of length due to the current must be exactly equal to the fall of magnetic potential in that element due to the reluctance of the iron. If the iron is of uniform permeability and constant cross section, and the ampere turns on the solenoid be uniform throughout the length, then the rod will be of uniform magnetic potential.

If, however, between *A* and *B* (Fig. 1) the iron is of lower permeability than the average or its section is a little less, the fall of magnetic potential will be a little greater between *A* and *B* than

² Ewing's *Magnetic Induction in Iron and Other Metals*; 3d ed., p. 362.

otherwise, and B will be at a lower magnetic potential than A . The result is a leakage of lines from A to B . On the other hand, if the iron between A to B has a higher permeability than elsewhere, or has a slightly larger section, then the fall of potential from A to B will be less than otherwise, and the magnetic potential at B will be higher than at A and the leakage will be in the reverse direction from B' to A' . In like manner, if the iron is perfectly uniform in section and permeability, but the winding of the magnetizing coil is a little more open between A and B than elsewhere, so that there is a slightly less number of current-turns per cm there than elsewhere, then B will be at a lower magnetic potential than A and a leakage will occur from A to B . If, however, the wire is wound closer, then B will be at a higher magnetic potential and the leakage is reversed from B' to A' . Thus,

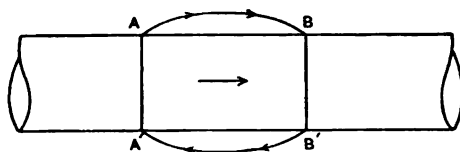


Fig. 1.

in general, in such a magnetic circuit, or in any magnetic circuit, in order that there shall be no magnetic leakage the magnetomotive force due to every element of winding must be just sufficient to overcome the reluctance of that element of the magnetic path within the winding. This requires in practice extreme care in the preparation of the magnetizing coils for the case of uniform rods, and very nice adjustment of the magnetizing windings in the case of nonuniform circuits, which are, of course, what we generally have.

Fig. 2 illustrates this distribution of magnetomotive force. We have here a uniform straight rod with its ends joined by a yoke of low magnetic reluctance, and magnetizing turns wound uniformly over the rod and distributed in proportion to the reluctance over the yoke and the joints. We thus secure the same uniformity of induction and magnetizing force for the straight element of the magnetic circuit that we had in the infinitely long rod or in the uniformly wound ring. We may now calculate the magnetic force

at any point of the straight bar from the magnetomotive force of a unit length of the surrounding solenoid.

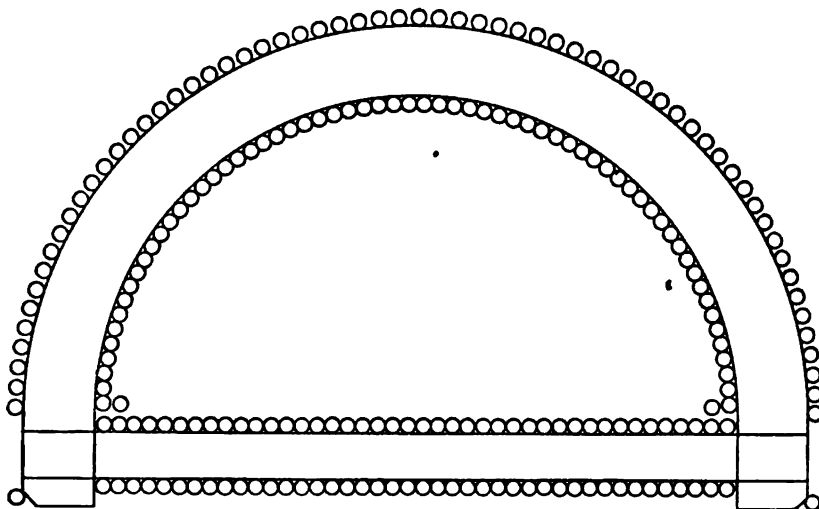


Fig. 2.—Illustrating a Distributed Magnetomotive Force.

The present paper has to do with magnetic circuits of this kind in which the reluctance of various parts of the circuits is overcome by properly placed current-turns.

III. PRELIMINARY INVESTIGATIONS.

1. ON YOKE RELUCTANCE.

As this method involves the distribution of magnetomotive force in proportion to the reluctance of the magnetic circuit, it is necessary that this reluctance be known. Further, since magnetic reluctance at any part of a magnetic circuit without a proportional magnetomotive force will give rise to magnetic leakage, and leakage is a quantity susceptible of direct measurement while reluctance is not, we shall begin the investigation with a study of magnetic leakage. For this purpose two rods surrounded by two equal and uniform solenoids, joined together at their extremities by heavy soft iron rectangular yokes, formed the magnetic circuit. The joints fit closely, so that the circuit is practically that of the Ewing double yoke² and double bar arrangement. Test coils containing the same number of turns are placed around the middle section of one

² Ewing's Magnetic Induction in Iron and Other Metals; 3d ed., p. 362.

of the bars and one of the yokes. The bar surrounded by the test coil we shall call the test bar and the other the auxiliary bar. These test coils may be connected so that the electromotive forces developed on reversal of the magnetizing current oppose each other, or they may be used independently. In this way the induction and leakage for various magnetizing forces are readily obtained. Then a single turn of wire is wound around each yoke and connected in series with the main magnetizing coils. With the magnetizing current flowing through these compensating turns and the main solenoids in series, the inductions and leakages are again determined in the same manner as before. This operation is repeated

TABLE I.

Showing the effect of leakage in the double bar and yoke magnetic circuit.

H	Uncompensated B at center	Leakage	Per cent of leakage.	Leakage+H	Number of turns to compensate	Column (5) + Column (6)	Reluctance of Yoke in Terms of equivalent Length of Test Bar
10	13000	440	2.9	44	5.8	7.6	5.8
20	14680	620	4.2	31	4.3	7.2	4.3
30	15580	780	5.0	26	3.6	7.2	3.6
40	16200	820	5.1	20	3.0	6.7	3.0
50	16670	830	4.9	17	2.5	6.8	2.5

Constants of apparatus:

Length of rods, 27 cm.

Length between yokes, 12.85 cm.

Section of rods, .277 cm.²

Length of yoke between rods, 2 cm.

Section of yokes, 1.5 x 1.5=2.25 cm.²

Primary turns on each rod=100.

Secondary turns, 50 on bar and 50 on yoke.

The number of turns to compensate is determined by interpolation in Fig. 3. The equivalent reluctance of yoke (column 8) is computed by dividing the total magnetomotive force applied to one yoke by H . It therefore expresses the length of rod to which the yoke is equivalent.

with other numbers of compensating turns about the yokes. The results are shown in Table I and Fig. 3.

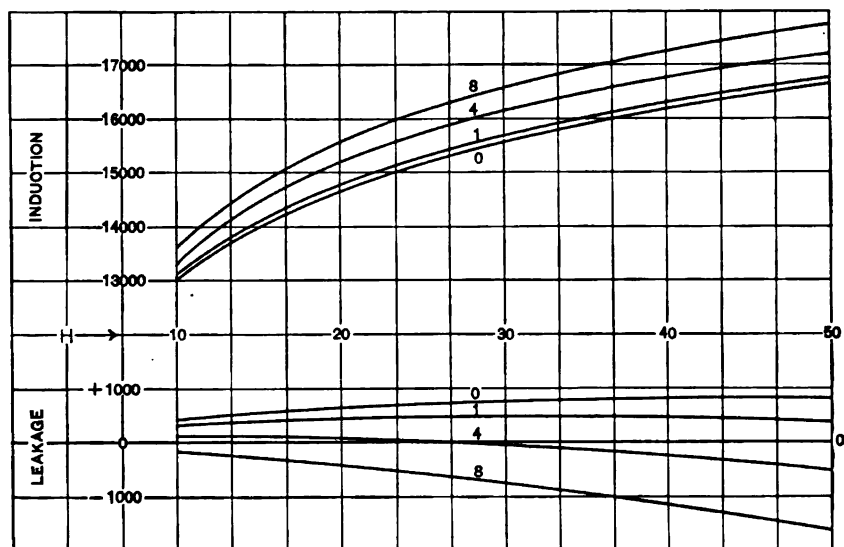


Fig. 3.—Showing the Induction at the Center of Test Rod, and the Leakage between Center of Rod and Center of Yoke for Various Inductions and Compensating Turns wound on the Yokes. (To accompany Table I.)

Curves 0 for no compensating turns.
 Curves 1 for one turn on each yoke.
 Curves 4 for four turns on each yoke.
 Curves 8 for eight turns on each yoke.

Table II shows the data of another experiment carried out with the same purpose as the preceding, but from a different point of view. In this set-up two pairs of long rods of similar material are used with two different sets of magnetizing coils whose lengths are in the ratio of 2 : 1, but whose other dimensions are equal. Test coils are placed one on the middle of the bar and one distributed with one-half over each end of the bar. Distributing the test coil tends to eliminate any irregularities due to the bar or solenoid. The compensation is effected by coils of 50 turns each placed over the yokes. Through these coils is passed a current which is adjusted independently of the main current.

From these two sets of data and the curve we notice that the leakage between the center of the bar and the center of the yoke,

and the compensating current-turns necessary to reduce this leakage to zero, are proportional and increase continuously as the magnetizing force increases. Furthermore, the reluctance of the yoke does not bear a fixed ratio to the reluctance of the test piece, but decreases relatively as the magnetizing force increases. Consequently the ratio between the compensating and main magnetizing turns varies, and as seen in the figure a fixed ratio of compensating turns which compensates exactly at some magnetizing force is too small for smaller magnetizing forces and too large for larger ones.

TABLE II.

Showing the magnitude of the compensation for leakage in the double bar and yoke magnetic circuit at two different lengths.

H	B Approximate	Permeability	l=42 cm		l=21 cm	
			Compensating Current	Equivalent Reluctance (cm of rod)	Compensating Current	Equivalent Reluctance
3	3110	1040	0.18	3.8	0.26	5.5
4	5370	1340	0.26	4.1	0.34	5.3
5	7240	1450	0.32	4.8	0.41	5.2
6	8800	1470	0.38	4.0	0.48	5.0
7	9750	1390	0.43	3.9	0.55	5.0
8	10750	1340	0.48	3.8	0.61	4.8
9	11400	1270	0.52	3.6	0.67	4.7
10	12080	1210	0.55	3.5	0.72	4.5
20	15010	750	0.82	2.6	1.08	3.4
40	16430	410	1.12	1.8	1.44	2.2
70	17350	250	1.53	1.4	1.90	1.7

Constants of apparatus:

Length of rod, 45 cm.

Cross section of rod, .277 cm².

Lengths between yokes, 42 and 21.

Yokes same as in Table I.

Test coils 50 turns on middle and 50 turns on end of bar.

Compensating turns on each yoke, 50.

The equivalent reluctance is obtained by dividing the total magnetomotive force about one yoke by H .

These data are quite in accord with what we might expect from a consideration of the relative values of the reluctances of the test bars and yokes. The yokes having a cross section 9 times that of the bars are worked under a much lower flux density always below that corresponding to the maximum permeability. The reluctance of the yokes therefore decreases as the magnetizing force increases. The reluctance of the test pieces passes through a minimum and then increases continuously. Consequently the reluctance of the yokes will be of less importance at the higher inductions.

2. FLUX DISTRIBUTION IN THE IRON.

In the preceding two experiments it has been shown that in a particular case of the double bar and yoke magnetic circuit the leakage between the centers of bar and the yoke amounts to from 2 to 5 per cent, and may be reduced to zero by placing about each yoke current-turns which vary from 5 to 25 per cent of the main magnetomotive force. It now remains to show to what extent the flux density varies in other parts of the magnetic circuit. For this purpose several test coils were distributed over various sections of the magnetic circuit. The magnetomotive force was applied through two solenoids over the rods and two yoke coils over the yokes.

Table III gives data on the distribution of total flux and leakage at different parts of the magnetic circuit for different values of the magnetizing force and for different lengths of rod. These three pairs of rods are of the same lot of low-carbon Bessemer steel, but nevertheless care must be exercised in making comparisons. The three sets of data may not be intercompared too closely, because no correction has been applied for slight changes in the galvanometer constant on different days.

Here we may note:

For any uncompensated system:

The leakage between the center of the bar and the center of the yoke, as well as the leakage between the center and end of bar, is not proportional to the total induction, but passes through a maximum roughly in the region of maximum permeability.

The maximum total induction in the iron occurs through the

middle section of the test bar for low inductions and near the end of the test bar for higher inductions. This anomaly will be discussed later (see p. 46).

TABLE III.

Showing magnitude of flux and leakage in the double bar and yoke apparatus, for different values of the main and compensating magnetomotive forces.

H	Deflections at C			Leakage				
	No Compensation	Compensated for		No Compensation		Compensated for		
		Yoke	End of Test Bar	Center—End	Center—Yoke	Yoke	End	
						Center—End	Center—Yoke	
>length=47	3	5.49	5.70	5.95	+0.29	+0.71	+0.16	—0.66
	5	20.30	20.86	21.43	+0.70	+1.62	+0.37	—1.78
	7	28.99	29.20	29.48	+0.73	+1.75	+0.40	—2.48
	10	34.54	34.58	34.60	+0.50	+1.45	+0.31	—2.84
	20	40.18	40.15		—0.07	+0.59	—0.09	
	40	43.31			—0.44	—0.16		
>length=25	3	3.96	4.13		+0.00	+0.27	+0.03	
	5	18.67	19.37	21.25	+0.10	+0.58	+0.10	—1.63
	7	27.81	28.23	29.31	+0.16	+0.76	+0.11	—2.27
	10	33.92	34.02	34.30	+0.15	+0.82	+0.09	—1.50
	20	40.12	40.06		—0.11	+0.56	—0.14	
	40	43.41			—0.42	—0.09		
>length=15	60	44.80			—0.66	—0.52		
	3	3.97	4.08	5.60	+0.09	+0.09		—0.60
	5	15.30	16.10	24.04	+0.25	+0.28	+0.24	—3.01
	7	25.10	25.72	30.45	+0.38	+0.41	+0.33	—3.70
	10	32.20	32.50	34.69	+0.46	+0.47	+0.39	—4.04
	20	39.05	39.09	39.61	+0.34	+0.32	+0.31	—3.81
>length=15	40	42.80		42.80	+0.04	—0.18		—0.60
	60	44.32			—0.20	—0.58		

NOTE.—Two test coils are placed over the test bar, one over the middle and one near the end. A third test coil surrounds the yoke. Compensation is secured by adjusting the current in fixed coils about each yoke.

For the higher inductions, the total flux through the middle section of the bar is less than the total induction through the end of bar, or through the yoke.

In the compensated system:

The change in total induction over the middle section of the bar, due to excitation of compensating turns, is greatest at lower inductions and practically negligible at higher inductions.

At low inductions a much greater induction results if the inductions through sections at the end and middle of the rod are brought to equality by compensation than if sections through the yoke and middle of the bar are used.

In raising the induction at the end of the bar up to that at the middle section a large overcompensation occurs at the yoke. This overcompensation passes through a maximum, and then diminishes rapidly with increase of induction. Other experiments on strips of transformer iron give similar results.

In seeking a solution of the anomalous flux distribution at high flux densities an examination was made of a great variety of magnetic circuits. Figs. 4 and 5 show some of the data thus obtained.

Fig. 4 shows the flux magnitude and distribution along a rod for different conditions of the magnetic return path outside the rod itself. The legend of the figure gives the condition under which the individual flux curves were taken. In this figure we can see how the total induction and the leakage varies with the yoke contact.

The compensated induction (curve 8) is practically constant throughout the length of the rod, and the other curves show, by comparison with it, the total reduction of flux due to the reluctance of the yokes and joints.

Fig. 5 shows the leakage of flux at all points between the center and end of a test specimen 40 cm long between yokes. The yoke consisted of a three-piece return circuit such as is shown in Fig. 10. Here the yoke effect is very marked, due to the great difference in cross section of specimen and yoke. It is to be observed that the uniformity is greater for $H=1$ and 3 than it is for $H=2$, so that a maximum nonuniformity occurs in the region of maximum permeability of the test piece.

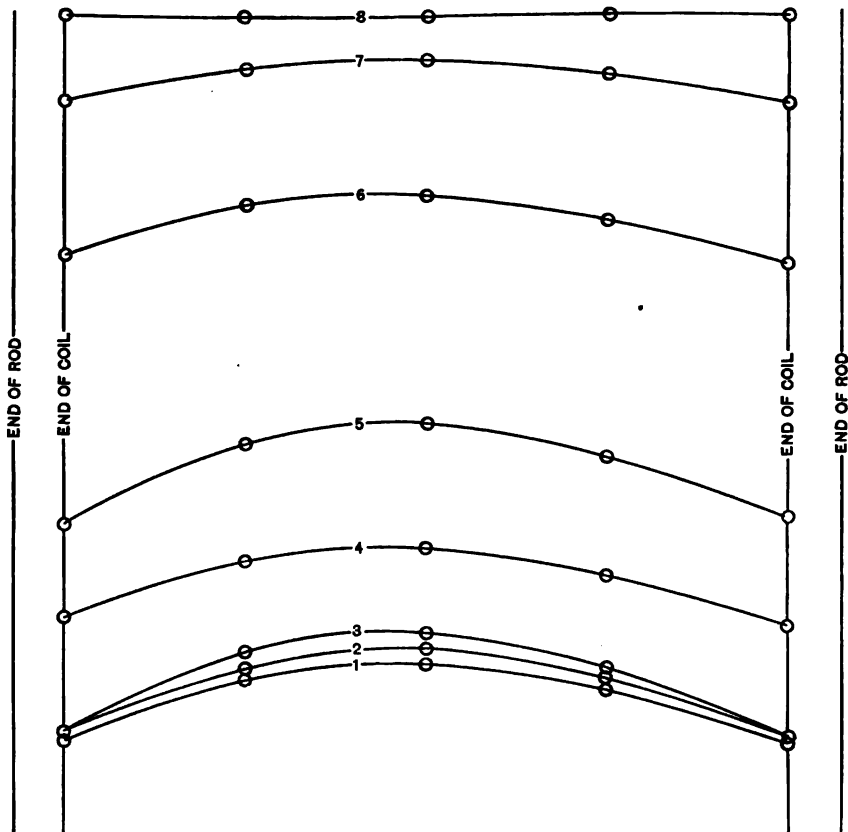


Fig. 4.—Showing the Magnitude and Distribution of Flux along the Length of a Rod 47 cm Long under Varying Conditions. Main Magnetizing Current Constant.

Curves 1–4 were taken with a second similar bar placed parallel to the first, and 6.5 cm distant.

Curve 2: Test bar alone is magnetized.

Curve 1: Both bars magnetized, with like poles adjacent.

Curve 3: Both bars magnetized, with unlike poles adjacent.

Curve 4: Same as 3, but with massive yokes butting against ends.

Curve 5: Same as 4, but with rods 2 cm apart.

Curve 7: Same as 6, but with better fitting rectangular yokes.

Curve 6: Same as 5, but with rods clamped in yokes.

Curve 8: Same as 7, but with compensating turns on yokes.

In each curve the leakage is less near the yoke than a little farther away, thus seeming to indicate the presence of a magnetomotive force in the yoke itself.

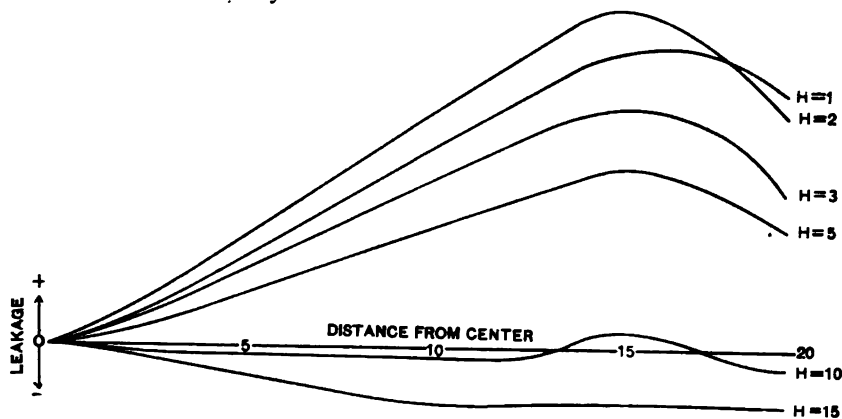


Fig. 5.—Showing Leakage under Various Magnetizing Forces in a Thin Strip of Transformer Iron Clamped in a 3-piece U-shaped Yoke.

For the higher magnetizing forces this magnetomotive force predominates and the middle section of specimen is a minimum for the total flux.

3. LEAKAGE FLUX IN AIR.

Up to this point the leakage has been studied as a loss of flux from the bar into the air. The distribution of this leakage in the air is of interest and was investigated by means of a small magnetic needle. In order that the needle might not disturb the field appreciably, it was made as small as it could be conveniently. It consisted of 3 mm of steel broken from the pointed end of a small sewing needle, and was suspended by a single cocoon fiber.

Fig. 6, (a), (b), and (c), shows by the arrowheads the direction of the magnetic field in air. The small arrowheads separated by dashes show the direction of flux in the iron. The circles indicate uncertainty in the direction of field. In (a) no compensating current was used, but in (b) and (c) a current was passed through fixed turns about the yoke till the flux was the same at the center and ends of bar. (b) and (c) differ only in the direction of magnetizing forces and leakage. Fig. 6 (d) shows more fully the flux in the air

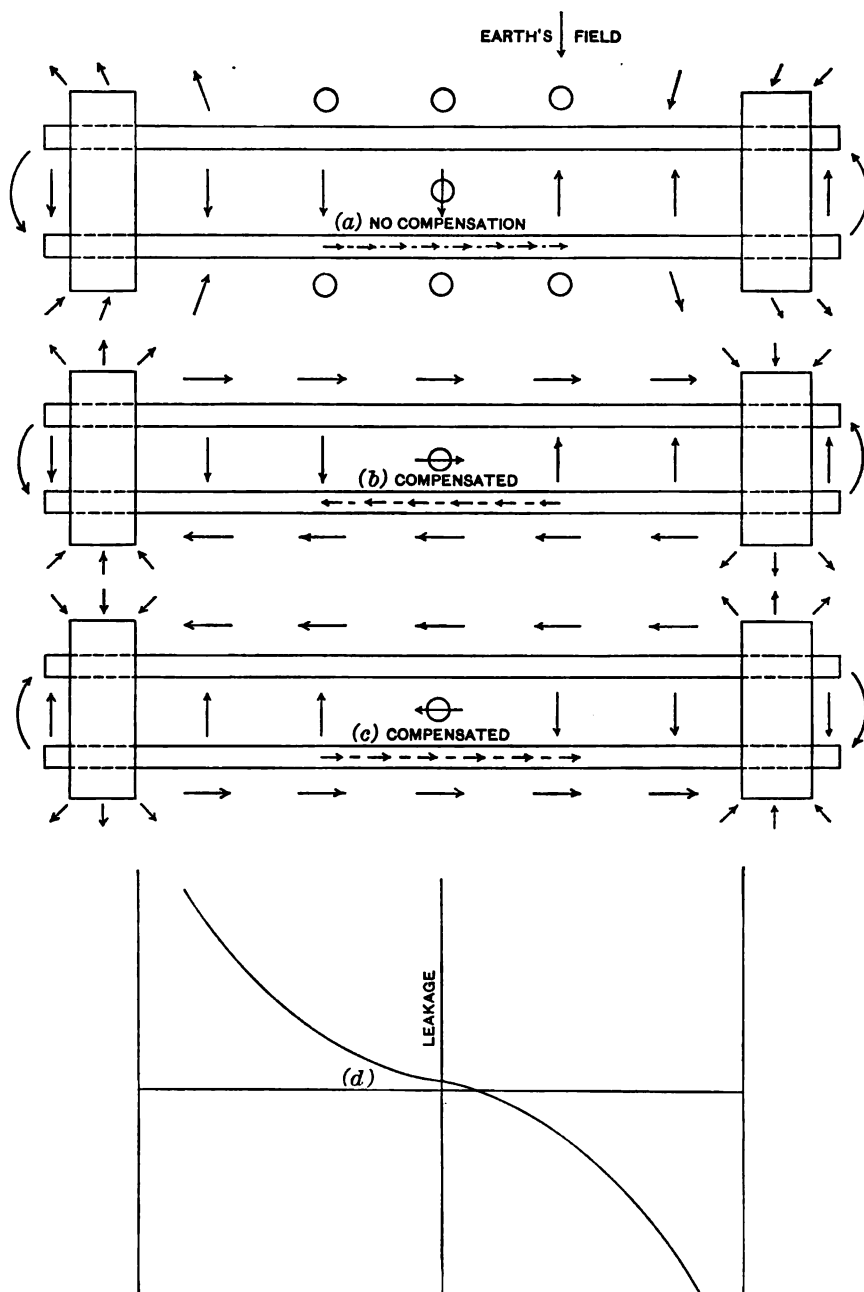


Fig. 6.—Showing the Distribution of Leakage in the Double Bar and Yoke Apparatus, both with and without Compensating Turns wound on the Yokes. Equal Solenoids over each Rod.

space between the two specimens. To obtain this last diagram, a test coil of 50 turns, 2.5 cm in diameter, was held at various positions while the magnetizing current was reversed and the ballistic throw noted. From these figures we observe that with no compensating magnetomotive force the external field is similar to that of two magnets placed with unlike poles together. The yokes act merely as enlargements of the ends of the test bars. The field outside the rectangle is weak and the pole length somewhat less than the distance between the yokes. When compensated by coils placed on the yokes so that the end of test rod carries the same flux as the middle section, the direction of field between the bars is reversed, while the field outside the rectangle and near the rods is strongly defined and in the same direction as the induction within the bars. In general, the external field is due to magnetization of the yokes by the compensating current.

It is evident that the field at the middle section of the test specimen is greater than that due to the magnetizing solenoid alone, and that this method of compensating is not free from objection. It is also evident that a magnetic needle is not a reliable indicator of the uniformity of flux within the test specimen.

4. REACTIVE FORCE OF THE YOKES.

From the preceding it is evident that the yokes do have an appreciable influence on the distribution of flux within the bar. This is further evidenced by a consideration of Fig. 7, which represents a portion of a magnetic circuit. When the magnetizing current is flowing the magnetic flux has the direction indicated by the arrows. The yoke is here magnetized to an intensity which varies in magnitude from point to point, but has the same general direction as indicated by the arrows. It is shown in works on the theory of magnetism³ that such a magnetized system creates a field which at any point in space has the value

$$H_i = \Delta \int dv \left(I \Delta \frac{1}{r} \right)$$

where $I = kH$ = intensity of magnetization.

³Abraham: *Theorie der Elektrizität*, vol. 1, p. 228, 3d ed.

Applying this equation to the system of Fig. 7 we see that the force due to the yokes has a component of the same sign as the impressed force for all points within the solenoid, which increases with the section of the coil, the magnitude of the impressed magnetizing force, and the susceptibility of the yoke, and decreases with increase in the section of the specimen and the distance from the yoke. These same conclusions are reached by supposing the faces of the yokes opposite the ends of the solenoids to be covered with free magnetism of the signs indicated in the figure.

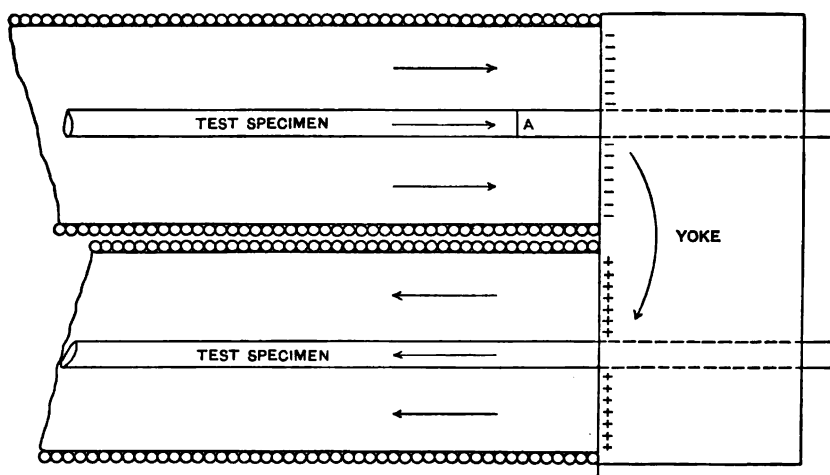


Fig. 7.

The resultant magnetic force is therefore composed of two parts—that due to the solenoid alone, which diminishes as the yokes are approached, and that due to the magnetization of the yokes, which increases as the yokes are approached. It would seem therefore possible, by varying the conditions, to make the force near the end of the specimen either greater or less than the force at the center. The anomalous flux distribution may therefore be accounted for by supposing that the yoke effect predominates. To settle the matter beyond doubt, other experiments using test specimens and yokes of various cross sections were made. In these experiments the test coils were wound close upon the test pieces and the magnetizing coils were free from any appreciable irregularity. To eliminate any effect due to differences in the two test pieces, the current through each solenoid was adjusted so that

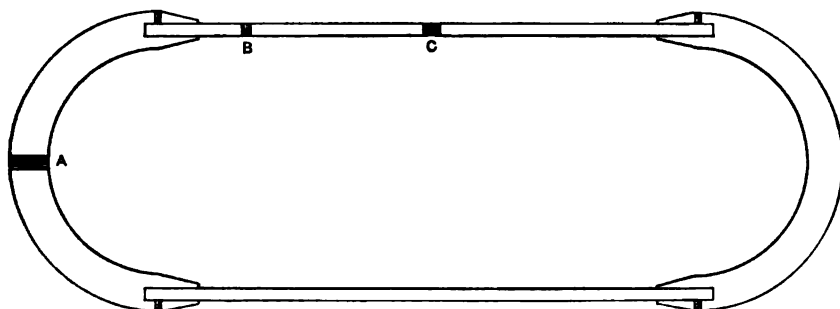


Fig. 8.—Showing the Arrangement of Magnetic Circuit and Test Coils as used in determining the Data of Tables IV and V. The Magnetizing Coils consist of a Uniform Solenoid over each Bar, Uniform Windings over the Yokes, and Concentrated Windings over the Joints. Test Coils are placed at A, B, and C.

TABLE IV.

Showing the flux and leakage distributions for double bar and yoke apparatus using curved yokes as shown in Fig. 8 (length between yokes = 40 cm).

H	Deflections			Leakage			
	No Compensation	Compensated for		No Compensation		Compensated for	
		Yoke C-A	End C-B	C-B	C-A	Yoke C-B	End C-A
1	0.57	0.57	0.57	0.03	0.18	0.00	0.00
2	1.10	1.14	1.14	0.06	0.29	0.00	0.00
3	5.86	7.16	6.76	0.31	1.25	- 0.15	+ 0.39
4	12.28	13.94	13.75	0.48	1.76	- 0.05	+ 0.34
5	18.41	19.92	20.31	0.76	1.92	+ 0.24	- 0.51
6	23.61	24.72	25.31	1.04	2.08	+ 0.42	- 1.12
7	27.30	28.14	28.69	1.22	2.22	+ 0.55	- 1.43
8	30.27	30.83	31.18	1.29	2.34	+ 0.57	- 1.47
9	32.27	32.68	32.98	1.21	2.25	+ 0.50	- 1.48
10	34.01	34.19	34.33	1.11	2.19	+ 0.48	- 1.54
20	40.26	40.21	40.26	0.29	1.23	+ 0.14	- 0.71
40	43.84	43.80	43.80	0.02	0.60	0.00	0.00
70	45.63		45.63	0.00	0.16		+ 0.16

Constants of apparatus as shown in Fig. 8.

the flux densities in the two specimens were equal. The investigation showed that for specimens of much smaller section than the solenoid, the excess of the flux density of the middle section over that of a section nearer the end decreased as the yokes of larger size were used, and finally became negative. For larger specimens, which filled the solenoids, this reversal did not occur.

5. CURVED YOKES.

With a view to keeping a greater proportion of the flux within the iron and reducing the disturbing leakage fields, curved yokes such as shown in Fig. 8 were designed. These yokes have the advantage of producing a flux which as it leaves the yoke has the same direction as the flux developed in the test rods. Furthermore, it is easy to so distribute magnetizing turns over the yoke that the sections of greater reluctance have a greater magnetizing force. In these yokes the winding over the main part of the yoke is uniform, while over each nose it is more closely wound. As before, the compensating current is adjusted independently of the main current. Table IV shows some data taken with this apparatus, using rods 40 cm long between the yokes.

With no compensating current the flux is a maximum at the middle of the test rod and a minimum at the middle of yoke. The anomalous flux distribution along the test rod, which was noted in the rectangular yokes, has here disappeared. When the compensation is adjusted so that the middle and end sections of the test bar are crossed by the same total flux, there is an overcompensation as far as the flux in the yoke is concerned. This gives rise to a field at the middle of specimen, in addition to that developed by the coil.

Table V shows the magnitude of the compensating currents required in the preceding experiment. Notice here that the compensating current does not increase continuously, but passes through a maximum for each form of compensation. The equivalent length of the yoke passes through a maximum, and at the highest magnetizing force used has the extremely low value of 1 mm. Experiments on other lengths of specimen give results substantially in accord with the preceding.

TABLE V.

Showing the compensation required and the equivalent reluctance of the curved yokes.

H	Compensating Current			Equivalent Length of Yoke
	When C=A	When C=B	Sum	
1	0.010	0.010	0.020	1.31
2	0.020	0.020	0.040	1.31
3	0.081	0.057	0.138	3.0
4	0.096	0.084	0.180	2.9
5	1.07	1.38	2.45	3.2
6	1.13	1.85	3.08	3.5
7	1.25	2.12	3.37	3.1
8	1.29	2.19	3.48	2.6
9	1.30	2.21	3.51	2.6
10	1.23	2.23	3.46	2.3
20	0.80	0.92	1.72	0.6
40	0.34	0.00	0.34	0.1

Fig. 9 shows the distribution of flux along one-fourth the length of the magnetic circuit, consisting of two rods and two curved yokes. The circles show where measurements were taken. In each curve there are two circles containing crosses indicating that at these two points the fluxes are equal. Curve 2 of this figure shows the normal flux distribution under the action of the main magnetizing coils alone. The other curves were obtained by adjusting to equality the fluxes at the two sections indicated. Here we observe that when the system is magnetized by the main solenoids and no compensation is used, the maximum of flux (in this particular case) occurs between the center and the end of test rod. To secure equality of flux across the middle section and any other section between it and the section of maximum flux it is necessary to reverse the current in the coils surrounding the yokes. The resulting indications at all points are lowered and the nonuniformity throughout the magnetic circuit beyond the sections of compensation is increased. To secure equality of flux across the middle section and any section beyond the section of maximum flux, direct compensation is applied. As a result the total

induction at all points is raised and the uniformity is improved. A comparison between the dotted lines representing the induction under full compensation and the other curves shows the relative magnitude of the true and apparent inductions under the various conditions.

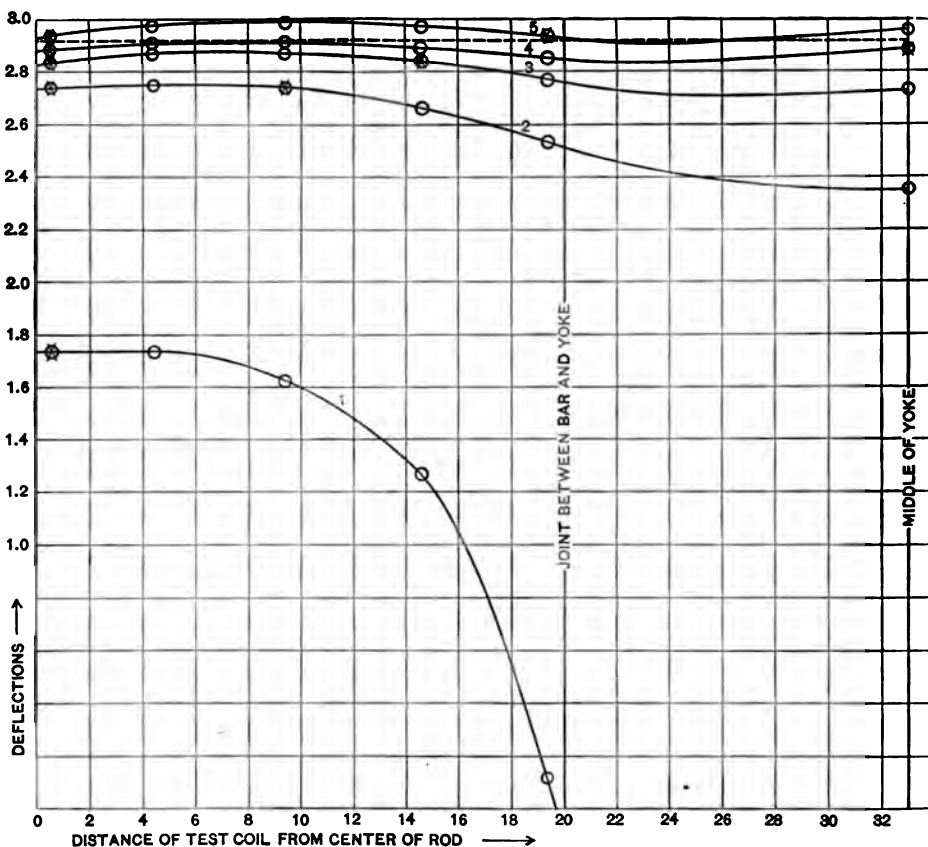


Fig. 9.—Showing the Distribution of Flux along a Double Yoke and Rod Magnetic Circuit with Various Degrees of Compensation. The Crosses on each Curve indicate the two Cross Sections, the Fluxes through which are made equal. The Solid Curves are for Compensation on the Yokes only. The Straight Dotted Line is, for Compensation on the Yokes and over the Joints, Adjusted Separately.

6. DOUBLE COMPENSATION.

From what has preceded, it is quite obvious that it is impossible to secure the same flux across every cross section of a ferromagnetic circuit by means of a main magnetizing current and a single

compensating current of the same relative distribution for all inductions. To secure better uniformity of flux, the compensating turns were divided into two sections and the current through each portion was adjusted separately. With one compensating coil wound over the yoke and a second wound over the test piece as close to the joint as possible, a quite uniform flux was secured throughout the length of the rod. If the rods were of identical magnetic properties, the compensating turns about the joints gave results practically as good as the double compensation. The difference in permeability between the two rods which are supposed to be alike (i. e., cut from the same stock length or from adjoining portions of the same sheet) and the fact that the bars constitute the greater portion of the reluctance of the circuit, may cause as much trouble in compensating as the total reluctance of the yoke. For this reason it was found desirable to divide the total applied magnetomotive force into three sections: (1) a uniform solenoid over the test piece, (2) a similar uniform solenoid over the auxiliary bar, and (3) a set of four short coils wound over the ends of the rods and connected in series. These three components of the total magnetomotive force are capable of independent adjustment. The currents are adjusted until the fluxes across the middle sections of the test and auxiliary bars and across the end sections of the test bar are equal. When these adjustments have been made it is found that the flux through the yokes does not differ materially from uniformity and the change in induction at the center of the test piece when this outstanding nonuniformity is compensated for is inappreciable. From a practical standpoint, it is too laborious to adjust more than three independent currents in securing uniformity of flux. With two adjustable currents we have seen that the flux is not sufficiently uniform when the magnetizing coils about the test and auxiliary rods are in series and adjusted as a unit, while a compensating current about the yokes is adjusted separately. Other experiments, not recorded, show that a little better uniformity is secured if the compensating coils are placed over the ends of the rods as near as possible to the joints instead of being wound over the yokes. With three adjustable currents, we may have the two main magnetizing coils in series for one adjustment and yoke and joint coils for the other two adjustments, or we may adjust the

two main magnetizing coils independently and make the third adjustment on a joint coil. The latter gives better satisfaction than having the third adjustable coil wound over the yoke.

7. VARIOUS FORMS OF YOKES.

When the magnetizing currents about the two rods are adjusted separately, it becomes immaterial whether the rods are of the same magnetic properties. In fact, they are seldom of the same properties; hence the necessity of the separate adjustment. We are thus independent of the relative properties of the test and auxiliary rods. We may therefore replace the auxiliary rod by something more convenient. This was done in a variety of ways.

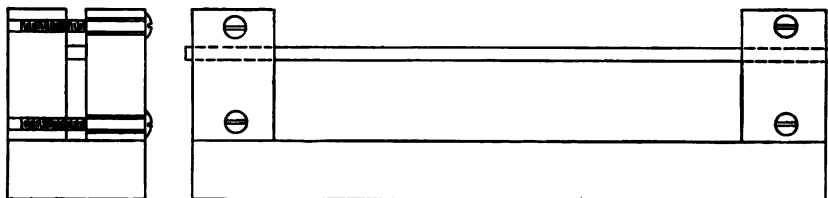


Fig. 10.—Showing a Magnetic Circuit in which the Ends of the Test Piece are Clamped in Two Small Yokes which in turn are Clamped to a Massive Soft Iron Base.

Figs. 10, 11, and 12 show three forms that were examined quite carefully. Fig. 10 is modified from the double yoke and bar apparatus by replacing the auxiliary rod by a massive bar faced off on its upper surface. A set of ordinary double yokes are faced off on one end so as to form a close joint with the yoke base. The test bar is clamped between the yokes in the usual manner. Magnetizing turns were distributed over the yoke base and horns, roughly proportional to the reluctance. This was not difficult, as they were made from the same large bar of Norway iron. A second compensating magnetizing coil was wound around the end of the specimen. Test coils for the adjustment of the compensation were placed over the middle and end of bar and middle of yoke base. This arrangement gave fair satisfaction, and was adjustable to any length of rod. Knowing that the projecting corners of the yoke horns are the sources of disturbing fields, the yoke horns were modified to the form shown in Fig. 11. Here the joint between base and horn is improved by increasing the surface. The jaws within which the specimen is clamped project

over the specimen in such a way as to permit the winding of compensating turns concentrically over yoke and rod. This tends to

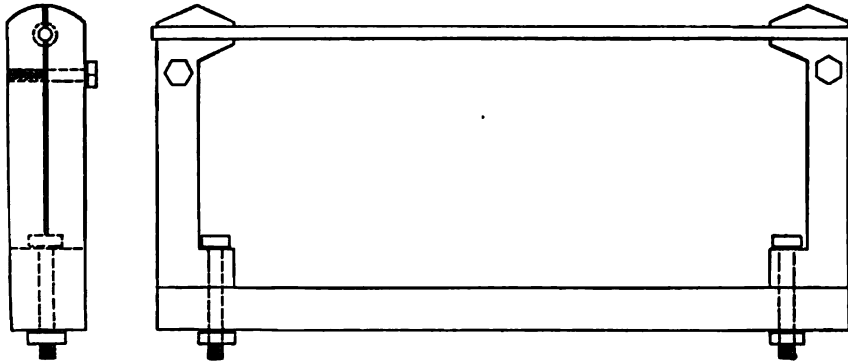


Fig. 11.—A Modification of the Yokes of Fig. 10 in which the Objectionable Corners have been Removed.

reduce the leakage field through the air. A third form is shown in Fig. 12. Here the yoke consists of two equal rectangles, with rounded corners, placed one over the other. Between these two yokes and along the line of the greater diameter is clamped the specimen. This apparatus is confined to specimens of one length only, but has the advantage of a more uniform yoke, unbroken transversely.

While these forms of magnetic circuit were all fairly satisfactory, they do not offer any marked advantage over the simple double bar and yoke circuit. Many forms of small yokes were tried. Here the aim was to improve the magnetic contact and to lessen the disturbing field due to local poles on the yokes.

Fig. 13 is one of a pair of the ordinary small yokes, such as used by Ewing. Here the contact is made by pressing the test bar against the walls of the yoke holes. Usually the rod is a trifle smaller in diameter than the hole which receives it, and the result is that the thumbscrew in forcing a good contact on one side of the rod destroys the contact on the other. Fig. 14 illustrates this point nicely. To reduce this trouble, yokes of the form of Fig. 15 were made. These yokes are in two pieces, and fitted together so that heavy machine screws draw the parts together. The bar comes in contact with the yoke along two surfaces, and the air space is less with these yokes than with the preceding, when made with the same care. A very good contact was obtained with yokes

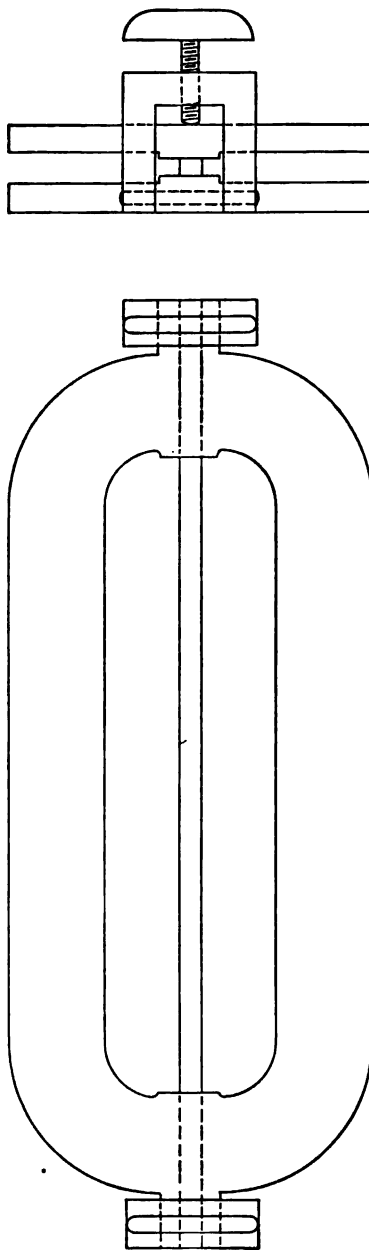


Fig. 12.—Another Modification of the Idea Involved in Fig. 10.

Here the uniformity of yoke reluctance permits an easy adjustment of compensating turns.

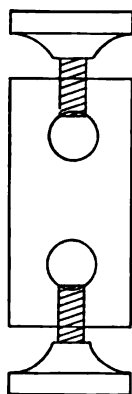


Fig. 13. —
Small Ewing
Yoke (to be
used in
pairs).

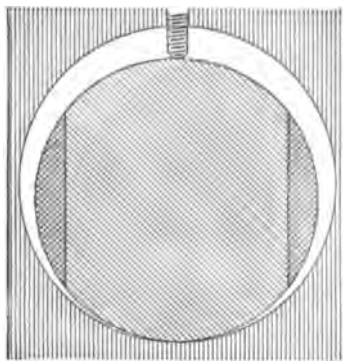


Fig. 14. — Showing the Contact of a Round Rod in a Round Hole, and held in Place by a set Screw. This Form is used in the Ewing Double Yoke, Ewing Bridge, etc. The Double Cross Hatching shows the Portion that may be Removed to Form Flat Surfaces for Parallel-jawed Yokes.

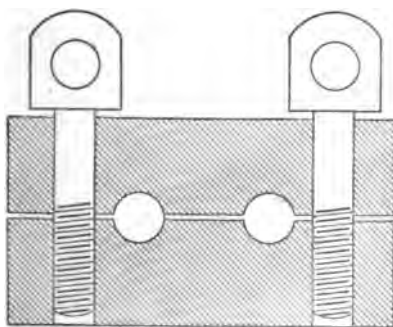


Fig. 15.

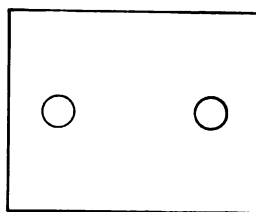


Fig. 16.

Modifications of Ewing Small Yoke to secure better Magnetic Contact.

of the form of Fig. 16. The holes of this set of yokes were made to fit a particular set of rods, and were very carefully machined to a driving fit. No set screws were used. Of course this is not practical for general use. Good contact in the curved yokes used in the set-up of Fig. 8 is secured by having the bottom and sides of the hole carefully machined. The rod, too, if necessary, is dressed off on the end. We thus have a contact on end and sides. In order to avoid denting the specimen, the set screw does not work directly against the specimen, but is separated from it by a small iron disk. This, from the purely magnetic standpoint, is the most satisfactory yoke used. It requires, however, a test specimen of a definite diameter and length. This is a practical objection.

In an effort to make the reluctance of the whole circuit as uniform as possible, semicircular yokes were made, having the same cross section as the test piece. These yokes did not offer any advantages to warrant their use, principally because the contact surfaces were too small.

If a round joint is made to fit any particular size of rod it is necessary to use bushings if a smaller rod is to be tested. In fact, some pieces of commercial apparatus are supplied with bushings for all sizes expected to be used. The use of bushings in that part

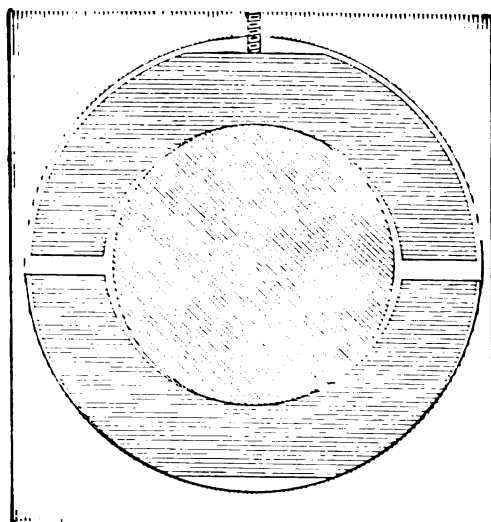


Fig. 17.—Showing the Contact of a Round Rod held in a Round Hole by means of Bushings.

of the magnetic circuit whose reluctance is to be small seems entirely unwarranted.

Fig. 17 shows the form of contact in one quite widely used commercial permeameter. As shown here, one bushing is entirely out of contact with the body of the yoke, and the other bushing has a very limited surface of contact. Between the specimen and the bushing, likewise, the contact is quite limited. Such unnecessary increase in the number of joints in the critical part of the magnetic circuit may be avoided by using a specimen of rectangular section. Fig. 18 shows a yoke designed for specimens with par-

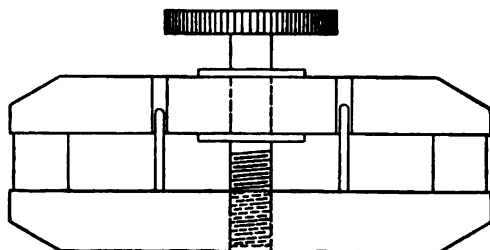


Fig. 18.—Showing the Perfect Contact of a Rectangular Specimen held between Plane Faces of a Parallel-jawed Yoke.

allel sides. Here a good contact is made regardless of the size of the specimen. The joint between the two parts of the yoke does not increase the magnetic reluctance because it is parallel to the direction of flux. The edges have been chamfered off in order to eliminate, as far as possible, disturbing pole effects. The single screw is an advantage over two screws, as it reduces the time required to insert and clamp the specimen. The two pins in one half of the yoke working in holes in the other half keep the two halves of the yoke in alignment. The shoulder soldered to the under side of the screw holds the jaws of the yoke apart for the reception of the test rod.

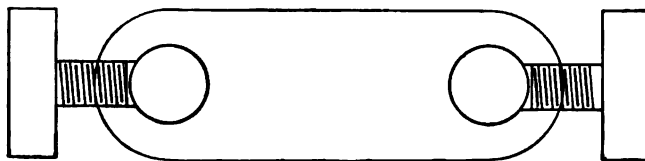


Fig 19.—Showing the Form of Yoke used with Round Specimens. The Thumb Screws are of Brass.

If round specimens must be used Fig. 19 shows a very satisfactory form. It is made of round iron, and the ends are turned down to hemispherical surfaces. The thumbscrews should be of brass.

It is possible, however, to use the same rod in a piece of apparatus which requires a circular section, and in another which requires a section of parallel opposite surfaces. A circular-sectioned rod may have two parallel plane surfaces machined on it. These plane surfaces would form the contact surfaces in apparatus of one kind, and the curved surfaces would give practically as good contact in round holes as the whole cylinder.

In Fig. 14 the double hatching shows a part of the rod that is to be removed. It is not necessary to machine the whole length of the cylindrical rod. Those portions of the ends which form the contacts are sufficient.

Transformer iron is tested in the form of strips. The shearing of the metal modifies the properties of the iron, and results obtained with narrow strips are misleading. Extermely wide test pieces would be free from any appreciable influence of the shears, but would present other practical difficulties. Strips 5 cm wide

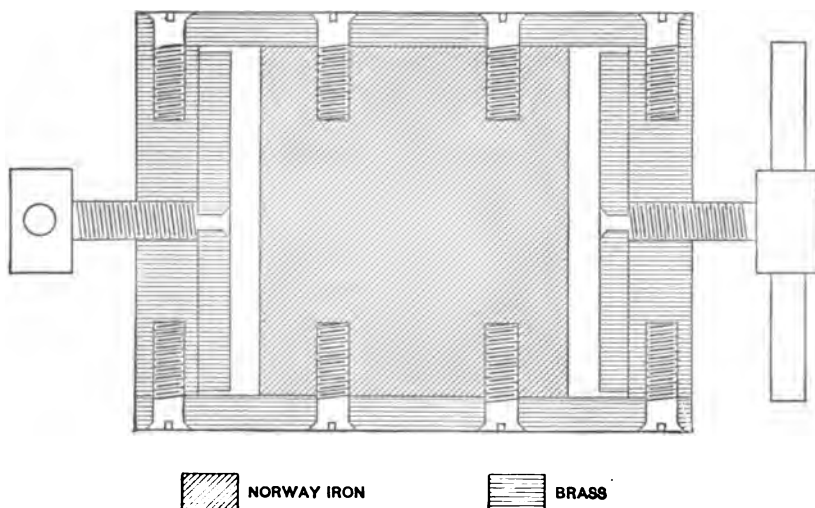


Fig. 20.—Showing the Form of Yoke used with Strips of Sheet Metal.

can be readily handled, and will give a fair approximation to the magnetic properties of the unsheared metal and are recommended as a standard width of test strip.

A convenient form of yoke for such strips is shown in Fig. 20. Here the outside portions of the yoke which form the clamp are made of brass. If made of iron it would produce a greater disturbance of the leakage and probably not improve the magnetic joint appreciably.

8. UNIFORMITY OF SPECIMEN.

There may be irregularities in the flux distribution other than those caused by the yokes and joints. Fig. 21 shows the flux distribution of a rod under different conditions of the complete magnetic circuit. In the curves of this figure the points of maximum induction have been shifted up or down along the vertical axis until the same point represents the induction at one end of the rod under all conditions. The scale unit is the same for all curves. The data for these curves were obtained by noting the deflection, on reversal of the magnetizing current, caused by the differential electromotive force in two test coils, one near the end of the bar and the other movable so that it occupied successively all points along the rod.

Instead of a single point of maximum induction, there are two such maxima separated by a well-defined minimum. The minimum is most marked under conditions of the lowest reluctance in the yokes and joints. When the bar is reversed (curve V) while the yokes and coils remain unchanged, the section of minimum flux changes with the rod, so that this irregularity in flux is due to the rod itself and not to magnetizing solenoid or the yokes. An examination of the surface of this test specimen showed that an identifying number had been stamped very lightly upon the rod at the place where the minimum of flux appeared. Thus it would seem that the stamping had produced a hardening of the metal at this point, and a consequent lowering of the permeability. It suggests that identifying marks should not be stamped on those portions of the test rod which are to be assumed homogeneous in the test.

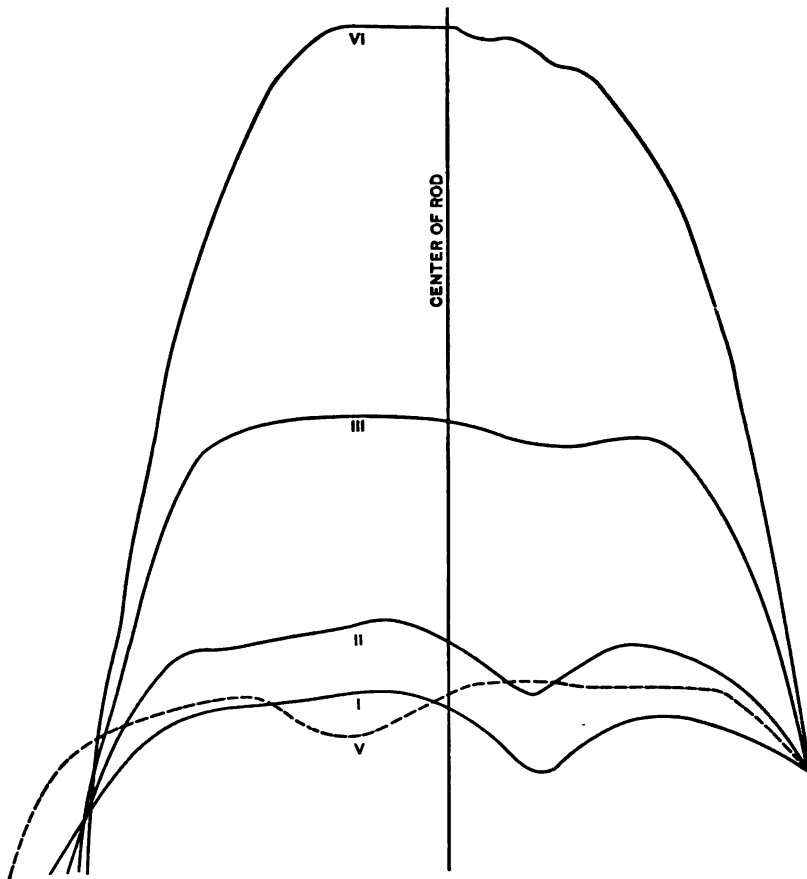


Fig. 21.—Showing Irregularities in the Flux due to Irregularities in the Test Bar itself in a Magnetic Circuit consisting of Two Rods and Two Yokes excited by a Constant Magnetomotive Force.

Curve I with yokes close to solenoids.

Curve II with yokes 1.5 cm from ends of solenoids.

Curve III with yokes removed.

Curve IV with test bar alone (yokes and auxiliary bar removed).

Curve V same as 1, but with test bar reversed.

Fig. 22 brings out the same main points in a different way. A test rod was placed in a double bar and yoke magnetic circuit, and the flux along its length measured. Then the yokes and solenoid were moved along 5 cm, relatively, to the rod, reclamped, and the flux distribution again measured. A second displacement of 5 cm was made, and another measurement taken. In the upper curve in the middle region of the bar, which here is surrounded by the middle of the magnetizing coils, are clearly

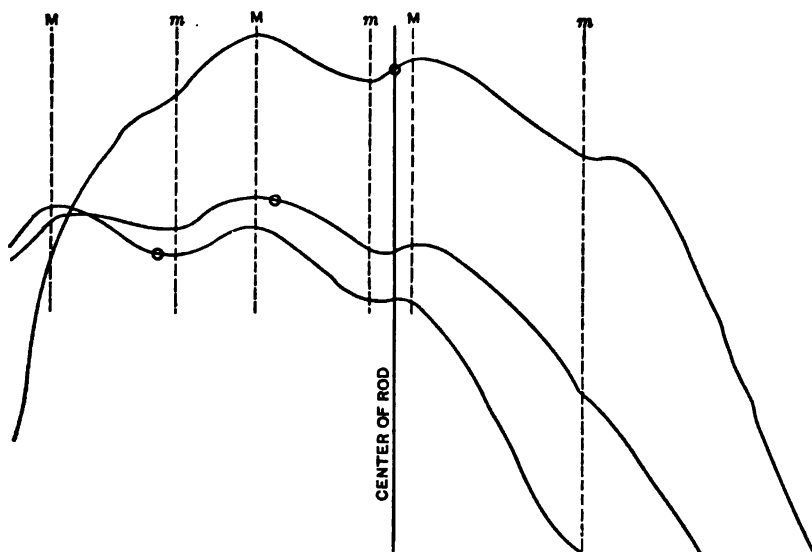


Fig. 22.—Showing the Flux Distribution for a Portion of a Rod 65 cm long, used in a Double Yoke Apparatus with 42-cm Coils for Three Different Relative Positions of Rods and Yoke.

The small circles show position of the midpoint of magnetizing coil.

seen two flux maxima and three flux minima. In the lower curves, where the magnetizing coils have been shifted, relatively, to the left so as to bring these points nearer the yokes, these same maxima and minima persist, but are less marked as they are approached by the yokes. Here again the general conclusion is that the rod produced some flux irregularity independent of that due to yokes or coils. A careful examination, visually, of this specimen failed to reveal any peculiarities at the points of maxima and minima.

Finally a rod was found which showed no marked irregularities.

This rod was stamped in three places with steel numbering dies, and then reexplored for variations in its flux distribution. Fig. 23 shows the flux distribution before and after stamping. The irregularities in flux distribution are here very marked.

From what has just been shown regarding flux distributions, it is quite evident that specimens which are to be accurately measured and preserved as standards must be handled carefully. It need not be surprising that a bar has slightly different magnetic properties after it has been dropped on the floor. Identifying numbers should not be stamped on the middle portion of the

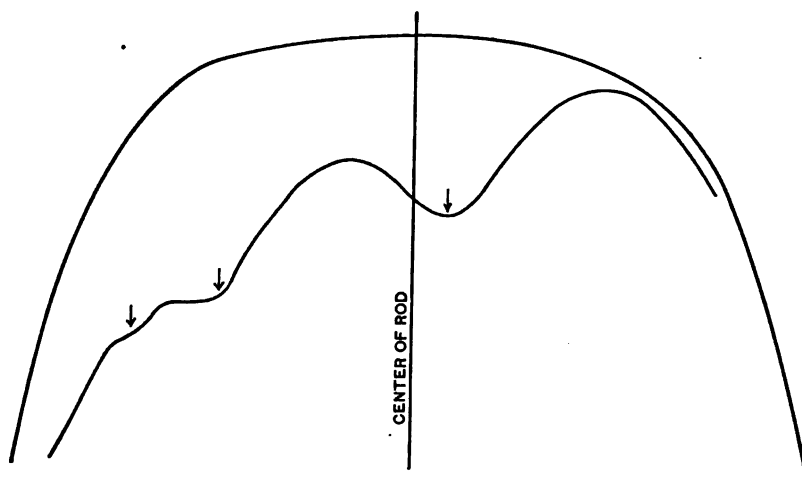


Fig. 23.—Showing Irregularities in Distribution of Flux in a Rod which has been Rendered Nonhomogeneous by Stamping Numbers at the Points Indicated by Arrows.

The upper curve shows distribution in homogeneous rod before stamping.

standard. It is objectionable to stamp them on the surfaces which are to form the magnetic joint with the yoke. The end of the rod seems the best place for the number. Placed here it has the further advantage of being visible when clamped in the yokes.

9. DISTRIBUTION OF THE TEST COILS.

Even with care in the selection and preservation of standard rods, irregularities are sure to occur. Consequently, the data obtained must be of necessity of the nature of mean values. To secure a reliable mean value, the test coils should be distributed over an appreciable length of rod.

It is found in practice that if uniformity of induction is secured over the middle section of the test and auxiliary rods, and over a section near the end of the test rod in the manner indicated, the small irregularity of flux that may remain in the yoke may be neglected. It is well, however, to place what we have called the end test coils far enough away from the yoke to avoid the local irregularities in the immediate vicinity of the joint.

A very satisfactory distribution of the three test coils is as follows: The main test coil is wound closely over the middle quarter of the test specimen. On each side of this, midway between it and the yokes, are wound the two halves of a second coil. The third coil, similar to the first, is wound over the auxiliary rod. The three magnetizing coils, one over the test rod, a similar one over the auxiliary rod, and the third distributed in four parts over the four ends of the rods, are adjusted until the fluxes, linked by the three test coils, are equal. These test coils may conveniently be wound on thin cores made of paper, tracing cloth, or slotted metal. They should fit as closely as possible so as to have only a small correction for the flux that is within the test coil, but outside the iron. They may, of course, be wound on the same cores as the main magnetizing coils either inside or outside of the solenoid, but this disposition necessarily gives a larger correction for the flux linked by the coil but not passing through the iron, and is not to be recommended.

IV. DIRECT READING METHOD OF MEASURING THE MAGNETIZING FORCE.

Having thus secured uniformity of flux in the test specimen, we may treat our magnetic circuit, as far as the middle section of the test bar is concerned, as though we had a uniform bar of infinite length surrounded by a uniform solenoid of equal length. In such a system the magnetizing force and magnetic induction may be readily calculated.

This assumption of equivalence to an infinite solenoid introduces no appreciable error. In Fig. 24 the magnetizing force at the center of the magnetizing solenoid is due to the main solenoid, the compensating turns, and whatever free poles may be developed in the iron itself. When the compensation is adjusted for uniformity of flux, the last force, of course, reduces to zero.

That due to the coils is approximately equivalent to an indefinite uniformly wound solenoid. To determine how closely this approximation holds, let us consider the corrections which must be applied to the actual field at the center of the finite solenoid to produce the ideal field of the infinite solenoid.

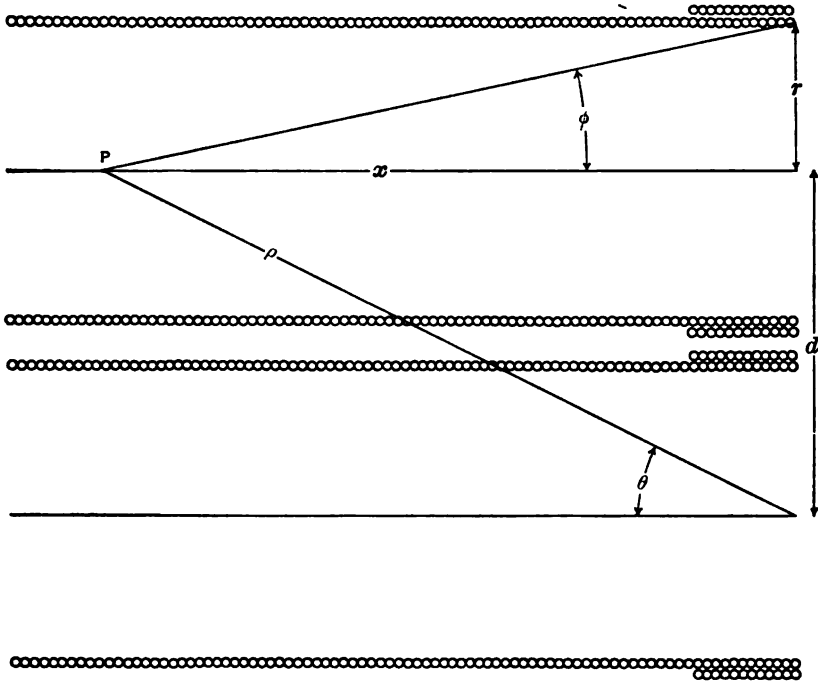


Fig. 24.

It may be shown that the correction due to one end of the solenoid A is $-\frac{1}{2}(1 - \cos \phi) H_0$ where H_0 is the field due the infinite solenoid. Expanding this (see Fig. 24).

$$\begin{aligned}
 -\frac{1}{2}(1 - \cos \phi) H_0 &= -\frac{H_0}{2} \left(1 - \frac{x}{\sqrt{x^2 + r^2}} \right) \\
 &= -\frac{H_0}{2} \left(1 - \left[1 - \frac{1}{2} \left(\frac{r}{x} \right)^2 + \frac{3}{8} \left(\frac{r}{x} \right)^4 - \frac{5}{16} \left(\frac{r}{x} \right)^6 + \dots \right] \right) \\
 &= -\frac{H_0}{2} \left[\frac{1}{2} \left(\frac{r}{x} \right)^2 - \frac{3}{8} \left(\frac{r}{x} \right)^4 + \frac{5}{16} \left(\frac{r}{x} \right)^6 - \dots \right] \quad (1)
 \end{aligned}$$

To get the force along the same axis due to one end of the other solenoid, we replace x by ρ and multiply each term by the appropriate zonal harmonic of $\cos \theta$. As the second solenoid develops a field opposite to H_0 we have for the correction due to one end of solenoid B

$$+ \frac{H_0}{2} \left[\frac{1}{2} \left(\frac{r}{\rho} \right)^2 P_1 - \frac{3}{8} \left(\frac{r}{\rho} \right)^4 P_3 + \frac{5}{16} \left(\frac{r}{\rho} \right)^6 P_5 - \dots \right] \quad (2)$$

For the correction at the center of the solenoid A , due to one of its ends for a particular set of coils in which

$$r = 1.6 \text{ cm}$$

$$x = 15 \text{ cm}$$

$$d = 6 \text{ cm}$$

$$\theta = 22^\circ$$

Eq. (1) gives

$$\begin{aligned} & - \frac{H_0}{2} \left[\frac{1}{2} \left(\frac{1.6}{15} \right)^2 - \frac{3}{8} \left(\frac{1.6}{15} \right)^4 + \dots \right] \\ & = - \frac{H_0}{2} [0.00568 - 0.00005 + \dots] \\ & = -0.0028 H_0 \end{aligned}$$

Eq. (2) gives

$$\begin{aligned} & + \frac{H_0}{2} \left[\frac{1}{2} \left(\frac{1.6^2}{15^2 + 6^2} \right) 0.9272 - \frac{3}{8} \left(\frac{1.6^4}{(15^2 + 6^2)^2} \right) 0.6019 + \dots \right] \\ & = + \frac{H_0}{2} [0.00454 - 0.0002 + \dots] \\ & = +0.0023 H_0 \end{aligned}$$

The total correction due to the four ends is therefore

$$2 (-0.0028 + 0.0023) H_0 = -0.0010 H_0$$

To determine the force at the center of a main solenoid due to the compensating coils, substitute the appropriate constants for each end of the compensating coil. In the particular case considered—

For inner end.

$$r = 2.5 \text{ cm}$$

$$x = 13 \text{ cm}$$

$$d = 6 \text{ cm}$$

$$\theta = \tan^{-1} \frac{6}{15} = 22^\circ$$

For other end.

$$r = 2.5 \text{ cm}$$

$$x = 15 \text{ cm}$$

$$d = 6 \text{ cm}$$

$$\theta = \tan^{-1} \frac{6}{13} = 25^\circ$$

$$\begin{aligned} \frac{1}{2} \Delta H &= -\frac{H_0}{2} \left[\frac{1}{2} \left(\frac{2.5}{16} \right)^2 - \frac{3}{8} \left(\frac{2.5}{15} \right)^4 + \dots \right] \\ &\quad + \frac{H_0}{2} \left[\frac{1}{2} - \left(\frac{2.5}{13} \right)^2 - \frac{3}{8} \left(\frac{2.5}{13} \right)^4 + \dots \right] \\ &\quad + \frac{H_0}{2} \left[\frac{1}{2} \frac{2.5^2}{231} 0.9272 - \frac{3}{8} \frac{2.5^4}{231^2} 0.6019 + \dots \right] \\ &\quad - \frac{H_0}{2} \left[\frac{1}{2} \frac{2.5^2}{175} 0.9063 - \frac{3}{8} \frac{2.5^4}{175^2} 0.5016 + \dots \right] \\ &= \frac{H_0}{2} \left\{ -(0.0139 - 0.0003) + (0.0185 - 0.0005) \right. \\ &\quad \left. + (0.0125 - 0.0002) - (0.0161 - 0.0002) \right\} \\ &= +0.0004 H_0 \end{aligned}$$

where $\frac{1}{2} \Delta H$ is the force at the center of one of the solenoids due to the compensating coils at one end for the same current turns per cm. as the main solenoids. As there are two such sets of compensating coils, and experiment has shown that at times they carry twice the current turns per unit length of the main solenoids, we may expect a maximum field at the center of main solenoids, due to these four compensating coils of $+0.0016 H_0$. Combining this correction with that due to the ends of the main solenoids we find a maximum correction of less than 0.1 per cent.

For a shorter set of coils 20 cm long, but with the other dimensions the same, we find for the corrections due to the ends of the magnetizing solenoids and to the compensating turns $-0.005 H_0$ and $+0.007 H_0$, respectively, so that these short coils may be used in work of a 1 per cent tolerance.

The remainder of this paper will be devoted to an account of certain modifications of the usual methods of measurement

which have been found convenient, especially where the apparatus may remain permanently set up.

Magnetizing coils, for use in laboratory work, are very frequently wound with no reference to ease of calculation of the magnetizing forces. The magnetizing force is given by the formula $H = 0.4\pi nI$ where n the number of turns per cm. is fixed, and I is variable. With haphazard values of n , the factor of proportionality between field strength and current will be an awkward number to handle. The computations are much simplified if the number of turns per centimeter is so chosen that this factor of proportionality is a power of 10. This is accomplished by making $n = 7.958$ turns per cm, whence $H = 10I$. For $n = 79.58$, $H = 100I$. Both these values of n are used in various commercial permeameters, and while very convenient, they offer considerable difficulty to the mechanician.

Another method of avoiding the labor of a long computation is to vary the current by a number of fixed steps in the regulating rheostat. If the total emf. impressed on the coil and regulating rheostats is constant, each value of the regulating resistance corresponds to a fixed value of magnetizing force. If, further, the divisions of the rheostat are adjusted so that integral values of H are given, we have a very desirable arrangement. Some means of maintaining constant potential difference must be employed. This and the lack of flexibility more than counterbalance the advantages of this method of regulation.

We may vary the magnetizing force by maintaining the current constant and varying the number of active turns. This method has the same advantages and disadvantages as the preceding and has the further objection that irregularities in the magnetizing force occur unless special precautions are used in winding. This method of varying the magnetizing force is used in some commercial apparatus.

In any given case we may determine the strength of the magnetizing current by measuring the fall of potential over a standard resistance of suitable value. The difference of potential can be measured to a high degree of precision by means of the potentiometer and standard cell. This method has great flexibility and with the following modifications offers many advantages.

The magnetizing force is given in terms of the fall of potential, E , over a given resistance, R , by the formula: $H = \frac{0.4\pi}{R} nE$.

If several magnetizing coils are to be used, from time to time, we can not expect n to be the same for all coils. Even for the same nominal value of n there may be variations of more than one-half per cent from this value unless the coil be wound with greater care than is ordinarily given to such work. We must, therefore, consider n as a variable as well as E . If $R = 0.01257 n$ then $H = 100 E$. By arranging a slide wire in parallel with a resistance coil of such value that the combined resistance is 1.257 ohms, the drop of potential over $\frac{n}{100}$ of the slide wire will be E . The reading of the potentiometer thus gives H by simply shifting the decimal point. This arrangement is shown in Fig. 25, where R_0 is a slide-

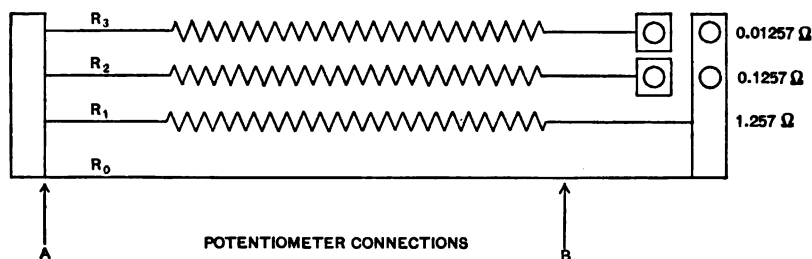


Fig. 25.—Showing Diagrammatically a Resistance Suitable for use with Potentiometer in Measuring the Magnetizing Force.

R_0 , a slide wire divided into 1,000 parts.

R_1 , R_2 , and R_3 resistances of such value that the total parallel resistances over R_0 and R_1 over R_0 , R_1 , and R_2 , and over R_0 , R_1 , and R_3 are 1.257, 0.1257, and 0.01257 ohms, respectively.

wire bridge wound on a drum and divided into 1,000 numbered parts, each subdivided into halves. The resistance, R , is adjusted to such a value that the total parallel resistance over R_0 and R_1 is 1.257 ohms. If, now, the potentiometer is connected to the fixed contact A , and the sliding contact B , and this sliding contact is set at a point $= \frac{10 n}{1000}$, we shall measure E . This setting needs to be made only once for each coil. The magnetizing force is now $H = 100 E$, where E is the potentiometer reading.

It may happen that a magnetizing current is required which would give a potential drop beyond the range of the potentiometer. In this case a second resistance, R_1 , is added in parallel, so that the total parallel resistance becomes 0.1257 ohm. For extreme cases a third parallel resistance may be added to reduce the total resistance to 0.01257. This last resistance would be required only for magnetizing coils of few turns per centimeter, where the large current required would produce excessive heating of the higher resistance.

In case all work is to be done with one set of magnetizing coils or with several sets all wound with the same number of turns per centimeter, it may be desirable to have a fixed resistance of the proper value to make the potentiometer direct reading.

V. THE MAGNETIZING COILS.

If the solenoid is wound in a single layer, it offers several advantages. It is more easily wound and its uniformity more readily determined. If we wish $H = 10 I$, we must have 7.958 turns per centimeter. No. 18 wire gives approximately this value.⁴ Such a coil will carry 5.41 amperes for a 30° C. rise in temperature; that is, with a single layer values of H up to 54 may be used continuously. For short intervals, long enough to take a measurement, double this value may be used. A solenoid of ten layers of the same wire will stand continuously 1.71 amperes, which means $H = 171$. Such a coil may be used for magnetizing forces up to 350 for short periods only.

The coils used in the precision magnetic work of the bureau have been made with great care. No. 18 double-cotton-covered copper wire was used, the first layer being wound on a screw thread cut by a lathe. Succeeding layers wound in the same direction, each wire being guided by the depression between adjacent wires in the previous layer and the ends of each layer being separately secured. There are ten layers in all. As each layer was finished it was examined very carefully, and in no case had any deviation from 8 turns per centimeter been observed. When the winding was finished, the ten layers were connected in series. We may safely assume that the field at the center of this coil can be calculated to within

⁴ Foster's Electrical Engineer's Pocket Book, p. 89, ed. 1902.

0.1 per cent. These coils give at the center a value of magnetizing force $H = 100,53 I$. As this current is measured by the fall of potential E over a resistance $R = 1.0053$ ohms, we have the working formula

$$H = 100 E$$

VI. MEASUREMENT OF THE INDUCTION.

Formula for the induction.—When measured by a ballistic galvanometer, calibrated by a mutual inductance, the formula used to calculate the magnetic induction is

$$B = \left(\frac{MI10^8}{N_2 A d_0} \right) d - \frac{a - A}{A} H$$

where

B = Flux per square centimeter.

M = Mutual inductance in henrys.

I = Current in amperes in the mutual inductance.

N_2 = Number of turns in test coil.

d_0 = Galvanometer deflection due to mutual inductance.

d = Galvanometer deflection due to test coil.

A = Cross section of iron.

a = Cross section of test coil.

H = Magnetizing force.

The Galvanometer.—The galvanometer must be of sufficiently long period that the full effect of the induced electromotive force of the test coils is transmitted to the moving system before it has moved very far from zero position. For rapidity of working, it is desirable that the moving system be aperiodic and the zero constant. The Thomson galvanometer is open to the objections that the zero is shifting continually, and often erratically, and that its sensibility varies with every change in the earth's horizontal intensity, and finally that the damping is tedious. The D'Arsonval is a more satisfactory instrument, as it is open to none of the above objections. It does have a lower sensibility, but this is sufficiently high for most ballistic purposes. The ease with which its sensibility is controlled and critical damping secured makes it a very desirable instrument in magnetic testing. For some work

an instrument of the flux-meter type, in which the suspension exerts no control, is desirable, but at the present time mechanical imperfections and difficulties leave much to be desired in these instruments.

Calibration of the Galvanometer.—Several methods of calibrating a ballistic galvanometer have been given, but I have obtained the best results by using a mutual inductance. This has the very great advantage that its secondary coil may remain a part of the galvanometer circuit, not only during calibration but also during the regular ballistic measurements. Thus the galvanometer is calibrated under normal working conditions. Furthermore, a standard of mutual inductance is inexpensive, simple in construction and use, permanent in its value, and if the primary is a long solenoid its value may be readily calculated from its dimensions.

Simplicity in calibration corrections may frequently be obtained by using only the lower part of the scale. In some of the earlier work of the present investigation the galvanometer sensibility was reduced until 1 cm deflection corresponded to 1000 in B . This gave a maximum deflection in the average practical test of 17 cm corresponding to $B = 17000$. The least reading, 0.1 mm., corresponds to 10 gauss. With this contracted range the calibration corrections were always less than 0.5 per cent and were usually negligible, thus permitting a great saving of time.

Unfortunately a galvanometer does not retain its calibration. The actual value of the induction corresponding to a particular deflection as well as the form of the calibration curve vary within small limits. Temperature changes may modify the strength of the magnet, the torsion coefficient of the suspension, the length of the suspension, the cross section of the coil, as well as the total resistance of the circuit. These variations are of sufficient importance to warrant a determination of the ballistic constant at the beginning and at the end of each run. Obviously, any mechanical adjustment of the suspension or of the galvanometer will produce still greater variations.

Another source of trouble in deflections taken with a galvanometer having an iron core is the effect of eddy currents and hysteresis. If the secondary of a mutual inductance is connected

in series with a galvanometer, it is found that the deflection due to the reversal of a given primary current is less for very rapid reversal than for a somewhat slower one. To prove that this difference is due to the iron in the galvanometer core, the galvanometer circuit was made to include a solenoid into which a bar of iron could be placed. In this case the galvanometer showed a smaller deflection when the iron was within the solenoid than when it was not, even when the rate of reversal and other details of the experiment were unaltered.

In one case the difference with and without iron, in the solenoid, was 0.25 per cent. The explanation of the difference in deflection in these two cases is probably to be found in the increase of effective resistance of the galvanometer circuit due to hysteresis and eddy currents. These latter will each be greater for the greater secondary current flowing, and since the integral of the electromotive force due to a given change in the primary current of a mutual inductance is constant, the instantaneous values of the induced electromotive force and, consequently, the current will be greater the shorter the time of integration, or in this case the time of reversal. The hysteresis and eddy currents thus reduce the galvanometer deflection. However, if the same methods of manipulation are followed in the determination of the constant and in the test, no serious error will result.

The Mutual Inductance.—The factors which enter into the value of B can not all be determined to the same precision. The mutual inductance may easily be determined to 1 part in 10000. As an illustration, one pair of coils gave for the value of the mutual inductance, as determined on different days, values of 15.2480 and 15.2486 millihenrys, respectively, a difference of 1 in 25000. The mutual inductance of these coils has not changed by a measurable amount since they were first made and measured three years ago. In using the mutual inductance it is necessary that all leads to both primary and secondary be twisted in pairs in order that each circuit may neither envelop nor develop any accidental magnetic flux. This precaution is particularly important close to the mutual inductance. This mutual inductance must be placed far enough away from the magnetic circuit under test, so that no stray field may act upon its secondary and

thus cause an extra electromotive force to be generated in the galvanometer circuit at the instant the ballistic measurement is made. Errors may be introduced into the ballistic readings if from any cause there is a sudden change in the magnetic field at the place where the mutual inductance is located. This source of trouble was forced upon my attention while working with a standard of mutual inductance approximately 20 centimeters in diameter and having several hundred turns in the secondary. With this coil in circuit with the galvanometer every movement of an elevator some 30 meters distant was indicated by the galvanometer. This disturbance was probably due to the change in vertical length of the steel supporting cable as it coiled and uncoiled upon the cable drum. The lower end of the cable is undoubtedly magnetized as a north-seeking pole by the terrestrial field, and as the elevator moves in its shaft the flux from this moving magnet sweeps through the surrounding space. The trouble disappeared when the plane of the coil was placed vertical instead of horizontal. This suggests that any disturbance due to a varying field may be eliminated by placing the coil so that its turns do not encircle the varying flux. This position can readily be ascertained by trial.

The Current.—The current through the primary of the mutual inductance may be determined with any precision required. Throughout the present work all current measurements were made by measuring the fall of potential over a standard resistance by means of a potentiometer.

Test Coils.—No appreciable error need appear in the value of N_2 if the leads are twisted. There may be loops in the leads and some stray flux may thus be caught. Increasing the number of test turns reduces any such effect. Care must be taken that there is no short circuiting of secondary turns.

The Cross Section of Specimen.— A , the cross section of the test specimen, may be an important source of error in the determination of the induction, especially in samples of thin transformer iron. As the specimen comes from the lathe, the milling machine, or the rolls and shears if in sheet form, it is not of uniform section but varies by as much as several hundredths of a millimeter. If this variation tapers from one end to the other the error is mostly self-compensating. Sheared metal furnishes the

most objectionable irregularities and milled metal the least. The actual dimensions of the cross section can easily be measured to a hundredth of a millimeter. This would mean about 1 or 2 per cent for thin transformer iron and one-tenth as much for iron several millimeters thick. The difficulty does not end here, however, as the specimen is often grooved and pitted. The micrometer measures the maximum dimensions and hence may indicate a larger cross section and a correspondingly smaller induction is computed. The error due to this last irregularity is eliminated by getting the specific gravity, weight, and length, and from these calculating the cross section. In one set of specimens 1 cm wide by 0.035 cm thick and 51 cm long the density method indicated a cross section 6 per cent less than that of direct micrometer measurement.

Air Flux.—The correction $\frac{a-A}{A}H$ is for the flux which passes through the test coil but outside the iron. This should be made small by winding the test coils close to the specimen, in the case of small specimens. If the specimen is of large cross section and nearly fills the magnetizing coil, and this is of a single layer, the test coil may be wound outside this primary coil. As an illustration of the magnitude of this correction, suppose the test coil 1.5 cm in diameter and the specimen 0.5 cm in diameter. Then the correction is $8H$ in B , or 8 in the permeability. For a sheet of transformer iron .03 by 1 in the same test coil, the correction is $58H$ in B , or 58 in the value of μ . An error of .01 cm in the diameter of the test coil would introduce an error in B of 0.8 H . This is equivalent to an error of 1 per cent at those parts of the induction curve where the permeability is 80. Consequently it is inadvisable to have the test coil wound outside the primary when specimens of very small section are to be tested.

It is possible to simplify this correction by having a second magnetizing coil and test coil arranged similarly to the working coils but without any iron inserted. The same magnetizing current flows through each primary coil and the two secondaries are so connected that the induced emfs. oppose each other. In such an arrangement the galvanometer is not affected by any change in the magnetizing circuit until the iron rod is inserted in one coil

and its deflections are due to the increase of flux due to the iron alone. To the value of B thus determined must be added the magnetizing force, but as this correction term is always small and requires no calculation it is not at all troublesome. As these coils constitute a mutual inductance, any form of adjustable inductance might be used and the proper value determined by noting when the galvanometer showed no deflection without iron in the magnetizing coil.

ZERO METHOD.

In discussing the influence of errors in the galvanometer deflections and scale calibration, it was noted that these errors were minimized if the deflection due to the mutual inductance in getting the galvanometer constant was equal to the deflection produced by the change of induction in the iron. This suggests that it might be desirable to impress both these electromotive forces in opposition simultaneously on the galvanometer. The galvanometer would indicate the equality of the two integrated electromotive forces by a zero deflection. If this is done the induction, neglecting the correction term, becomes

$$B = \frac{MI_1 \times 10^9}{N_s A}$$

As N_s and A are constant for any given test coil and specimen, this formula becomes

$$B = MI \times \text{constant} \quad (3)$$

If we use this equation and vary both M and I it is desirable to have the same current I flow through the magnetizing coil and the primary of this balancing inductance. Since further

$$H = I \times \text{constant} \quad (4)$$

we get by dividing (3) by (4)

$$\mu = \frac{B}{H} = M \times \text{constant}$$

so that in this method the values of permeability are obtained very readily from M after a single calculation.

Instead of varying both M and I we may keep I constant and vary M alone. This offers some advantages when the same sized specimen is always used. In this case we have

$$B = KM$$

The inductance may be calibrated to read B directly.

We may, however, keep M constant and vary I . The formula (3) now becomes

$$B = I \times \text{constant} \quad (5)$$

If, further, M is adjustable through a wide range it may be given such a value that the constant becomes a power of 10. In this case the induction is obtained directly from the current I by merely shifting the decimal point.

The value of mutual inductance required is

$$M = \frac{N_z A}{10^4} \times \text{constant}$$

In the present work the constant chosen is 10^4 , so that $M = \frac{N_z A}{10^4}$ and $B = 10000 I$. Since $N_z = 100$, $M = 0.01 A$.

As the current measurement is made by determining the fall of potential over a standard resistance, the above formula becomes $B = 10000 \frac{E}{R}$, or for a resistance of 1 ohm $B = 10000 E$. Thus the flux density determination depends upon a potentiometer setting.

First among the several advantages of this manner of measuring the induction is the fact that it is a zero method and as such shares all the merits usually possessed by zero methods. It is independent of the galvanometer scale calibration and the galvanometer may be worked at its maximum sensibility, usually many fold that permitted by the deflection method. It permits one to set the mutual inductance current at a value corresponding to any chosen value of the induction, after which H may be adjusted to correspond. This is of particular importance in the double yoke method, in which the two magnetizing forces used to produce the same induction in different lengths of specimen are compared.

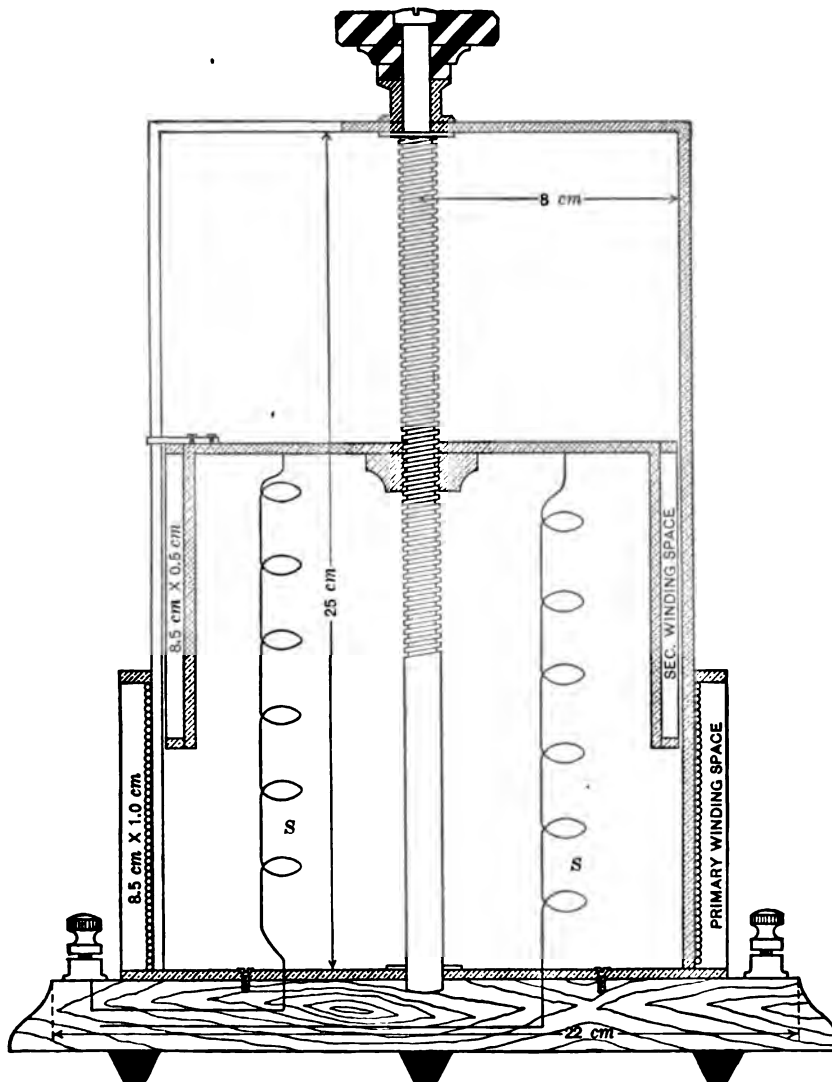


Fig. 26.—Sectional View of a Variable Mutual Inductance designed especially for use in Magnetic Testing.

By the deflection method it is necessary to plot the full curve for each length of rod and then measure the abscissa differences and finally lay off the resultant curve, which is the only one desired. By the zero method the balancing current is set to the required value of the induction for each length and the differences in magnetizing forces obtained directly from the data without the necessity of plotting the two auxiliary curves. In fact it is not necessary to plot any curve unless so desired.

It is also a decided advantage to know the corresponding values of B and H during the course of the experimental determination. Without this knowledge it is difficult to determine the proper spacing of the points of observation. By the deflection method it frequently happens that too few points are taken on the steep part of the curve and more than necessary in the upper ranges. In cyclic curves the points most frequently desired are the maximum induction, the residual induction, and the coercive force. The zero method enables one to set on these points directly, while in the deflection method the curve must be plotted before the coercive force is known.

VARIABLE MUTUAL INDUCTANCE.

Fig. 26 is a sectional view of a variable mutual inductance designed for use in magnetic testing. The primary is wound in six layers on one end of a brass cylinder slotted so as to reduce eddy currents. The terminals of this primary and tap wires from two intermediate points are brought out to binding posts mounted on the base. The secondary is wound on an inner cylinder which is drawn along the axis of the primary coil by a threaded shaft of phosphor-bronze. A small index moving in a slot cut in the outer cylinder prevents the inner spool from turning on its axis and also indicates its position. The terminals of the secondary are brought to binding posts in the base, through the spirals S .

The secondary coil has 1200 turns of No. 30 D. S. C. magnet wire, having a total resistance of 180 ohms at 22° C. The primary has three sections of No. 20 D. C. C. magnet wire, with the following constants:

Designation of Section	Number of Turns	Maximum Mutual Inductance	Minimum Mutual Inductance	Resistance
A	6	Mh. 0.89	Mh. 0.13	Ohms. 0.17
B	55	7.8	0.8	1.07
C	543	75	7	9.13

Fig. 27 gives the calibration curves for the three primaries A, B, and C.

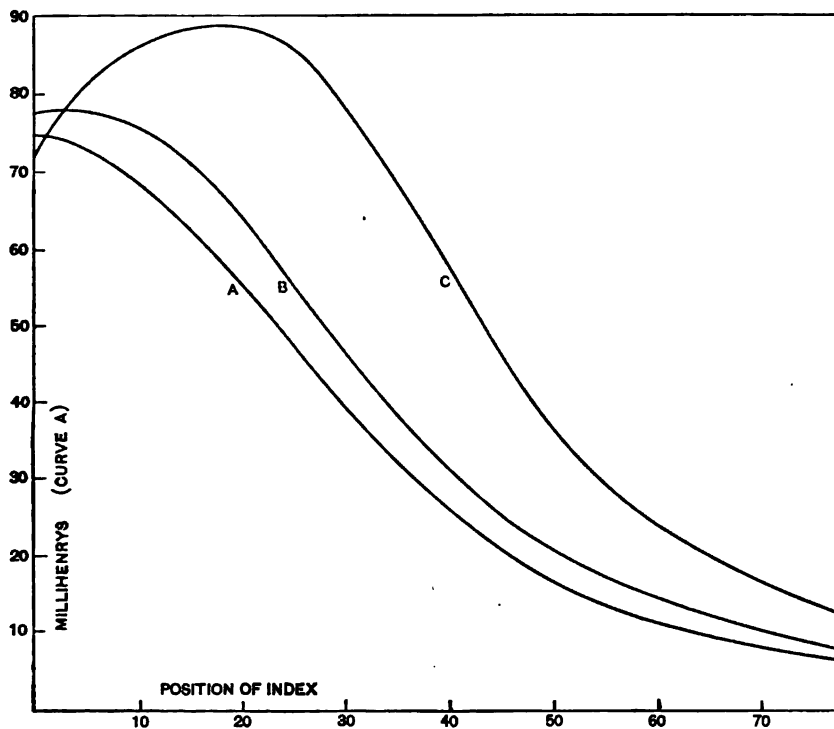


Fig. 27.—Showing Calibration Curves for the Variable Mutual Inductance shown in Fig. 26.

Curve A is with the full 543 turns of primary winding.

Curve B is with 55 turns of primary and the vertical scale is 0.1 that indicated on the left.

Curve C is with 6 turns of primary and the vertical scale is 0.1 that indicated on the left.

From these curves we see that the ratio of the change in inductance to the displacement of the coil is a maximum when the coil is in its middle position. Even here, however, a change of 0.1 per

cent in inductance corresponds to a rotation of a little over 7° of the shaft by which the coil is moved. The peculiarity in the shape of the calibration curve for the 6 turns of primary is due to the fact that these turns are wound on the upper end of the primary so that they do not become most effective until the secondary has moved up far enough to be directly opposite these turns.

This variable mutual inductance can be set, with an accuracy of 0.1 per cent or better, to any value greater than 0.13 millihenry and less than 75 millihenrys. As the current through the primary is to be related to the flux density in the iron by the formula $B = 10000 I$, it is necessary that a current of 2 amperes produce no excessive heating if a flux density of 20000 is desired. A current of this value maintained for one hour produced a rise of 33° in temperature as measured by the change in resistance of the primary. Consequently the current-carrying capacity is ample.

VII. FULL SET-UP AND OPERATIONS.

After the test bars are inserted in the yokes, the circuit is thoroughly demagnetized by a cyclic magnetizing force of one period per second, which is gradually reduced from an initial intensity which carries the induction well beyond the point of maximum permeability to a final value somewhat lower than the lowest induction to be studied.

After demagnetization in this way, the lowest force to be used is applied and reversed many times until the iron is brought to a cyclic magnetic state, that is, until the reversal of the magnetizing force reverses the direction of magnetization without changing its magnitude. During this process of reversals the adjustment for uniformity of flux may be made, as the adjustment involves only small changes in the magnetizing force.

Having made the adjustments necessary to secure uniformity of flux and brought the iron to the cyclic state, the induction is measured by balancing the electromotive force induced on reversal of the magnetizing current against that induced in the secondary of a mutual inductance on reversal of its primary current, which is adjusted to give the required balance. In order to exclude any influence due to mechanical vibrations the entire magnetic circuit is mounted on a layer of felt. These lower values of the

induction produced by small magnetizing forces are easily altered by incompleteness of demagnetization or failure to reduce to a cyclic state. As a check, therefore, after determining this first point on the induction curve, it is well to carry the iron again through the demagnetizing process, readjust the compensation, reduce to a cyclic state, and redetermine the point.

If the second determination does not yield the same result as the first on the initial induction, the full process should be repeated more carefully until concordant results are obtained.

Having thus secured satisfactory data for the lowest magnetizing current, it is not necessary to demagnetize again. The remaining points on the induction curve may be obtained by passing to the next higher magnetizing force, adjusting the main and compensating currents, repeating the reversals till a cyclic condition is reached, and so on till the required data are secured. The curve determined by these points is called the normal induction curve.

Before determining the hysteresis loop the iron is demagnetized in the same manner as before the normal induction test, although the demagnetization may be omitted if the specimen has not been subjected since its last demagnetization to forces greater than the maximum used in the hysteresis cycle. The maximum magnetizing force is applied and the compensation adjusted as for a point on the normal induction curve. By suddenly inserting a separately adjusted rheostat in the main magnetizing circuit the magnetizing force can be reduced to any desired value. This, in general, means that new adjustments of the compensating currents must be made in order to maintain uniformity of flux at this second point. This is done most conveniently without disturbing the original adjustment by inserting in the proper circuits suitably adjusted series or parallel resistances, according to whether the particular current is to be lowered or raised. This adjustment is made by means of the same test coils and in a similar manner as the original adjustment. Before each change from the maximum magnetizing current to the lower value, however, the iron must be brought back to a cyclic state. A double reversal of the maximum current usually suffices for this. After this compensation is made the change in induction produced by the change in the magnetizing force is measured in the same manner as before,

and from this value the induction corresponding to the lower magnetizing force is readily calculated.

Data with negative values of the magnetizing force are obtained by reversing the currents in addition to making the adjustments already described. The points on the hysteresis curve may be taken in any order.

In this method of obtaining hysteresis data the measured quantity is the change in induction when the magnetization is changed at one step from a maximum to any other given point on the hysteresis loop. This method is comparatively free from the irregularities due to the slow creep that occurs when a magnetizing force is applied slowly or changed by small steps. It is also free from irregularities due to variations in the size of step in the "step-by-step" method. For these reasons and the fact that the errors are not cumulative, it seems the better method for standard measurements.

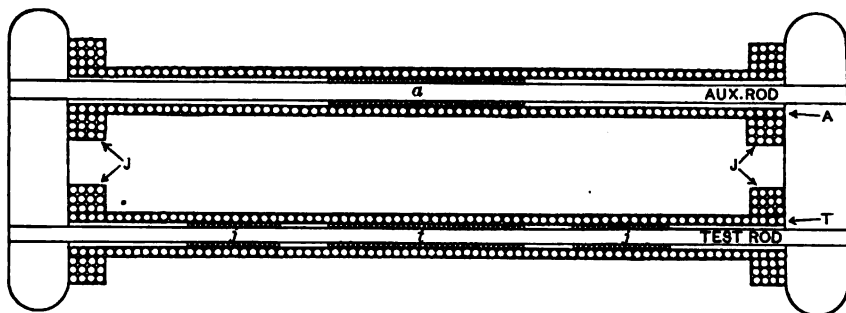


Fig. 28.—Showing the Relative Positions of Magnetizing and Test Coils.

T and A represent the main magnetizing solenoids.

J, the compensating turns about the four joints.

t and a are the two test coils surrounding test and auxiliary specimen respectively.

j is the end coil distributed with one-half over each end of the test rod.

Fig. 28 shows the relative positions of the magnetizing and test coils when double yokes and double rods are used. The lower rod is the one under test. The upper rod is an auxiliary rod of approximately the same magnetic properties as the test specimen. T and A are the two main magnetizing coils, one wound over each rod. Over the four joints are wound four short coils each about 1.5 cm long. These are connected in series and used as a single coil.

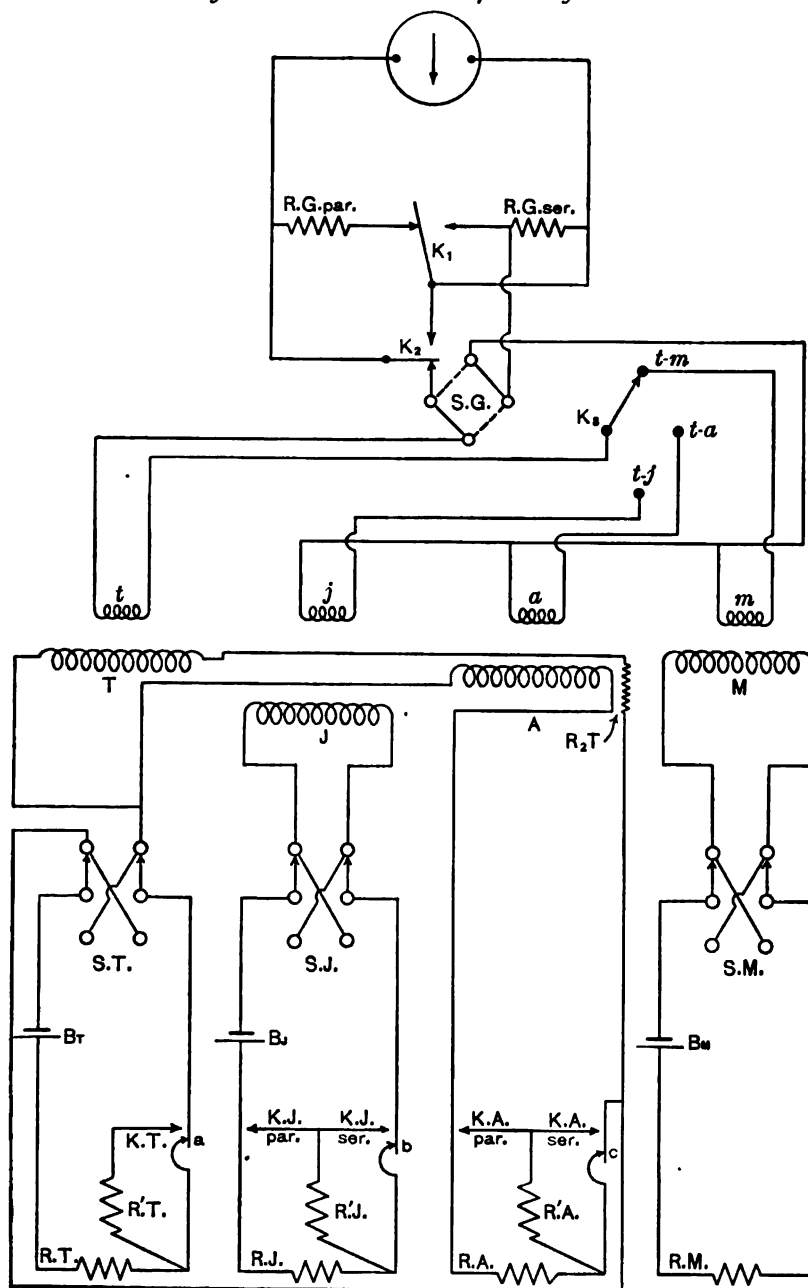


Fig. 29.—Showing Diagrammatically Fethull Battery and Galvanometer Connections used in Precision Magnetic Measurements.

T refers to the main magnetizing circuit about the test bar, J to the compensating circuit, A to the magnetizing circuit about the auxiliary rod, and M to the variable mutual induction circuit.

The test coils t and a are wound over the middle portions of the test rod and auxiliary rod, respectively. Over the two ends of the test rod are wound the two halves of a third test coil j . These three test coils have the same number of turns and are spread over a considerable length of rod, so as to prevent any irregularities which may exist in the iron from exerting a preponderating influence. The end test coils are far enough away from the yokes and joints so that there is little irregularity from these causes. These test coils are wound on a separate form inside the solenoid, so that the correction for the flux between the rod and coil may be as small as possible.

Fig. 29 shows, diagrammatically, the full scheme of electric circuits both primary and secondary. The coils T , J , A , t , j , and a are the same as those of the same letters in the preceding figure. M and m are the primary and secondary of the variable mutual inductance used to balance the emf. induced in the test coil.

The four reversing switches and the five keys in the various magnetizing circuits are all of the mercury type and are mounted on a single base and arranged side by side similar in size and position to the white keys of a piano, so that by using the fingers of both hands any combination may be operated. This compact arrangement of keys and switches should be borne in mind while reading the following description. On removing the fingers the keys are drawn back to their normal positions by means of small springs. The connections shown in the figure form a permanent set-up of considerable flexibility.

FOR RINGS AND UNCOMPENSATED BARS AND YOKES.

If this permanent set-up is to be used in magnetic measurements on rings or on uncompensated bar and yoke magnetic circuits, the magnetizing system consists solely of the circuit containing the battery B_1 . The circuit of battery B_2 is not used. The induced electromotive force of the test coil t acts on the galvanometer and produces a deflection when the switch ST is reversed. If we wish to use the zero method, the switch SM is reversed simultaneously and the resistances adjusted until the deflection on reversal is reduced to zero.

In the galvanometer circuit the resistance RG ser. determines the sensibility, and the resistance RG par. adjusts to critical damping. The key K_1 is in the ordinary working position. If it is desired to use the galvanometer at its maximum sensibility, the key K_1 is thrown to the right, thus removing the shunt RG par. and short circuiting the series resistance RG ser.

The key K_2 is in its working position on the lower point, as shown. When raised to the upper contact, the test coils are thrown out of circuit and the galvanometer short-circuited. The reversing switch in the galvanometer circuit is very convenient in many cases.

If we are determining normal induction data in the above case the reversing switch ST is the only one to be manipulated. To secure hysteresis data the change in induction when the flux in the iron passes from a maximum to some other value, positive or negative, is measured.

To lower the current from the maximum value to some other value the resistance $R'T$ is inserted in series with the main resistance RT by means of the key KT . The action of this key is to insert the resistance $R'T$ across the contact a and then open this contact. To change the current to a lower value of opposite sign the above operation is performed simultaneously with the reversal of switch ST .

COMPENSATED BAR AND YOKE.

In the determination of normal induction data in the bar and yoke magnetic circuit with compensation for yoke effects and dissimilarity in the test and auxiliary rods, all the circuits shown in the figure are used. The switches ST and SJ are reversed repeatedly, and the resistance RA and RJ adjusted until the three test coils, t , j , and a , indicate the same change in flux on reversal of the magnetizing currents. With the key K_3 on the point $t-a$, the equality of flux in the test and auxiliary rods is secured first. Then with the key K_3 on the point $t-j$ the flux near the magnetic joints is adjusted to uniformity. To measure the induction, K_3 is moved to the point $t-m$. A reversal of the magnetizing forces produces an impulsive electromotive force acting on the galvanometer, which may be measured as a deflection or may be compensated for as before by reversing simultaneously a

suitable current through the variable inductance M . These operations are not so complicated as they appear at first sight. They have been much simplified by means of a set of special rocking mercury switches held in their normal position by suitable springs. These switches resemble in size, position, and operation the keys of a piano, and are operated as readily. Five of these keys can be operated with one hand with little more effort than is required for one. What complications exist are in the permanent apparatus. The operations themselves may be performed in less time than it takes to describe them.

To secure hysteresis data in the compensated bar and yoke method it is necessary to change the currents flowing in the coils T , J , and A from the values necessary for adjustment for uniformity at the maximum induction to corresponding values at other inductions. The resistances and keys of the T circuit are manipulated as in the case of a ring specimen. In circuit J , however, the second current may be greater or less than the first. For this reason it is necessary to be able to insert the resistance $R'J$ either in series or in parallel with the initial resistance RJ . By depressing the key KJ ser. first the resistance $R'J$ is shunted around the contact b and then the contact b is broken. By depressing the key KJ par. one throws the resistance $R'J$ in parallel with RJ . In a similar manner the resistance $R'A$ may be thrown either in series or in parallel with the resistance RA . Thus the resistances RT , RJ , and RA determine the magnitude and distribution of mmf. for the tip of a hysteresis loop, while $R'T$, $R'J$, and $R'A$ determine the corresponding quantities for any other point of the cycle. By this means it is possible to adjust to uniformity of flux in any case that may arise. The variable mutual inductance may be used to reduce the measurement to a zero method or the direct deflections may be used.

Having secured uniformity of flux throughout the magnetic circuit we may measure the magnetizing current about each rod. If, further, the auxiliary rod is a standard of known properties, we have at the same time a comparison method as well as a direct measurement. Thus the one set of operations furnishes a check on itself, and no duplicate observations are necessary.

The following is a sample set of data:

Formulae use—

$$M = 0.01 A$$

$$B = 10000 E_m$$

$$H = 100 E$$

Bars marked.....	11	12
Length	35 cm	35 cm
Diameter	1.000 cm	1.000 cm
Cross section	0.7856 cm ²	0.7856 cm ²
Solenoid used	5343 T.	5343 A.
Mutual inductance setting	P 543 S 72.1.	
B	H	
2400	1.61	1.83
Demagnetized.		
2420	1.62	1.84
5076	2.14	2.45
8295	3.32	3.59
10690	5.08	5.18
12180	7.05	6.96
14330	13.8	12.7
16330	35.6	33.4
17810	84.6	83.8
19960	233.4	232.3

VIII. GENERAL CONCLUSIONS.

To sum up the general conclusions of the preceding investigations:

A close approximation to uniformity of flux along the test rod may be secured by using properly constructed specimens and yokes and properly distributed magnetizing coils.

The double bar and yoke form of magnetic circuit seems to offer the greatest number of advantages and the fewest disadvantages. The reluctance of the yoke and joints should be small.

This is accomplished by having the yokes short and of moderately large cross section, and making the surface of contact between specimens and yoke of considerable area. The corners of the yokes should be rounded off so as to avoid, as far as possible, disturbing end effects.

The magnetomotive force may advantageously be distributed in three sections, a uniform solenoid over the test specimen, a second one over the auxiliary rod, and the third section subdivided into four short coils and placed over the four ends of the rods as near to the yokes as feasible. These three magnetomotive forces should be capable of independent adjustment and simultaneous reversal. The test coils to be used as indicators should be long enough to avoid errors due to local irregularities due to inhomogeneity of the rods. One coil should be wound under the middle section of each magnetizing solenoid and the third should be distributed with one-half over each end of the test bar, but not too near to the yokes.

In the measurement of the magnetizing force and the magnetic induction these quantities are obtained by zero methods directly from potentiometer readings.

As to accuracy, it was hoped at the beginning of this investigation that the magnetizing force required to produce any induction between 1,000 and 20,000 gaussses might be obtained to within 1 per cent. Investigations now in progress indicate that this accuracy has been secured.

WASHINGTON, May 1, 1909.

A METHOD FOR CONSTRUCTING THE NATURAL SCALE OF PURE COLOR.

By P. G. Nutting.

The color sensation is known to vary far from uniformly with the wave length of the exciting radiation. In a normal spectrum the variation is much more rapid in the yellowish-orange and bluish-green regions than in the midgreen or in the extreme red and violet. Hence, a color scale of say one unit for each 10 $\mu\mu$ difference in wave length would represent far from equal color steps. The difference in wave length just perceptible as a difference in color is roughly 5 $\mu\mu$ in the two most sensitive regions, 15 $\mu\mu$ in the midgreen, and much greater in the violet and red. The method here described makes use of data on this difference limen.

Given the least perceptible difference (difference limen) for an eye throughout the visible spectrum, the reciprocal will be proportional to color sensibility as a function of wave length. But sensibility is the derivative¹ of a scale-reading curve, in this case the color scale desired. A method based on this principle will be applied to some of the best recent data on difference limen to illustrate the construction of a color scale.

Steindler² has recently published data on the difference limen of twelve subjects having normal color vision. The characteristics of the color limen curve are shown in the figure. It has two deep minima at about 490 $\mu\mu$ and 580 $\mu\mu$, a maximum in the green at 535 and a slight maximum and minimum at either end. The location and heights of these seven maxima and minima are given for each of the twelve subjects in the following table:

¹ This Bulletin 5, p. 266, 1908.

² Sitz. Ak. Wiss. Wien, 115, IIa; Jan., 1906.

	First Minima		First Maxima		Second Minima		Second Maxima		Third Minima		Third Maxima		Fourth Minima	
	λ	$\partial\lambda$	λ	$\partial\lambda$	λ	$\partial\lambda$	λ	$\partial\lambda$	λ	$\partial\lambda$	λ	$\partial\lambda$	λ	$\partial\lambda$
Dr. O. St.	435	23.6	454	37.6	495	11.6	535	32.8	585	7.6	626	40.0	638	31.0
Dr. E.	434	25.7	450	38.0	488	11.0	532	36.4	587	8.0	630	47.0	651	34.5
Prof. E.	430	14.5	444	18.0	480	5.0	523	34.2	586	9.0	624	34.2	637	28.6
Dr. Sch.	435	16.0	462	21.6	498	12.0	535	20.4	582	12.4	612	26.0	628	24.0
Dr. A. St.	446	26.0	465	34.0	494	25.5	540	49.5	585	22.4	620	38.0	637	32.0
Dr. Ma.					488	15.9	520	22.0	572	9.1	622	24.0	638	17.2
Dr. Me.	436	13.8	448	24.2	478	5.2	545	37.2	598	11.2	630	36.0	646	20.0
Dr. Bi.	447	18.0	462	24.0	505	12.0	540	22.2	583	11.2	608	21.2	614	19.2
Hr. B.	454	30.4	462	35.5	492	19.6	530	46.0	568	24.0	605	44.0	618	43.0
Frl. M.	442	14.0	468	24.4	490	16.0	540	32.0	572	20.0	633	60.0	642	45.5
Dr. H.	450	51.0	460	54.6	497	22.4	530	46.0	583	19.2				
Dr. G.	437	39.3	435	39.3	501	7.6	536	22.1	571	12.8	625	42.4	640	34.6
Means.	440	24.7	455	29.3	492	13.6	534	33.4	581	13.9	621	37.5	635	30.0

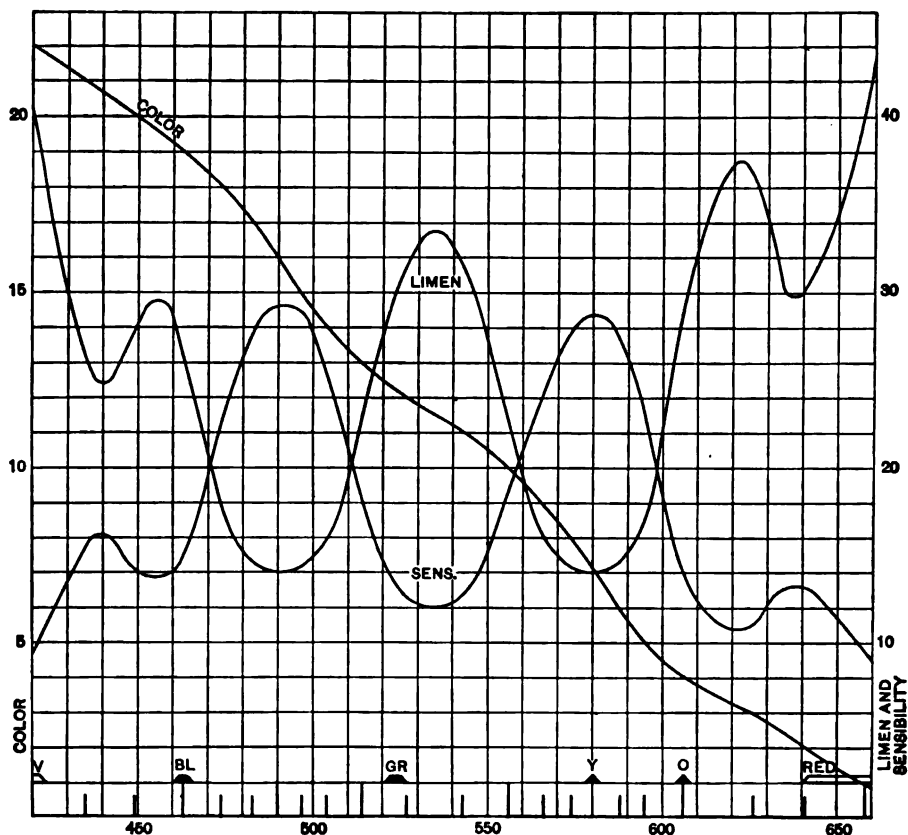


Fig. 1.

While variations from the mean positions of the maxima and minima are not excessive they are so large that a simple average of all the curves would be smoother than any of them. Adjusting the positions of the maxima and minima of each curve to the means, the following data were obtained for the properties of average (of these twelve) eyes.

TABLE II.

Wave Length ($\mu\mu$)	Color Limen ($\mu\mu$)	Sensibility	Ordinate Area	Color Scale
420	41.0	244	5.45	21.96
425	36.0	278		
430	30.0	333	7.51	21.41
435	26.1	384		
440	24.7	405	7.55	20.66
445	26.3	380		
450	28.5	351	6.90	19.91
455	29.3	341		
460	28.8	347	8.00	19.22
465	24.6	407		
470	20.6	485	11.60	18.42
475	17.0	588		
480	15.2	658	13.12	17.26
485	14.0	714		
490	13.7	730	14.44	15.95
495	13.7	730		
500	14.5	690	12.04	14.50
505	16.5	607		
510	19.4	515	8.70	13.30
515	22.9	436		
520	27.4	365	6.56	12.43
525	30.9	324		
530	32.7	306	6.04	11.77
535	33.4	300		
540	32.5	308	6.60	11.17
545	30.3	330		
550	27.1	369	8.80	10.51
555	22.7	440		
560	19.2	520	11.72	9.63
565	17.1	588		
570	15.4	650	13.90	8.46
575	14.3	700		
580	13.9	720	14.00	7.07
585	14.1	709		
590	15.2	658	11.36	5.67
595	17.3	578		
600	22.2	450	7.32	4.53
605	28.2	355		
610	32.8	305	5.58	3.80
615	35.6	281		
620	37.5	267	5.48	3.24

TABLE II—Continued.

Wave Length ($\mu\mu$)	Color Limen ($\mu\mu$)	Sensibility	Ordinate Area	Color Scale
625	36.9	271		
630	33.5	298	6.48	2.69
635	30.1	332		
640	30.3	330	6.24	2.04
645	32.0	313		
650	35.6	281	5.21	1.42
655	38.6	259		
660			9.00	0.90

The ordinates of the curve of color as a function of wave length are given in the last column. They were obtained by integration of the sensibility curve, the partial areas (divided by two on the scale of the figure) being given in the fourth column. Each number in the last column is the sum (divided by 10) of the ordinate areas of the fourth column added from the bottom to and including that wave length. The color-wave length curve is plotted in the figure.

A difference of one unit in the color scale represents a difference in color that is just easily perceptible, hence forms a convenient natural unit, although any other subdivision might be used. In Fig. 1, each unit of the color scale is indicated on the wave-length axis and just above are indicated roughly the positions of six spectral hues.

To test the theoretical color curve, a normal spectrum was projected on a black screen in which had been cut slits spaced according to the wave lengths of the color units, the slits being covered with ground glass. No departure from uniformity in the color steps could be detected by the ten or more individuals who carefully examined them.

The wave lengths of each of these color steps is given in Table III.

These computations have been carried through merely to illustrate the method. They may easily be made for any eye for which the sensibility curve is known.

If the sensibility curves of a large number of subjects were known, the properties of an average normal human eye might be deduced and a scale of color constructed and adopted.

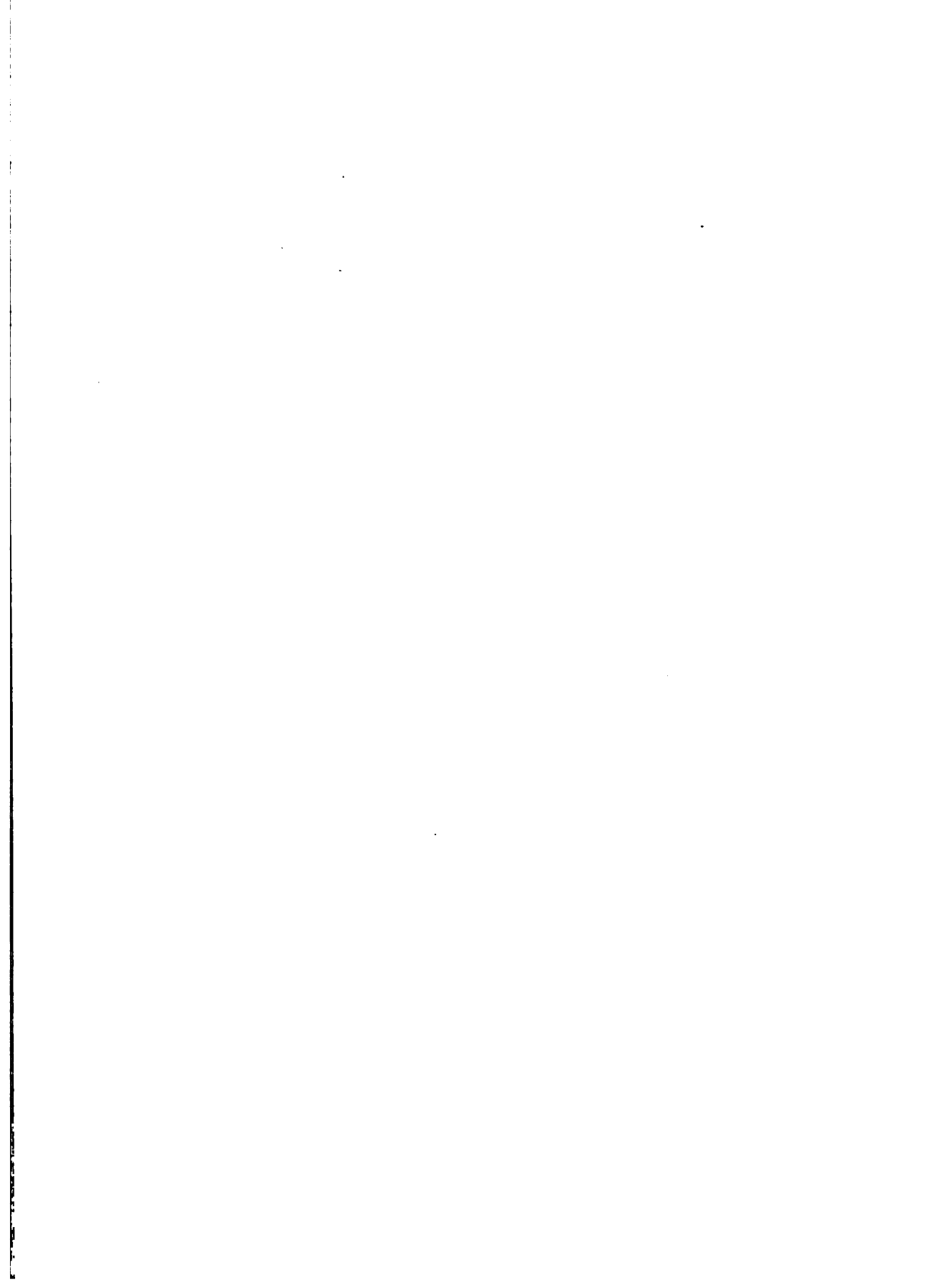
TABLE III.

Color	Wave Length
1	420 Violet
2	435
3	449
4	463 Blue
5	474
6	483
7	490
8	497
9	504
10	514
11	527 Green
12	543
13	556
14	566
15	574
16	580 Yellow
17	588
18	595
19	606 Orange
20	626
21	641
22	658 Red

There is a widespread demand for reference standards of color in terms of which other colors may be specified. Such standards may easily be prepared of any desired hue or shade, but the great difficulties are in choosing rational and uniform divisions on the one hand and in obtaining dyes and pigments that are permanent on the other. Both difficulties would be largely obviated by the adoption of a fixed rational chromatic scale for use as a primary standard.

WASHINGTON, April 27, 1909.

2192—No. 1—09—7



AN APPROXIMATE EXPERIMENTAL METHOD FOR THE ANALYSIS OF EMF WAVES.

By P. G. Agnew.

1. INTRODUCTION.

It is often of importance, both in physical and in engineering work, to obtain a knowledge of the harmonics present in the electromotive force wave of a generator. While several types of apparatus are commercially obtainable, notably oscillographs and curve tracers, which trace the wave with various degrees of accuracy, most of such apparatus is elaborate and expensive. After the curves are obtained the ordinates at a specified number of points are accurately measured and from the results the amplitudes of the various harmonics are calculated. The arithmetical computation of these Fourier coefficients has been greatly simplified within the last few years by methods developed by a number of different investigators. In some forms of apparatus the curve is not obtained automatically, but the ordinates are given directly by point to point methods. Of some 20 methods summarized by Orlich¹, all but one, the method of resonance due to Pupin,² require highly specialized apparatus.

Pupin passed the current to be analyzed through a noninductive resistance, around the terminals of which he connected his tuning circuit, consisting of a condenser and an adjustable inductance. By adjusting for resonance with any desired harmonic, and by observing the rise of potential across the condenser by means of an electrostatic voltmeter, it is possible to calculate the amplitude of the harmonic. But even in this case it is necessary to have at hand a relatively wide range of adjustable self-inductances and electrostatic voltmeters with a considerable range of sensitiveness.

¹ Orlich, *Aufnahme und Analyse von Wechselstromkurven*, Braunschweig, 1906.

² *Am. Journal of Science*, 48, pp. 379, 473; 1894.

The experiments here described were undertaken to determine whether it is possible to get an approximate value of the first and second harmonics present in an emf. wave by means of ordinary portable instruments and calibrated condensers.

A distorted electromotive force wave will pass more current into a condenser than a pure sine wave having the same effective

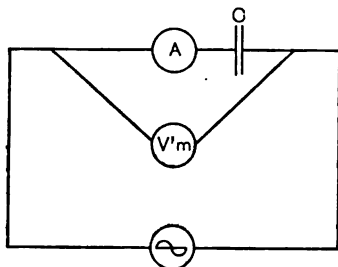


Fig. 1.

value, since the effective rate of change of potential across the terminals of the condenser is greater for the distorted wave. We ought, then, to be able to use this fact to determine the amplitudes of the harmonics to which the distortion is due. For simplicity, suppose the third harmonic only to be present, and let E be the reading of the voltmeter and I that of

the ammeter (Fig. 1). If E_1 and E_3 represent the emfs of the fundamental and harmonic, then we may consider that they cause the respective currents i_1 and i_3 to flow in the condenser. If C is the capacity and p is 2π times the frequency of the fundamental, and if we neglect the resistance of the ammeter,

$$i_1 = E_1 pC;$$

$$E^2 = E_1^2 + E_3^2$$

$$i_3 = E_3 \cdot 3pC;$$

$$I^2 = i_1^2 + i_3^2$$

we thus have four equations connecting E_1 , E_3 , i_1 , i_3 , from which these four quantities may be calculated. This is the inverse of the process sometimes employed in correcting for wave form, for example, in the 3-voltmeter method of measuring inductance. Theoretically, it could be extended to any number of harmonics. Preliminary experiments on waves, obtained by combining emfs. of different frequencies in known proportions, indicated that the method was limited, by the unavoidable errors, to the determination of one harmonic, and that it did not give good results even then. Of course, account has to be taken of the resistance and inductance of the ammeter. It is evident that it is extremely

difficult to obtain indicating instruments whose error is, for example, not greater than 0.1 per cent, under the conditions of frequency and wave form implied.

2. THE METHOD USED.

The necessity of knowing the absolute values of the current and voltage may be avoided by substituting a resistance in place of the condenser and adjusting its value to give the same settings on ammeter and voltmeter as were observed with the condenser. This is really making the ultimate measurement that of an impedance rather than that of an emf., as will be seen by developing the formula for the ratio of the harmonic to the fundamental emf. Suppose the third and the fifth harmonics present. Let L (Fig. 2)

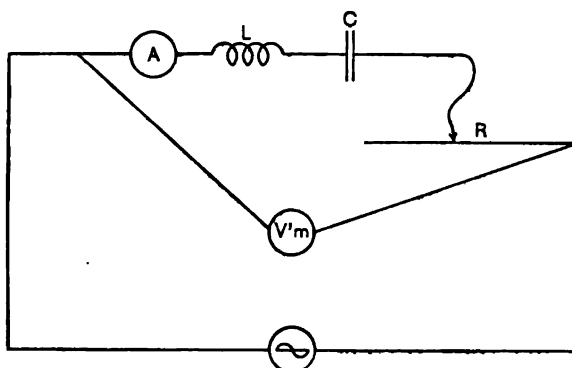


Fig. 2.

be the total inductance, and R the total resistance of the circuit. Then the three components of current are

$$\frac{E_1}{\sqrt{R^2 + \left(pL - \frac{1}{pC}\right)^2}}, \quad \frac{E_3}{\sqrt{R^2 + \left(3pL - \frac{1}{3pC}\right)^2}}, \quad \frac{E_5}{\sqrt{R^2 + \left(5pL - \frac{1}{5pC}\right)^2}}$$

So that

$$I^2 = \frac{E_1^2}{R^2 + \left(pL - \frac{1}{pC_1}\right)^2} + \frac{E_3^2}{R^2 + \left(3pL - \frac{1}{3pC_1}\right)^2} + \frac{E_5^2}{R^2 + \left(5pL - \frac{1}{5pC_1}\right)^2} \quad (1)$$

where subscripts have been added to R and C to indicate that they are corresponding values. Now choose another value of C and adjust R so as to obtain the same settings of voltmeter and ammeter, giving

$$I^2 = \frac{E_1^2}{R_1^2 + \left(pL - \frac{1}{pC_1}\right)^2} + \frac{E_3^2}{R_1^2 + \left(3pL - \frac{1}{3pC_2}\right)^2} + \frac{E_5^2}{R_1^2 + \left(5pL - \frac{1}{5pC_3}\right)^2} \quad (2)$$

Finally, short circuit the condenser and again adjust R to obtain the same settings, giving

$$I^2 = \frac{E_1^2}{R_1^2 + p^2 L^2} + \frac{E_3^2}{R_1^2 + 9p^2 L^2} + \frac{E_5^2}{R_1^2 + 25p^2 L^2} \quad (3)$$

Equations (1), (2), and (3) suffice for the calculation of the ratios E_3/E_1 and E_5/E_1 . Writing for brevity $a_1, a_2, \dots, a_9$ for the squares of the impedances in the nine denominators in the order given above, and comparing (3) successively with (1) and with (2),

$$\left. \begin{aligned} \frac{E_1^2}{a_1} + \frac{E_3^2}{a_2} + \frac{E_5^2}{a_3} &= \frac{E_1^2}{a_7} + \frac{E_3^2}{a_8} + \frac{E_5^2}{a_9} \\ \frac{E_1^2}{a_4} + \frac{E_3^2}{a_5} + \frac{E_5^2}{a_6} &= \frac{E_1^2}{a_7} + \frac{E_3^2}{a_8} + \frac{E_5^2}{a_9} \end{aligned} \right\}$$

Collecting terms, these become

$$\begin{aligned} \left(\frac{1}{a_2} - \frac{1}{a_8}\right) E_3^2 + \left(\frac{1}{a_3} - \frac{1}{a_9}\right) E_5^2 + \left(\frac{1}{a_1} - \frac{1}{a_7}\right) E_1^2 &= 0 \\ \left(\frac{1}{a_5} - \frac{1}{a_8}\right) E_3^2 + \left(\frac{1}{a_6} - \frac{1}{a_9}\right) E_5^2 + \left(\frac{1}{a_4} - \frac{1}{a_7}\right) E_1^2 &= 0 \end{aligned}$$

Replacing the six coefficients in the parentheses by $m_1, m_2, m_3, m_4, m_5, m_6$, respectively, we get

$$\left. \begin{aligned} m_1 E_3^2 + m_2 E_5^2 + m_3 E_1^2 &= 0 \\ m_4 E_3^2 + m_5 E_5^2 + m_6 E_1^2 &= 0 \end{aligned} \right\} \quad (4)$$

Solving for the ratios of the harmonics to the fundamental,

$$\frac{E_3}{E_1} = \sqrt{\frac{m_3 m_5 - m_2 m_6}{m_2 m_4 - m_1 m_5}} \quad (5)$$

$$\frac{E_5}{E_1} = \sqrt{\frac{m_1 m_6 - m_3 m_4}{m_2 m_4 - m_1 m_5}} \quad (6)$$

In case it is desired to make the assumption that only one harmonic is present, we have only to put E_3 or E_5 equal to zero in the first equation of (4) giving, for the presence of the third harmonic only,

$$\frac{E_3}{E_1} = \sqrt{\frac{-m_3}{m_1}} \quad (7)$$

and for the presence of the fifth alone,

$$\frac{E_5}{E_1} = \sqrt{\frac{-m_5}{m_1}} \quad (8)$$

Theoretically the method might be extended to any number of harmonics, but the limitations of the accuracy of the readings and the greatly increased labor of computation limit it practically to two.

Should it be desired to solve for other harmonics instead of for the third and fifth, as in certain cases of polyphase work, the changes necessary in equations (1), (2), and (3) are obvious.

A vector diagram brings out the relations more clearly. The terms on the right of equation (1) are the squares of the component currents.

$$I^2 = i_1^2 + i_3^2 + i_5^2$$

Let the fundamental be represented by the x coordinate of a system of space vectors (Fig. 3), i_3 by the y and i_5 by the z coordinate. The resultant vector, I , then represents the combined current. The relative magnitudes of the component currents are next changed by altering the constants of the circuit. The vector I remains of the same length, but rotates to a new position. The constants of the circuit being again changed, the end of the vector I again changes to a third position on a sphere of radius I . Hence the above algebraical solution is mathematically analogous to the

solution of the spherical triangle determined by the three positions of the end of the vector I .

This graphical representation shows that the analysis of a much distorted wave will give values which are not only relatively, but absolutely, more accurate than that of a wave which is nearly sinusoidal. We are determining the sines of small angles by the changes in the cosines of the angle, and so the method can not be expected to yield reliable values of harmonics which are present

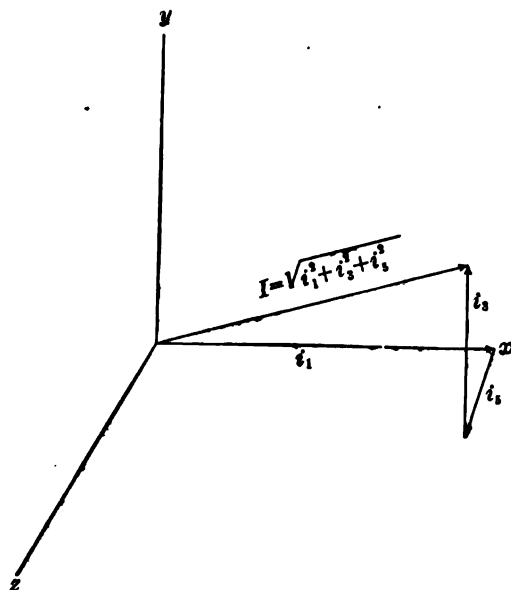


Fig. 3.

to the extent of only a per cent or so. Accordingly, if in a numerical calculation a component comes out as a small imaginary of *the above order of magnitude*, it should be taken to be zero so far as the accuracy of the method indicates.

3. EXPERIMENTAL CONDITIONS FOR GREATEST ACCURACY.

The quantities which enter into the calculation are resistance, inductance, and capacity, which depend directly on bridge measurements, together with the frequency. It is to be noted that the only thing required of the voltmeter is sensibility, as, in each case, it is set at the same point and is working on the same frequency and wave form. The same is true of the ammeter with

respect to scale errors and the effect of frequency on its readings, but the wave form of the current changes slightly between readings. Yet, if the ammeter is affected by wave form, the error enters only differentially.

Two values of capacity are required for the determination of two harmonics. It has been found by experiment that the ratios of the two values of capacity employed may conveniently lie between 1.5 and 2, for the cases of nearly sinusoidal and much distorted waves, respectively. With such a ratio, and if R_1 is kept as low as possible, R_2 will, roughly, be half of R_1 , thus giving good spacing. As shown below, the use of good paper condensers and soft-iron ammeters will give fairly accurate results on badly distorted waves, but for good waves it is better to use standard mica condensers with a dynamometer-ammeter, and the accuracy may be still further increased if an approximate value of the phase angle of the condensers is known. In case an alternating current mil-ammeter is not available, the series resistance of an alternating current voltmeter, preferably of the dynamometer type, may be short-circuited, giving a mil-ammeter which requires from 0.05 to 0.1 amperes for full scale deflection. The recently introduced type of thermo-ammeter has the advantage of having a negligible inductance and extremely low resistance, but the needle is so slow in assuming its final position that the source must be very steady to get consistent readings. Yet some inductance is of advantage in that it may be used to neutralize the reactance of the condenser, and also to reduce the effect of the residual higher harmonics of which the formulas do not take account. It is necessary in analyzing very good waves that either the inductance of the circuit be made negligible, or that it be made of such a magnitude that its frequency of resonance with the capacity used shall be at least as low as the highest harmonic of which the formulas take account, this being the fifth in equations (5) and (6), above. If this is not done, resonance may occur in the case of some neglected higher harmonic, bringing it out strongly in comparison with the lower ones of which the formula does take account, even though its amplitude in the emf. wave may be negligibly small. It is usually easy and convenient in practice to bring the resonance point between the two harmonics of which account is taken,

or even, roughly, to approach the conditions of resonance for both C_1 and C_2 ,

$$\begin{aligned}9p^2C_1L &= 1 \\25p^2C_2L &= 1\end{aligned}$$

with a single value of L . This is the ideal condition for accuracy, although for other practical considerations it is better to take C_1 and C_2 rather closer together than this condition requires. In fact, nothing is gained by attempting to make the tuning close.

By making settings on the ammeter and voltmeter well up on the scale, and allowing them to stand some little time with current on, so as to insure steady conditions, the impedance settings may be made to 0.1 per cent, or slightly better under favorable conditions. Hence, the calculations should be carried to at least as great an accuracy, but this need be done only in the case of a_1 , a_4 , a_7 , m_3 , and m_6 , as these are the computed quantities on which the accuracy chiefly depends.

4. EXPERIMENTAL RESULTS.

A few other details will be considered in connection with some actual analyses.

In the first case, a badly distorted wave was obtained by connecting in series three machines from a harmonic alternator set, in which the machines are all rigidly connected to the same shaft. The actual ratios of harmonics could be determined separately. The values given are corrected for slight impurities known to be present in the fundamental. Paper condensers and an ammeter of the soft-iron type giving full scale for 0.5 ampere were used in this case. In the following table of the data of the experiment, the harmonics are expressed in per cent of the fundamentals:

$C_1 = 10.00$ microfarads.	$R_1 = 10.2$ ohms.
$C_2 = 19.32$ microfarads.	$R_2 = 85.0$ ohms.
$L = 10.6$ millihenrys.	$R_3 = 140.6$ ohms.
Freq. = 61.2 cycles per second.	$E = 33$ volts.
	$I = 0.235$ ampere.

	E_1	E_2
	<i>Per cent.</i>	<i>Per cent.</i>
Calculated.....	33.6	17.1
Actual.....	32.2	16.7

With the same instrument and the same type of condenser, the method was tried with a machine giving a wave approximately sinusoidal. When computed without allowing for the phase angle of the condensers, which averaged $14'$ at 60 cycles, E_2 came out a fairly large imaginary, and E_1 about 5 per cent. When computed, allowing for the phase angle of the condensers by adding to R the equivalent series resistance, $r = \tan \theta / pC$, due to the dielectric loss, better results were obtained. Here, as in the cases which follow, they are compared with the results of careful analyses, by Dr. M. G. Lloyd, some of which were obtained with a Rosa curve tracer, and others by means of a special rotating commutator designed by Dr. Lloyd and Mr. J. V. S. Fisher.³

	E_1	E_2
	<i>Per cent.</i>	<i>Per cent.</i>
Calculated values.....	6.6	Small imaginary.
Curve tracer values.....	7.0	2.0

The phase angle of the condensers used was determined for 60 cycles. The same correction for phase was applied to a_1 , a_2 , and a_3 , similarly the same to each of a_4 , a_5 , a_6 , no attempt being made to distinguish between values of R_1 for the different frequencies. In fact, a careful examination of the equations, or of Fig. 3, will show that the final accuracy depends largely on the determination of a_1 , a_4 , and a_7 , and consequently on the calculation of m_3 and m_6 , which depend directly upon them, so that the value of r corresponding to the fundamental may safely be used to correct the higher frequency terms also.

³ This Bulletin 4, p. 467; 1908.

The following case is that of a composite wave from a 3-machine motor-generator set, the third being the only harmonic added:

$$C_1 = 1.0037 \text{ microfarads.}$$

$$R_1 = 161 \text{ ohms.}$$

$$L = 61.2 \text{ millihenrys.}$$

$$R_2 = 2082 \text{ ohms.}$$

$$E = 198 \text{ volts.}$$

$$I = 0.08 \text{ ampere.}$$

Frequency = 61.2 cycles per second.

	Per cent.
E_s (calculated).....	24.6
E_s (actual).....	22.5

The condenser was a mica standard, having a phase angle at 60 cycles of $4'$. The calculated value of E_s is not changed by allowing for the phase angle of the condenser. The mil-ammeter used was of the dynamometer type, full scale reading 0.1 ampere, resistance 160 ohms, inductance 61.2 millihenrys.

When used on a machine giving a good emf. wave, 200 millihenrys were added to the circuit containing this mil-ammeter and the mica condensers in order to suppress the residual higher harmonics, as explained above.

	E_2	E_3
	Per cent.	Per cent.
Calculated values.....	7.2	0
Curve tracer values.....	5.5	1

Allowance for dielectric loss was necessary.

A still more rigorous test of the method was made by attempting to analyze the emf. of a machine in which the third harmonic is practically negligible.

	E_2	E_3	E_7
	Per cent.	Per cent.	Per cent.
Values by rotating commutator.....	0.6	1.6	1.0
Calculated for third and fifth.....	Small imaginary.	Small imaginary.	
Calculated for third only.....	4.8		
Calculated for fifth only.....		2.1	

Evidently the most the method is capable of in the case of so good a wave is to indicate that the harmonics are extremely small.

But in a large number of engineering problems this is all that is required. As explained above, the small imaginaries indicate that the wave is so nearly sinusoidal that a separation into three components is beyond the accuracy of the method. Either of the above values, rough though they are—namely, 4.8 per cent for the case where all the residuals are lumped together and computed as the third, or 2.1 per cent similarly obtained for the fifth—really gives a more adequate description of the wave than to call it exactly sinusoidal. At first sight it seems strange that an equally good or an even better result should be obtained by solving for only one harmonic than by trying to take account of two, but it is to be noticed that in the case of one harmonic only the result depends on but two readings, in one of which the condenser is short circuited while in the other the current is so nearly in quadrature that a small change in the resistance component of the impedance is negligible. Like instances, where an analysis for one harmonic only gives better results than one including two harmonics, may occur in machines of higher frequency.

The following refers to an 180-cycle machine:

	Thrd	Fifth	Seventh
	<i>Per cent.</i>	<i>Per cent.</i>	<i>Per cent.</i>
Curve tracer values.....	4.0	0.4	0.4
Calculated for third and fifth.....	4.6	Imaginary.	
Calculated for third only.....	4.3		

In this, as in the other cases of very good wave form, calculation for one harmonic only has the advantage of being nearly independent of the energy loss in the condenser, the latter making in this particular case a difference of only 0.1 per cent.

For still higher frequencies, where small values of capacity can be used, it would be advisable to employ an air condenser in order to avoid the uncertainty of correcting for the capacity and phase angle with frequency. In all other cases the condenser should be calibrated at the frequency used.

5. CONCLUSION.

From the foregoing we may say that the accuracy of the method is intermediate between that of the curve tracer and that of the oscillograph, and may be taken as 2 or 3 per cent. of the fundamental.

It has the disadvantage of giving no idea of the phase relations, and hence of the form factor, and of not lending itself to the analysis of current waves, unless a considerable amount of energy is available.

Its chief advantages are that no special apparatus is required and that the calculations are simple. In case it is desired only to know whether or not the wave is nearly pure all the harmonics may be lumped together and computed as if the third were alone present, thus still further simplifying the observations and computations. In special cases a fairly complete knowledge of the impurities may be gained, for example, in the case of a star-connected, 3-phase generator. If the wave be analyzed for the fifth and seventh, the first four odd harmonics are taken account of, since the third and ninth are automatically eliminated by the star connection.

But for the accuracy necessary in commercial work engineers have usually found that the third and the fifth are the only harmonics large enough to be of importance.

Another advantage of the method is that neither direct access to the generator nor the use of a synchronous motor is necessary, so that observations may be taken at a distance from the machine.

I am indebted to Messrs. F. W. Grover and H. L. Curtis for the determination of the inductances, and the capacity and power factor of the condensers used, and to Mr. Louis Cohen for many valuable suggestions. In fact, the method was suggested by an investigation by Mr. Cohen of the differential damping out of harmonics in an alternating magnetic field, by a copper slab placed in it.

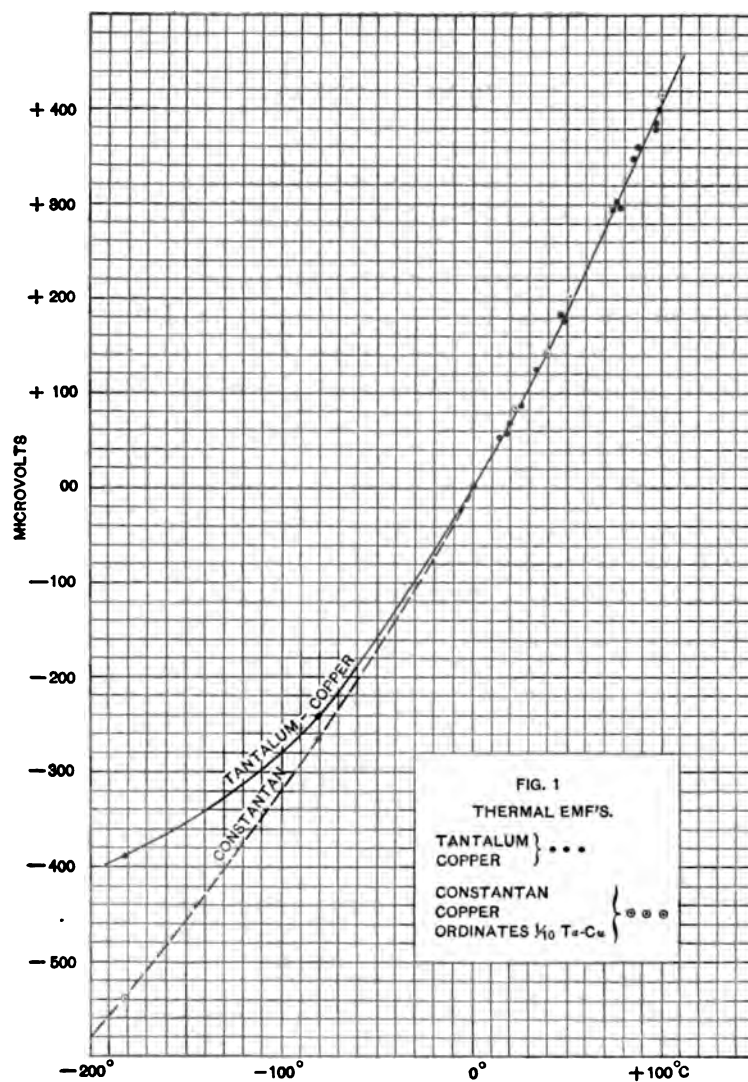
WASHINGTON, April 5, 1909.

NOTE ON THE THERMOELECTRIC PROPERTIES OF TANTALUM AND TUNGSTEN.

By W. W. Coblentz.

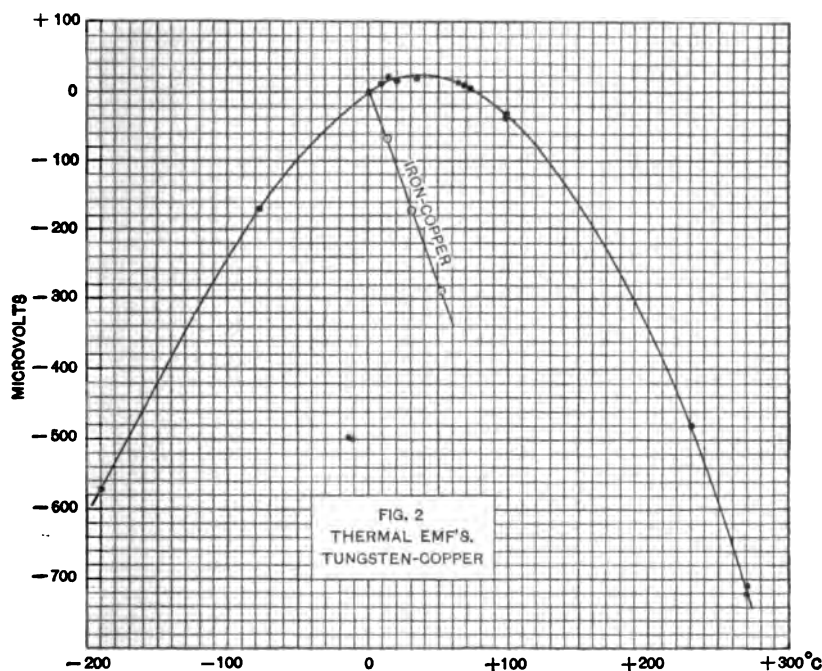
Tantalum is obtainable in fine wires; it is pliable and does not oxidize readily at ordinary temperatures, and hence suggests itself as a substitute for iron (steel) in connection with constantan wire in thermopiles. The following observations on the thermoelectric behavior of tungsten and tantalum may be of interest. The tantalum wire employed was 25 cm long and 0.048 mm in diameter. It was taken from an unused incandescent lamp. The resistance was 0.85 ohm per cm, which was about 0.9 that of a sample of steel wire. The ends of the tantalum wire were wound tightly around copper wires, about 0.1 mm diameter, and then soldered. The copper wires were joined to a potentiometer, by means of a mercury reversing switch, and the comparisons made against a standard cell. Observations were also made on copper constantan and on iron constantan couples made of wires 0.0513 mm in diameter. The temperature was varied from 10° to 100° , and an attempt was made to eliminate the lag of the mercury thermometer. Observations were also made at the temperature of liquid air and of solid carbon dioxide. The results are shown in Fig. 1, in which the black dots indicate the observations on the tantalum-copper couple drawn to scale.¹ The observations at 10° , 24° , 32° , and 47° , respectively, belong to the first series. The observations at 17° , 50° , and higher temperatures were made on a new couple and show no marked variation from the preceding. The circles show the observations on the constantan-copper couple, the ordinates (microvolts) being drawn to one-tenth the scale of the tantalum couple.

¹ The thermoelectric power of a tantalum-copper couple found by Pearson (Electrician 63, p. 594, 1909) is only about one-tenth this value. Since no check measurements were made on known elements the discrepancy may be due to an error in recording the results. His observations were made by joining the couple with a galvanometer.



The results show that throughout the range investigated the thermoelectric power of tantalum (against copper) is about 4.1 microvolts per degree, which is one-tenth that of the copper-constantan couple. The direction of the current is the same as in the copper-constantan couple.

The tungsten wire used in the present experiments was taken from a large "series" lamp. It was about 11 cm long and 0.26 mm diameter. To the ends were wound copper wires 0.1 mm diameter and covered with solder. Tungsten is very brittle, and the sample was accidentally broken before the observations were completed, so that the observations at 231° and 269° (the freezing points of tin and bismuth, respectively) were made with a length of wire less than 6 cm long, the copper wires being fused to the tungsten. It was found that this did not affect the results.



The thermal electromotive force curve of tungsten is given in Fig. 2, and is rather unusual in that its inversion temperature occurs at about 40° C. Other combinations, which have an inversion

temperature near 0° , are cadmium-iron at 170° and copper-zinc at about 30° C. From its appearance it is inferred that the thermoelectric curve is not a true parabola, as was also observed by Dewar and Fleming² on numerous other metals. For temperatures below -100° C the thermoelectric power (against copper) is about 3.5 to 4 microvolts per degree, while above $+200^{\circ}$ C the thermoelectric power is about 4.5 microvolts per degree, so that tungsten, like tantalum, shows no apparent advantage in thermoelectric work.

WASHINGTON, January 15, 1909.

²Dewar and Fleming, *Phil. Mag.* 40, p. 95, 1895.

THE ESTIMATION OF THE TEMPERATURE OF COPPER BY MEANS OF OPTICAL PYROMETERS.

By G. K. Burgess.

The experiments here described are a part of a proposed series of determinations of such thermal properties of materials, at high temperatures, as are of considerable technical interest.

The measurement of the temperature of inaccessible masses of hot metal, whether liquid or solid, is of prime importance to the metallurgist, and it is generally recognized that an optical or radiation pyrometer, sighted upon a clear surface of liquid metal, such as iron or copper, indicates too low a temperature if the pyrometer has been calibrated in the usual way in terms of the radiation from an inclosure approximating a black body.¹ It was in order to furnish the data needed for an approximately correct estimate of the temperature of liquid and solid copper by means of optical pyrometers using monochromatic light and by means of pyrometers using total radiation that the following experiments were undertaken. Some preliminary observations have also been made on iron, but further work has been postponed until the greater facilities of the new heat laboratory now in course of erection are available. It is also hoped to make similar measurements on aluminum, brass, bronzes, and special steels.

It is a very difficult matter to determine accurately by any method the true temperature of the surface of a mass of metal, although in the case of metals which are good conductors and at the same time poor radiators, as is liquid copper, the uncertainties are reduced to a minimum. As what is generally desired,

¹ See "Optical Pyrometry," C. W. Waidner and G. K. Burgess, this Bulletin 1, p. 189, 1905, and "Pyrometer Testing and Heat Measurements," Cir. Bureau of Standards, No. 7.

however, is not the temperature of the surface of the metal, but of its mass as a whole, this difficulty is of minor importance, since the corrections to the optical pyrometer sighted on the surface may be given so as to reduce its readings to the temperature at some definite region within the mass of the metal. A complete experimental study of the temperature distribution within a considerable bulk of cooling copper has not as yet been undertaken, but a preliminary attempt is here made to determine, to a degree of accuracy sufficient for most technical needs, the relation between the readings of the two types of radiation pyrometers above mentioned and those of a thermocouple whose junction is placed just beneath the surface of the metal.

Arrangement of Experiments.—The experiments were carried out in a darkened room during September and October, 1908, using an American Gas Furnace Company's No. 3 melting furnace for melting the copper, which was contained in shallow crucibles, both magnesia and graphite, of 8-cm inside diameter. With the cover of the furnace removed, the gas and air supply could be so regulated that either a clear copper surface or one of oxide could be obtained at will on the liquid copper, the crucible being so set in the furnace that the flames did not play directly on the metal surface in line with the optical pyrometer.

The crucible was set so high in the furnace that radiation from the inside walls of the furnace could not be reflected into the pyrometer, which was pointed at the copper surface at an angle of 25° to 30° with the vertical. In the case of the rough oxide surface diffuse reflection from the furnace walls may affect the optical readings, especially if the walls are hotter than the oxide surface. This effect was shown to be small, in the experiments where monochromatic light was used, by changing the height of the crucible in the furnace; but it becomes serious when using the whole spectrum. When sighting on the pure copper liquid surface, which forms a mirror, there is only specular reflection from bright objects to contend with, and in this case the metal may be at the bottom of a deep furnace, and no error can arise from reflection from furnace walls or other incandescent sources unless they can actually be seen in the pyrometer telescope.

In order to measure the true temperature of the layer of copper immediately beneath the surface, a thermocouple was inclosed in a thin porcelain tube bent into the form of a crook, in the tip of which was placed the junction of the couple. This method of introducing the thermocouple had the advantage of leaving virtually the whole surface of the copper open to the view of the optical pyrometer, besides eliminating any error in the temperature readings arising from conduction along the thermocouple wires.

The Pyrometers.—The temperature readings of the platinum, platinum-rhodium thermocouple were taken with a Siemens and Halske pyrometer galvanometer of 400 ohms resistance. The optical pyrometer used was of the Holborn and Kurlbaum type, using the Morse principle of disappearing lamp filament, although the results obtained apply equally well for any type of optical instrument using the same red-colored light, such as the Wanner and Féry-absorption pyrometers and the Morse thermogage when the last is provided with a suitable red glass between the lamp and eye, as in the Holborn-Kurlbaum type of this instrument. There will still be a small correction, however, for the slightly differing reds used in these various instruments. In order to bring out the effects of selective emission in the radiation from copper, measurements were also taken with green glasses in place of the red between the eye and the pyrometer lamp. The total radiation instrument used was a Féry reflecting telescope, with gold mirror. It is probable that other total-radiation pyrometers would give results differing slightly from those found here, and even two Féry instruments would probably not give identical results.

All the instruments used were calibrated in terms of the standard temperature scale, on which the melting points of pure zinc, antimony, and copper are taken as $419^{\circ}.0$, $630^{\circ}.5$, and 1084° C, respectively. The temperature of the copper freezing point is lowered by the presence of cuprous oxide, and in a gas furnace, unless a reducing atmosphere is maintained, the copper usually begins to freeze out on cooling at temperatures between 1075° and 1080° C. That the thermocouple whose junction was just beneath the surface of the copper was really indicating a tem-

perature not far from the true temperature of this free surface of metal was evidenced by the fact that, as the crust began to form, the thermocouple never read higher than 1080° , and usually within 5° of 1075° C.

MEASUREMENTS WITH THE OPTICAL PYROMETER.

On Clear Liquid Copper.—To a sufficient degree of accuracy for all practical purposes the relation between the apparent temperature of the clear surface of the metal free from oxide, as given by an optical pyrometer using monochromatic light, and the actual temperature of this surface is linear.

For liquid copper possessing a mirror surface free from oxide and haze we find the following equations to hold for red and green light, respectively:

$$(1) \text{ Red light } (\lambda = 0.65\mu) : t = 1.515 r - 359$$

$$(2) \text{ Green light } (\lambda = 0.55\mu) : t = 1.515 g - 477$$

where t is the true temperature centigrade of the copper and r and g the apparent temperatures given by the optical pyrometer using red and green light, respectively. It will be seen that at the melting point of copper the pyrometer using red light reads 130° C too low, and that the apparent temperatures with green light are higher than those with red light by a constant amount over the whole temperature range, namely, by 78° C.

The precision attained in these measurements is shown in Table I, in which the computed temperatures satisfy equations (1) or (2).

In deducing equations (1) and (2) a slight allowance was made for the fact that the optical readings would tend to be high, due to the slightest impurity on the copper surface.

On Cuprous Oxide.—When the copper freezes, or when an oxide coating forms on the liquid, the readings of the optical pyrometer will be higher than when sighted on the pure metal. For a thin coating of oxide floating on the liquid the optical pyrometer using red light reads approximately 100° C higher than when sighted on the clear metal. When green light is used,

the change from metal to oxide increases the apparent temperature by only about 35° C, indicating the relatively greater intensity of the green in the light emitted by liquid copper. This greenish appearance persists in incandescent solid copper, as may be seen by adjusting the gas feed to remove the surface oxidization.

TABLE I.

Temperatures of Molten Copper with Optical Pyrometers.

With red glass: $\lambda=0.65\mu$			With green glass: $\lambda=0.55\mu$		
Temperature Centigrade		Comp.-obs.	Temperature Centigrade		Comp.-obs.
Computed	Observed		Computed	Observed	
1073°	1077°	— 4°	1082°	1087°	— 5°
1075	1075	0	1083	1087	— 4
1075	1075	0	1092	1101	— 9
1075	1081	— 6	1107	1104	+ 3
1077	1075	+ 2	1115	1119	+ 4
1085	1085	0	1154	1153	+ 1
1085	1081	+ 4	1154	1141	+13
1085	1096	—11	1159	1172	—13
1090	1079	+11	1159	1159	0
1090	1093	— 3	1182	1188	— 6
1090	1096	— 6	1187	1195	— 8
1094	1096	— 2	a. d. of a single obs..... <		

Corrections to Optical Pyrometer Readings.—Table II and Fig. 1 give approximately the true temperatures in terms of the readings of optical pyrometers using red light when sighted upon either the clear copper or the oxide. It will be noticed that for the oxide the relation between the optical readings and the true temperatures is not quite linear. If the liquid surface is hazy, encircled by flames, or only partly oxidized, the optical readings will lie somewhere between the two curves *B* and *D*.

TABLE II.
Temperatures with Optical Pyrometer.

[Red Light: $\lambda=0.65\mu$.]

Liquid Copper		Cuprous Oxide	
Pyrometer Reading	Temperature Centigrade	Pyrometer Reading	Temperature Centigrade
950°	1082°	900°	903°
975	1118	950	958
1000	1156	1000	1020
1025	1193	1050	1087
1050	1231	1100	1159
		1150	1233

MEASUREMENTS WITH TOTAL RADIATION PYROMETER.

The measurements of surface temperature by this method are much less satisfactory than with the optical pyrometer, as slight changes in the properties of the surface produce a much more marked effect on the readings of an instrument using the whole spectrum, visible and invisible, than of one using only a narrow spectral band.

With the Féry pyrometer used in these experiments, the difference in reading at the melting point of copper, when sighted, first on the clear copper and then on the oxide, was from 275° to 300° C, and the apparent temperature of freezing as given by the Féry instrument sighted on the clear liquid was slightly less than 600° C, instead of 1075° or 1080° C, as given by the thermocouple, so

that according to the condition of the copper surface, the corrections to the readings of this instrument lie between 480° and 175° C at the copper melting point. Even a very slight haze or fog over the copper surface will cause a rise of 25° or 50° in the Féry reading, and the presence of flames will change the readings by amounts uncertain both in quantity and direction. The depth of the crucible in the furnace and the effects of radiation from the furnace walls also influence the readings considerably, unless the copper surface is absolutely clear.

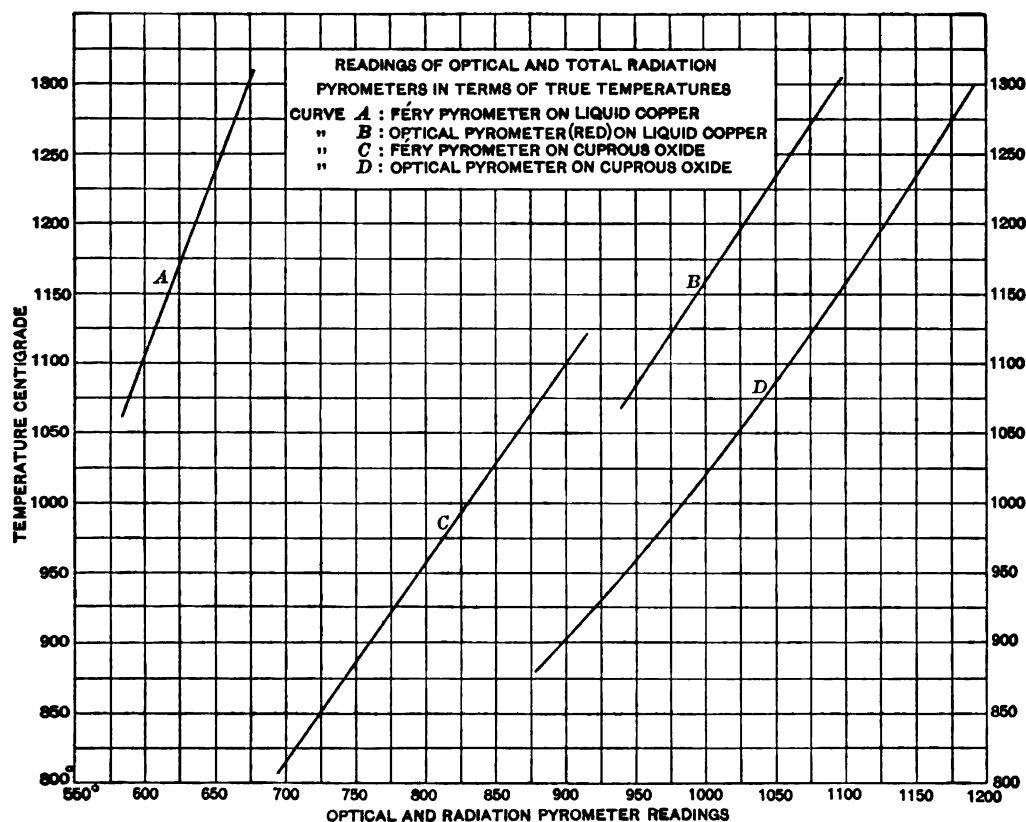


Fig. 1.

The readings of the Féry pyrometer, in terms of true temperature, both for the clear copper surface and for the oxide surface, are given in Fig. 1, lines A and C.

These observations satisfy the following equations:

$$\text{Liquid copper: } t = 3.55 F - 1018$$

$$\text{Cuprous oxide: } t = 1.41 F - 169$$

where t is the true temperature centigrade and F the reading of the F ry pyrometer.

EMISSIVITY OF COPPER AND CUPROUS OXIDE.

The results obtained from the temperature measurements on liquid copper and cuprous oxide may be expressed in terms of their emissivities; that is, by the ratio of the intensity of radiation, either monochromatic or total, for each of these substances to the corresponding intensity of radiation of a black body taken as unity.

TABLE III.

Emissivity of Liquid Copper and of Cuprous Oxide.

Liquid Copper				Cuprous Oxide			
Temperature Centigrade	e_x for—		E_x	Temperature Centigrade	e_x for—		E_x
	$\lambda=0.65 \mu$	$\lambda=0.55 \mu$			$\lambda=0.65 \mu$	$\lambda=0.55 \mu$	
1075°	0.17	0.47	0.16	800°			0.66
1125	.15	.38	.15	900			.60
1175	.14	.32	.15	1000	0.80		.56
1225	.13	.28	.14	1100	.60	0.68	.54
1275			.13	1200	(.49)	(.49)	

For monochromatic light we have, for the emissivity, e_x , of a substance:

$$\log e_x = \frac{14500}{\lambda} \log \epsilon \left(\frac{1}{T_x} - \frac{1}{T_b} \right)$$

where the subscript x refers to the substance under investigation and b to the black body, λ is the wave length of the light used, T is absolute temperature, and ϵ the Napierian base of logarithms.

For total radiation the emissivity, E_x , is given by:

$$\log E_x = 4 (\log T_x - \log T_b)$$

These quantities are given in Table III for both copper and cuprous oxide.

Thwing² also finds $E_x = 0.14$ for molten copper with his total radiation pyrometer sighted on a stream of the metal.

These experiments were carried out with the able assistance of Mr. J. J. Crowe.

WASHINGTON, June 17, 1909.

²C. B. Thwing: On the Emissivity of Molten Iron and Copper, *Phys. Rev.* **28**, p. 190; 1908.

THE RESOLVING POWER OF OBJECTIVES.

By P. G. Nutting.

The mathematical theory of the resolving power of lenses, developed long ago, gives a relation between the radius of the lens r , the wave length λ of the light used in illumination, and the least angular separation φ between the objects to be clearly separated in the image.

$$\varphi = a \frac{\lambda}{r}$$

The value of the constant a depends upon the quality of the definition required in the image.

This constant a has never been determined experimentally, although it is in its very nature an experimental constant. Theory tells us that when the image of one of the objects to be resolved lies in the first dark ring surrounding the other, $a = 0.61$; but it does not tell us how good a definition this means in practice nor to what quality of image other values of the constant correspond.

The work here described was undertaken to determine the constant a experimentally and to establish a relation between different specified qualities of definition in an image and different values of a . Incidentally we wished to find out how near to this theoretical resolving power different objectives could be operated and to study the differences between lack of resolution and residual coma and spherical aberration.

The prime requisite for such work is, of course, an object of homogeneous texture in one dimension, illuminated by monochromatic light. Such an object was found in one half of an uncemented half-tone screen. This consists of parallel rulings on glass, the rulings being etched and filled with opaque material and accurately equal in width to the spaces between them. The rulings run 50 to 200 to the inch, the finest being suitable for small objectives and laboratory distances.

In this work a screen whose spacing was 0.127 mm was placed over the second slit of a monochromatic illuminator lighted by a Nernst lamp. The image was viewed both with high-power objectives and low-power microscopes, care being taken to use only those of greater aperture ratio than the objective being tested; otherwise the effects observed might have been due to lack of resolving power in the observer's eye or in the ocular used.

This arrangement proved to be very sensitive. When near the limit of resolution, varying the distance from objective to screen by as little as 1 per cent, or shifting the wave length of the illuminating light by 15 $\mu\mu$, produced a very perceptible variation in the sharpness of the image. An auxiliary iris stop at the objective proved very convenient in testing for sensibility.

As the limit of resolution is approached, the bright lines become hazy at the edges and then broaden, filling in the dark lines chiefly *from the sides* apparently. Spherical aberration, on the other hand, appears to fill in the dark lines, not from the sides, but from the bottom, leaving the edges more or less sharp. Coma produces a shading off similar to lack of resolution but easily distinguishable from it by its dissymmetry.

Telescope objectives we find fulfill this theoretical resolving power even at full aperture. This is true even of small reading telescopes with objectives of the old cemented doublet type. On the other hand, none of the four photographic objectives tested, all of the best modern types, gave this theoretical resolving power except when stopped down to $F/10$ or smaller. This defect appears to be due to a slight residual spherical aberration that is sacrificed to the oblique corrections.

The chief advantages of this method lie in its sensibility and its concreteness. One disadvantage is the difficulty in obtaining precise numerical results. This is due, not to lack of sensibility, but lack of any fixed criteria of quality. The most easily determined fixed point is when all structure just leaves the image. This is for a value of the constant a of 0.48 or 0.50 with an uncertainty of not over 0.02. At $a=0.60$, outlines are well established in the image, but so rounded that it is barely possible to tell the nature of the object. With $a=0.70$, outlines are firm, and the image bears a close resemblance to the object but lacks sharpness

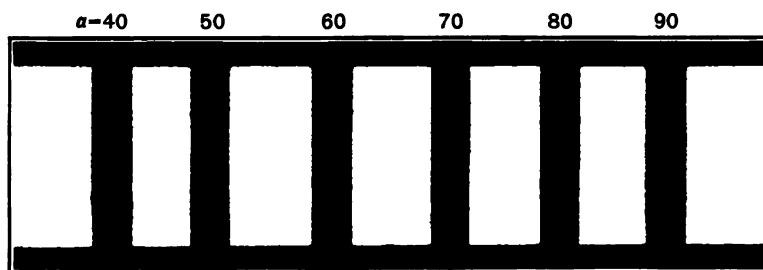


Figure showing variation in resolution in an image photographed near the limit of resolving power of an objective. Enlarged $2\frac{1}{2}$ times. Object 200 lines to the inch.

of definition. At $a = 0.80$ the image is just perceptibly deficient in definition, while at $a = 0.90$ it shows no defects. The uncertainty in a setting on the beginning of lack of definition is about 0.05, considerably larger than for a setting on disappearance of all structure.

The appearance of the image in various stages of resolution is shown in the cut. The image was photographed at about three diameters with a "Planar" lens of 20-mm focal length used as photographic ocular. Exposures were made at $a = 0.40$ (very diffuse), $a = 0.50$, 0.60, 0.70, 0.80, and 0.90. The changes are just easily perceptible from step to step. A variation in resolution from side to side of each image, due to varying wave length, may also be noted. The range of wave length in question was about 12 $\mu\mu$.

When a galvanometer is used with telescope and scale, the galvanometer mirror is in effect a stop half way between the object (scale) and the objective lens of the telescope. In this case the resolving power of the telescope is double what it would be were a stop the size of the mirror placed at the objective. The telescope objective need not be more than four times the diameter of the galvanometer mirror to obtain the best results with proper eye-pieces. Experimental tests of resolving power were made with stop placed in front of objective at a distance of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{7}{8}$ of the distance from the objective to object, and the theory verified with nearly the same precision as with full aperture.

In the resolving-power formula, $\varphi = a\lambda/r$, φ is the ratio of size to distance of the test object and hence applies to the image as well. Writing then $\varphi = \delta/F$ where δ is the size of the smallest resolvable image and F the equivalent focal length of the objective

$$\varphi = \frac{\delta}{F} = a \frac{\lambda}{r}$$

hence

$$2a A = \frac{\delta}{r}$$

or the smallest resolvable detail in the image, measured in wave lengths, is equal to the aperture ratio $A \left(= \frac{F}{2r} \right)$ of the objective.

This is a useful rule in designing or selecting objectives. $2a = 1$ represents the extreme limit of resolution, $2a = 2$ corresponds to a sharply defined image.

When an image is to be viewed by an ocular, δ and λ are necessarily the same for the ocular as for the objective, hence if the ocular has an aperture ratio A equal to or greater than that of the objective, then the resolving power of the system will equal that of the objective alone. It can not be greater, for the light cone is of the same angle for the ocular as for the objective.

CONCLUSIONS.

The resolving power of objectives may be determined experimentally with simple but sensitive apparatus.

The resolving power constant varies from 0.50 for an image showing no detail whatever to 0.90 for an image just free from defects. The theoretical value 0.61 corresponds to a very diffuse but recognizable image.

Every lens fulfills its computed resolving power, but this is masked by residual aberration except in the case of telescope and other objectives that are carefully freed from third-order axial aberrations. The camera objectives examined show best resolution at about $F/10$.

In any lens system to be used at full power, no lens should have a smaller aperture ratio than the objective.

WASHINGTON, August, 1909.

THE THEORY OF THE HAMPSON LIQUEFIER.

By Edgar Buckingham.

1. Introduction.—The action of the Hampson or single-circuit type of liquefier involves the production of cold by the continuous expansion of gas within the liquefier from a high to a low pressure, and the conservation of cold by the Linde-Hampson regenerator, in which the nonliquefied cold gas is passed back over the copper inlet worm, thus giving up its cold to the high pressure gas which is advancing toward the expansion valve, and being itself warmed nearly to the temperature of the incoming gas. If the velocities of the feed and of the exhaust escaping from the regenerator are so small that the kinetic energy is negligible, as is usually the case in practice, the cold of expansion is due solely to the Joule-Thomson or porous plug effect.

The total cold available, per gram of gas fed in, is determined by the initial temperature θ_1 of the gas at the point where it enters the regenerator, its initial pressure p_1 , and the final pressure p_2 , which, in the Hampson liquefier, is one atmosphere. It can be computed if we know the specific volume and the rate of thermal expansion at constant pressure as functions of the pressure between p_2 and p_1 at the constant temperature θ_1 . It is independent of the internal construction or arrangement of the liquefier, of the distribution of temperature in the regenerator coil, and of all other circumstances whatever, provided, as stated above, that the kinetic energies of the feed and the exhaust are negligible.

This total cold available is used in the following three ways: (a) to offset the heat that leaks into the liquefier from outside, either through the insulation or along the copper worm; (b) to cool the whole quantity of gas from its initial temperature θ_1 to

the slightly lower temperature θ_2 , at which the waste gas escapes from the regenerator; (c) to cool a fraction x of the gas from this exhaust temperature θ_2 to the normal boiling point θ_1 , and there condense it into liquid, the cooling and condensation taking place at atmospheric pressure.

The thermal leakage depends on the particular construction of the liquefier and can not be computed *a priori*. The incompleteness of the regenerative process can be made negligible by increasing the interchanging surface; or its effect can be computed and allowed for. These two terms represent losses which reduce the fractional yield or efficiency of liquefaction, x . If we know the latent and specific heats of the gas, together with the above-mentioned data for computing the total cold available, we can compute the ideal value of x which would be attained if the thermal insulation were perfect.

It is easy to prove the correctness of these statements by elementary thermodynamics. This will be done in §§ 2 and 3, while in §§ 4 and 5 it will be shown that the results of computation agree satisfactorily with the published results of experiments on the efficiency of the Hampson liquefier.

2. The total Cooling Effect Available in a Completely Irreversible Expansion.—Suppose the machine to have been working so long that a steady state has been established. Let the resistance offered to the nonliquefied gas in escaping through the regenerator be so small that the pressure on the low pressure side of the valve is sensibly the same as the outside atmospheric pressure. Let the kinetic energy of the eddy currents formed at the valve be all dissipated into heat inside the liquefier—if the valve is replaced by a fine-grained porous plug, this kinetic energy is small and is dissipated within a short distance of the plug.

In Figure 1, let A , at (p_1, θ_1) , represent on the p, v plane the initial state of one gram of gas as it enters the regenerator coil.

Let B , at (p_2, θ_2) , represent the state of the same gas just after it has issued from the expansion valve, the kinetic energy of eddy currents being assumed to be already dissipated. This assumption simplifies the reasoning, but is not essential. The eddy currents may be dissipated anywhere before the waste gas escapes at the exhaust without affecting the final result, except in so far

as slight changes in the distribution of temperature inside the liquefier may influence the thermal leakage in from outside. If any liquefaction at all is taking place, θ_1 is the normal boiling temperature at atmospheric pressure p_1 , and at B the substance is a mixture of liquid and saturated vapor.

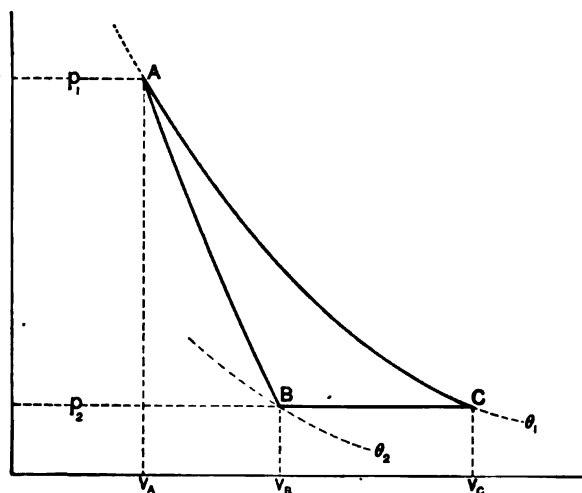


Fig. 1.

Let C , at (p_2, θ_1) , represent the state the gram of gas would have been in if, during the same fall of pressure through a porous plug or throttling valve, heat had been supplied so as to keep the temperature from changing.

Let Q_1 be this quantity of heat. Let Q_2 be the heat needed to convert the mixture of liquid and vapor at B into gas at C while the pressure remains constant at p_2 or one atmosphere. Let ϵ be the internal energy of one gram of the substance.

The work done by the compressor on the gram of gas as it enters the liquefier is $(pv)_A$; that done by the gas as it issues from the valve against the outside pressure p_2 is $(pv)_B$ at the temperature θ_2 , or $(pv)_C$ at the temperature θ_1 . We therefore have for the increase of internal energy during the isothermal process AC

$$\epsilon_c - \epsilon_A = (pv)_A - (pv)_C + Q_1 \quad (1)$$

For the adiabatic process AB , since no heat is supplied, we have

$$\epsilon_B - \epsilon_A = (pv)_A - (pv)_B \quad (2)$$

whence by subtraction and rearrangement we have

$$Q_1 = (\epsilon_c - \epsilon_B) + p_2(v_c - v_B) \quad (3)$$

the pressure p_2 being the same at B as at C , i. e., the adiabatic and the isothermal expansion having taken place between the same limits of pressure, so that $(pv)_c - (pv)_B = p_2(v_c - v_B)$.

Consider now the heat Q_2 needed to heat the mixture from B to C . This is equal to the increase of the internal energy or $(\epsilon_c - \epsilon_B)$ plus the external work done by the substance during this boiling off and warming up. The pressure being constant, this work is merely the pressure multiplied by the increase of volume, or $p_2(v_c - v_B)$. Hence we have

$$Q_2 = (\epsilon_c - \epsilon_B) + p_2(v_c - v_B) \quad (4)$$

Since by the first law of thermodynamics the difference in the internal energy of a system in any two states is dependent solely on the states themselves and not on how they are reached $(\epsilon_c - \epsilon_B)$ is the same however computed, so that equations (3) and (4) are equivalent to

$$Q_1 = Q_2 \quad (5)$$

We may interpret this result in words as follows: If we let the expansion through the valve take place isothermally at the initial temperature and then subsequently abstract from the expanded gas at this final pressure the same amount of heat that we had to put in during the isothermal expansion, the final state reached after this double process is the same as if we had let the expansion go on without supplying or withdrawing any heat at all, as is the case—except for the imperfect thermal insulation—in the operation of the Hampson liquefier.

The magnitude of this quantity of heat Q_1 , or the total cooling effect available per gram of gas, obviously depends only on the limits of pressure and on the initial temperature at which the expansion AC is supposed to occur and not on the details of anything that may happen inside the liquefier.

If we let ρ be the ratio of the heat absorbed by one gram of gas at θ_1 to its fall of pressure in expanding by an infinitesimal amount through a throttling valve or porous plug, with the kinetic energy of eddy currents all dissipated, we have—

$$\delta Q_1 = -\rho \delta p \quad (6)$$

whence

$$Q_1 = \int_{p_1}^{p_2} \rho dp \quad (\theta = \theta_1) \quad (7)$$

We also know that the quantity ρ , or the Joule-Thomson effect, satisfies the familiar equation¹—

$$\rho = \theta \left(\frac{\partial v}{\partial \theta} \right)_p - v \quad (8)$$

where θ is the absolute thermodynamic temperature. If we let v_0 be the specific volume at 0°C. , and $\alpha = \frac{1}{v_0} \left(\frac{\partial v}{\partial \theta} \right)_p$, we have by (7)

$$Q_1 = \int_{p_1}^{p_2} (\theta_1 \alpha v_0 - v) dp \quad (\theta = \theta_1) \quad (9)$$

Hence the total cold available per gram of gas fed into the liquefier may be found if we know the values of the coefficient of expansion α and the specific volumes v_0 and v at 0°C. and at θ_1 as functions of the pressure. It is, of course, not necessary to express these functions by equations; it suffices to plot the values of $(\theta_1 \alpha v_0 - v)$ as ordinates against p as abscissa, draw a smooth curve through the points, and find the integral between any two pressures by taking the area under the curve and between the two pressures in question.

3. Efficiency of the Hampson Process.—Let x be the fraction of the gas fed in that remains finally as liquid. Let θ_e be the temperature of the exhaust escaping from the regenerator. Let L be the latent heat of the boiling liquid and C_p the specific heat of the gas, both at the constant pressure of one atmosphere.

¹ Kelvin Papers Vol. I, p. 428. See also Phil. Mag. (6), 6, p. 518, 1903; and note at the end of the present paper.

The resulting product, at B (Fig. 1), of the actual expansion of one gram of gas fed in is a mixture of x grams of liquid and $(1-x)$ grams of gas. To evaporate the liquid and warm the resulting gas up to θ_3 would require a quantity of heat

$$x(L + \int_1^{\theta_3} C_p d\theta)$$

To warm the $(1-x)$ grams of gas from θ_3 to θ_2 requires no heat *from outside*, for this heat is supplied in the regenerator by more gas, which has already entered the liquefier. The whole gram of gas has now to be heated further, at p_2 , from θ_3 to θ_1 , which requires a quantity of heat

$$\int_{\theta_3}^{\theta_1} C_p d\theta$$

Let H be the heat that leaks in from outside per gram of gas fed in, the whole process being in fact not quite adiabatic. Then the total heat that would have to be supplied from outside, including H , to get the resulting product at B (Fig. 1) back to C is

$$Q_2 = x(L + \int_{\theta_1}^{\theta_3} C_p d\theta) + \int_{\theta_3}^{\theta_1} C_p d\theta + H \quad (10)$$

By comparing this with (5) and (9) we have

$$x = \frac{\int_{p_1}^{p_2} (\theta_1 a v_0 - v) dp - \int_{\theta_1}^{\theta_3} C_p d\theta - H}{L + \int_{\theta_1}^{\theta_3} C_p d\theta} \quad (11)$$

The specific heat of air at one atmosphere is very nearly constant to at least as low as -170°C . If therefore we set $C_p = \text{constant}$, we have as a sufficient approximation for purposes of computation

$$x = \frac{\int_{p_1}^{p_2} (\theta_1 a v_0 - v) dp - C_p(\theta_1 - \theta_3) - H}{L + C_p(\theta_3 - \theta_2)} \quad (12)$$

In these equations, the denominator is the "total heat" of one gram of the substance from liquid at its normal boiling point to

gas at the temperature of the exhaust. Hence the physical meaning of the right-hand side of the equations is clear. The first term is the total cold per gram fed in divided by the cold needed to liquefy one gram, i. e., it is the fraction of the gas that will be liquefied if there is no waste. The second term is the cold wasted by imperfect regeneration, divided by the same denominator, i. e., it is the liquid wasted, per gram of gas fed in, on account of the insufficiency of the regenerator surface. The third term, in like manner, is the liquid wasted on account of the lack of perfect thermal insulation.

This last term can not be computed in any general way. We may, however, say that starting with small rates of feed its importance at first decreases as the rate of feed increases, for the total thermal leakage will not increase so fast as the total rate of flow of gas into the liquefier. Hence the leakage per gram will decrease. Leaving this term out of account we have remaining only quantities on which we have some numerical data for air and hydrogen, and we may use the equation

$$x = \frac{\int_{p_2}^{p_1} (\theta_1 \alpha v_0 - v) dp - C_p (\theta_1 - \theta_2)}{L + C_p (\theta_2 - \theta_1)} \quad (13)$$

for computing x in specific cases. If the regenerating surface is so large that the exhaust has risen sensibly to the temperature of the feed, we have $\theta_2 = \theta_1$ and the last term vanishes, leaving for the ideal case of perfect regeneration and perfect insulation

$$x = \frac{\int_{p_2}^{p_1} (\theta_1 \alpha v_0 - v) dp}{L + C_p (\theta_1 - \theta_2)} \quad (14)$$

4. Application to Air at 16° C.—According to Witkowski² the specific heat of air from +100° C. to -170° C. appears to be about 0.237 calories per gram with, possibly, a slight increase at the lowest temperatures. We shall be close enough to the true value if we take 0.24 or, setting $J = 42 \times 10^6$, $C_p = 1.01$ [joules/gram].

² Phil. Mag. (5), 42, p. 1, 1896.

The latent heat of boiling air depends, of course, on the composition of the liquid and is somewhat uncertain. The experiments of Behn, Shearer, Estreicher, Richtmeyer, and Alt indicate a value of about 50 calories per gram, so that we shall take $L = 210$ [joules/gram].

Taking θ_1 , the boiling point at one atmosphere, to be $-191^{\circ}.5$ C. or $81^{\circ}.5$ K³ we have for the total heat of 1 gram of liquid air up to the exhaust temperature

$$L + C_p (\theta_3 - \theta_2) = 210 + 1.01 (\theta_3 - 81.5)$$

which gives us the denominator of equation (13) when θ_3 is known.

For the specific volume and coefficient of expansion we have data by Witkowski⁴ at several temperatures and up to 130 atmospheres. We have also data by Amagat⁵ up to very much higher pressures. Witkowski's data have been preferred because they are given in much greater detail. From 130 atmospheres up to 200 atmospheres Amagat's data have been used to guide the extrapolation of Witkowski's values.

We may illustrate the use of equation (13) by the data for air at 16° C. It is needless to give the details of the computation of the Joule-Thomson effect, but the results are shown in Fig. 2 in which the abscissas are pressures in atmospheres and the ordinates the values of the Joule-Thomson effect, the unit of $(\rho \times p)$ being $(1 \text{ atm.} \times 1 \text{ cm}^3)$ or 0.1013 joules, and the unit of mass being 0.001293 gram. The values deduced from Witkowski's data are marked by crosses. At low pressures, ρ appears as a small difference of two large quantities and is therefore subject to large errors. To help out, we have the value determined by Joule and Thomson by direct experiment for various low pressures. This value is marked by a circle.

Curve *A* has been drawn to represent $\rho = f(p)$. From 50 atmospheres down it is drawn somewhat arbitrarily, but the uncertainty in the total area is not great. These experimental values lie sur-

³The symbol K (Kelvin) is used to denote the thermodynamic absolute scale of temperature.

⁴Phil. Mag. (5), 41, p. 288, 1896.

⁵Ann. de Ch. et Phys. (6), 29, p. 68, 1893.

prisingly close to curve *B*, which shows what the Joule-Thomson effect would be at 15°.6 C., if the air followed van der Waals's

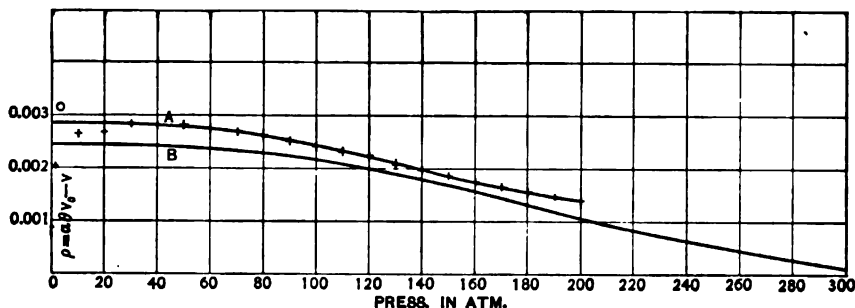


Fig. 2.

equation and had the critical constants $\theta_c = 133^\circ \text{K}$, $p_c = 39.3 \text{ atm.}$, $v_c = 2.85 \text{ cm}^3/\text{gm.}$

From curve *A* we can find the value at 16° C. or $\theta_1 = 289^\circ \text{K}$ of

$$\int_{p_1}^{p_2} (\theta_1 \alpha v_0 - v) dp \text{ between any two pressures up to 200 atmos-}$$

pheres. We can then, after reducing the values to joules per gram, substitute them in equation (13), together with the values of C_p and L already given. We thus get the value of the efficiency x for any initial pressure, up to 200 atmospheres, and for any value of the exhaust temperature, θ_2 , the thermal insulation being assumed to be perfect.

The resulting values of x are plotted in Fig. 3. Curve *A* is computed for $\theta_2 = \theta_1$, i. e., for perfect regeneration. Curve *B* is computed for $\theta_2 = \theta_1 - 5^\circ$. The change in x caused by small changes in $(\theta_1 - \theta_2)$ is nearly linear, as may be seen from equation (13), so that the value of x for any other small value of $(\theta_1 - \theta_2)$ may be found from these two curves by linear interpolation or extrapolation.

5. Comparison of Computed with Observed Values of x .—It is interesting to compare the computed values as shown by the curves of Fig. 3 with the values observed by Messrs. Bradley and Rowe.⁶ All but three of their experiments were made at an initial

⁶ Phys. Rev. 19, p. 330, 1904.

temperature of about 2°C. or $\theta_1 = 275^{\circ}\text{K.}$ It would thus have been better to compute the values of x for this temperature, but data were lacking, so that 16° has been used. There are, in all, ten experiments at about 181 atmospheres initial pressure, and six others at various lower pressures. To be comparable with the computed values these observations have to be reduced.

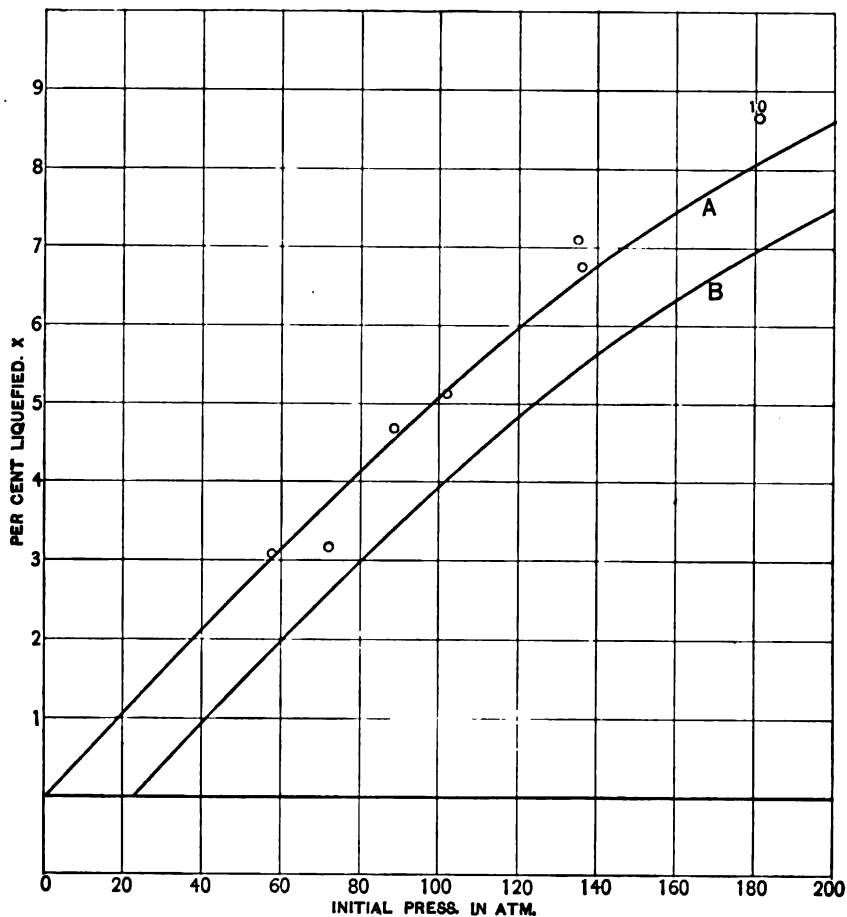


Fig. 3.

In the first place, the insulation of Messrs. Bradley and Rowe's liquefier was, of course, not perfect. To correct for this I have assumed that within the comparatively small variations of flow

used, the temperature gradients were not sensibly altered by changes in the rate of flow, so that the total heat leakage inward was the same in all cases, and therefore that the amount of liquid wasted per gram of air fed in, or the decrease in x due to thermal leakage, was inversely proportional to the rate of feed. Letting x be the observed value, x' the value that would have been observed if the insulation had been perfect, M the rate of flow, and A the decrease of x by thermal leak when the flow is unity, we have

$$x = x' - \frac{A}{M}$$

The value of A was found from the three, presumably strictly comparable, experiments made with special reference to the influence of varying rate of flow. Its mean value, when x is given in per cent and M in cubic feet per minute, is 20.6, so that we have as our reduction formula

$$x' = x + \frac{20.6}{M}$$

All the observed values of x have thus been corrected for the effect of thermal leak. They are shown in Table I, Column VI.

In the experiments at 181 atmospheres, the temperature of the exhaust was always very near that of the feed, so that the regeneration was sensibly perfect and no correction is needed on this score. At the lower pressures, however, the exhaust was colder than the feed. The necessary correction was found by linear interpolation between curves *A* and *B* of Fig. 3 and the values of x with this second correction applied are given in Column VII of the table. The sixteen values in this column are therefore, as nearly as we can compute them, the results which would have been obtained if the thermal insulation had been perfect and the regenerator surface large enough for perfect heat interchange even at the low pressures where the linear velocity of the air through the worm was greatest.

But these values still all refer to an initial temperature of about 2° C. We have finally to reduce them to 16° before comparing with the computed curve *A* of Fig. 3. To do this, we plot the

results of the four experiments made at $p_1=180$ atmospheres and at initial temperatures of 2° , 30° , 59° , and 92° C. When the observed values of x are corrected as above for thermal leak and plotted against θ_1 , they lie on a smooth curve. The ratio of the ordinate of this curve at 16° to that at 2° is 0.87. I have assumed that the ratio would have been the same if the pressure p_1 had been different from 180 atmospheres, and have reduced the corrected values of x for 2° to 16° by multiplying them by 0.87.

TABLE 1.

Summary of Observations by Bradley and Rowe.

I	II	III	IV	V	VI	VII	VIII
Ref. to Phys. Rev., vol. 19 pp.	p_1 (atmosph.)	M (R. per min.)	x observed	$\theta_1-\theta_2$	x corr. for thermal leak	x corr. for ($\theta_1-\theta_2$)	$0.87 x$ = x_{16}
335	181.3	14.9	8.6	0.00	9.98	9.98	
335	180.3	15.5	8.0	0.00	9.33	9.33	
335	180.3	14.4	8.1	0.00	9.53	9.53	
335	181.0	15.0	8.7	0.00	10.07	10.07	
335	181.6	13.6	8.8	0.00	10.32	10.32	
339	181.3	14.9	8.6	0.00	9.98	9.98	
339	180.3	14.4	8.1	0.00	9.53	9.53	
341	180.3	13.6	8.8	0.00	10.31	10.31	
341	180.3	10.8	8.3	0.00	10.21	10.21	
341	180.3	8.1	7.8	0.00	10.34	10.34	
.....	180.7					9.96	8.67
339	136.1	15.8	6.4	0.25	7.70	7.76	6.76
339	102.0	16.2	4.5	0.50	5.77	5.89	5.13
339	72.0	17.1	1.7	3.25	2.90	3.65	3.18
339	134.7	14.6	6.5	1.0	7.91	8.14	7.09
339	88.4	15.8	3.5	2.5	4.80	5.37	4.68
339	57.8	16.4	1.0	5.5	2.26	3.52	3.07

There are evidently some rather uncertain assumptions involved in this reduction as well as in the correction of observations at very different initial temperatures for thermal leakage; but it is the best we can do, and it is better to base the reduction on the

internal evidence of the work of Messrs. Bradley and Rowe than to use any formula, theoretical or otherwise, deduced from other and independent *a priori* considerations. The ten observations at 181 atmospheres have been averaged before reduction, so that we have in Column VIII of the table seven values which are, as nearly as we can tell, what would have been obtained at an initial temperature of 16° C., and the various initial pressures, under ideal conditions of insulation and regeneration. These seven points are plotted in Fig. 3 surrounded by circles.

When we remember that all the points except that marked 10 represent single experiments, we must, in view of the unavoidable experimental errors in x which may be estimated from the first ten values in Column VII of the table, consider the agreement of the observed and computed values to be sensibly perfect up to about 130 atmospheres, the upper limit of Witkowski's experimental data. The point marked 10, representing the mean of ten experiments, lies distinctly above the curve, which is here based on extrapolated data and may be too low, the probable error of the mean of the ten experiments computed in the usual way being only 0.08 while the curve passes about 0.5 below the point.

It has thus been shown, in the specific case of the best set of experimental values available, that the observed values of the efficiency or ratio of liquefaction x , when so reduced as to be comparable with values computed *a priori* from experiments on the properties of air at ordinary temperatures or at atmospheric pressure, do agree quite well with the computed values, so far as the latter can be considered reliable. We may congratulate ourselves on finding that none of the existing data we have had to use seem to be far wrong, in spite of their heterogeneous nature and the great experimental difficulties involved in obtaining them, especially in such work as that of Witkowski.

6. Methods of Increasing the Yield.—It has been shown that if the regenerator surface is large enough, the internal arrangement of the liquefier can influence the liquefaction ratio only in so far as it may influence the temperature gradients which determine the thermal leakage and the resulting waste of liquid. If a given liquefier has sufficient regenerator surface, the only thing that can be done to improve its yield, when fed with air at a given

pressure and temperature and at a given rate, is to improve the thermal insulation. The experiments of Messrs. Bradley and Rowe show, in agreement with equation (12), that the fractional yield or efficiency of liquefaction increases with the rate of feed. The most obvious improvement in operation is therefore the use of a larger compressor and higher rate of flow. There is a limit to this, however, for a point will be reached where the regenerator surface becomes so insufficient and the low temperatures are brought so near the inlet end of the coil that the losses offset the gains.

A second method of improving the yield, with a given mass of air passing per second, is to use higher initial pressures. At 16° C. the Joule-Thomson effect is about half as great at 200 atmospheres as at low pressures, to judge from the curve representing Witkowski's experiments, but there is still some advantage to be gained by increasing the initial pressure. How far this would continue we can not say, for we have no sufficient data; but from curve *A* of Fig. 2 it seems likely that ρ would pass through zero somewhere in the vicinity of 400 atmospheres, and that the total increase of available cold attainable by increasing the initial pressure to this point would be only about one-third of what we already have at 200 atmospheres. Amagat's observations at 0° and 15°.7 also indicate that for a mean temperature of 7°.85 the inversion pressure would be close to 400 atmospheres.

Curve *B* of Fig. 2, computed from the reduced van der Waals equation

$$\left(p + \frac{3}{v^2}\right)(3v - 1) = 8\theta$$

together with the critical constants already quoted, crosses the axis at 328 atmospheres. The curve computed from the equation

$$\left(p + \frac{3}{\theta v^2}\right)(3v - 1) = 8\theta$$

crosses the axis at 268 atmospheres but does not pass anywhere near the experimental points shown by the crosses in Fig. 2, being everywhere very much lower. A similar curve computed from the equation

$$\left(p + \frac{16}{3\theta v^2}\right)(4v-1) = \frac{128}{9}\theta$$

deduced by D. Berthelot⁷ for low pressures, lies close to curve *B* up to 80 atmospheres, but thereafter falls much less rapidly than *A* or *B* and crosses the axis at about 770 atmospheres. Up to 200 atmospheres the simple van der Waals equation seems to represent the facts fairly well and it is the best guide we have in guessing at what may happen outside the range of the available experimental data.

The third and most effective way to increase the yield of a given liquefier is to use precooling, i. e., to lower the initial temperature θ_1 at which the air enters the regenerator. That this gives a rapid increase of efficiency is shown for temperatures between 0° and 100° by the experiments of Messrs. Bradley and Rowe as well as by Joule and Kelvin's porous plug experiments. For lower temperatures we have only rather scanty data, but from Witkowski's figures, treated as before, we find, for air at an initial pressure of 130 atmospheres, the following ideal maximum values:

$t_1 = +16^\circ \text{C.}$	-35°C.	$-78^\circ.5 \text{C.}$
$x = 6.3 \text{ per cent.}$	9.2 per cent.	14.4 per cent.

Similarly for hydrogen,⁸ with an initial pressure of 60 atmospheres we find

$t_1 = -190^\circ \text{C.}$	-212°C.
$x = 8.3 \text{ per cent.}$	16.2 per cent.

These values are based on imperfect data, but doubtless give an approximate idea of the advantage to be gained by precooling.

Whether there is any corresponding increase in the mechanical efficiency, i. e., in the amount of liquid produced per kilowatt hour expended, is another question into which I shall not enter here. In general laboratory practice, energy is usually the cheapest of the things used and the problem is commonly how to get

⁷ Trav. et Mem. Bur. Int. Vol. XIII.

⁸ Witkowski: Bull. Cracow Ac. Sc., June, 1905.

liquid as fast as possible from the apparatus available, regardless of how much power is used. The whole problem of the mechanical efficiency of the Hampson or the more complicated Linde process could be treated by the principles already set forth so far as data on the thermal and mechanical properties of the gas to be liquefied are at hand.

7. On Dissipation at the Valve.—It is sometimes said or implied that friction at the valve is a source of loss of efficiency because heat is thereby generated, but this is a misconception if we are considering only the Hampson or Linde process.

The truth of this statement is most easily seen by considering extreme cases. Suppose the valve to be replaced by a nozzle designed, as in the case of a steam turbine nozzle, so as to allow the flow to be as smooth and free as possible. We shall now have a very high exit velocity with high kinetic energy. This kinetic energy represents a decrease in the internal energy of the gas as it expands and the gas *in the jet* is cooled a great deal. But if all this kinetic energy, except the small amount corresponding to the velocity needed to carry the exhaust away quietly through the regenerator, is dissipated *inside the liquefier*, the total result is precisely the same as if we went to the opposite extreme and used a very fine-grained porous plug so as to have almost no kinetic energy at all in the issuing gas. The result is the same in any intermediate case; with a high velocity of escape from the nozzle or valve, we get very great local cooling but an equivalent heating somewhere else. There is a definite total amount of cold available inside the liquefier, and if we get more at one place and so liquefy a higher percentage of the gas in the jet as it issues from the valve, we have to pay for it somewhere else, for we are developing an equivalent amount of heat somewhere else *inside* the liquefier. If we avoid dissipation at the valve by the use of a well-designed nozzle, and then convert the resulting kinetic energy, not into heat but directly into work which we lead out along a turbine shaft, we may get a highly increased efficiency. But we are then introducing a new source of cold and making a radical change in the principle of the liquefying process. This dissipation of energy inside the liquefier is an essential characteristic of

the completely irreversible expansion occurring in the Linde or Hampson process. Gas enters the apparatus by one opening at high pressure while the work done on it in forcing it in is the product of its volume by the high pressure. It issues by another opening doing work equal to the product of its new volume by the new pressure. The total heating or cooling effect is the sum of two independent parts, one due to the fact that the product (pv) changes with the pressure and the other to the fact that the internal energy also changes with the pressure. We have the resultant of external and internal work, both of which are zero, by definition, for the ideal gas, but may be either positive or negative in any other fluid, whether liquid or gas. The energy that is dissipated inside the liquefier is what might be obtained as work if the expansion could be carried out reversibly, instead of irreversibly.

If by the use of a motor, as in Claude's process,⁹ part of this work can be saved from dissipation inside the liquefier, the available cold and the amount of liquid produced are increased correspondingly. In either case, the Joule-Thomson effect remains the same. In the completely irreversible plug expansion, that is the only source of cold; in the other case there is an added source of cold in the abstraction of energy by the motor.

At an initial temperature of 16°C . and pressure of 40 atmospheres, insulation and regeneration being perfect, the amount of air liquefied with completely reversible expansion would be about 23 per cent instead of the 2.1 per cent attainable by the completely irreversible liquefier of the Hampson type.

8. Liquefaction before the Valve.—Although it seems sufficiently proven that the internal arrangement of the Hampson type of liquefier influences the yield only in so far as it influences the losses due to thermal leakage and imperfect regeneration, there is one more point that may be worth considering.

It has been found experimentally¹⁰ that if the last few turns of the coil are immersed in the liquid air produced, the net yield is

⁹ Journ. de Phys. (4) 5, p. 1, 1906.

¹⁰ Cottrell; Journ. Phys. Chem. 10, p. 265, 1906.

Bradley and Fenwick; *ibid.*, p. 275.

precisely the same as with the more usual arrangement where the valve is at the lowest point of the coil. In the ordinary case, the air when it reaches the valve is probably above its critical temperature of -140°C . At all events, if the pressure is as high as 100 atmospheres and no precooling is used, the temperature to which the advancing air can be cooled by the action of the regenerator is very considerably above -140°C ., although some further cooling may take place by conduction back from the valve along the worm. Assuming, then, that the air is above its critical temperature, it is naturally looked upon as a gas. When the last few turns of the worm are immersed in liquid air it may be assumed as highly probable that the air reaches the valve below its critical temperature. Hence we may naturally look upon it as a liquid in view of its high pressure. The distinction is in reality only a matter of definition, for so long as a fluid is subject to more than its critical pressure, no liquefaction in the sense of a separation into liquid and gaseous phases can ever take place, no matter how low the temperature may be.

After what has been said above regarding the method of computing the efficiency of liquefaction, this experimental result is not surprising. The view has, however, been expressed, that since the air reaching the valve is already liquid, the Joule-Thomson effect must be very small or zero, and the conclusion has been drawn that some other and unfamiliar source of cooling must come into play. This conclusion is unnecessary, for there is no sufficient reason for assuming that the Joule-Thomson effect is either negligible or small under the given conditions. The Joule-Thomson effect is a property not of gases alone, however the gaseous state may be distinguished from the liquid, but of all fluids or fluid mixtures, no matter what their density.

For most purposes the definition of the Joule-Thomson effect given in equation (6) is the most convenient; but if we prefer to define it by reference to an adiabatic expansion and let μ be the fall of temperature per unit fall of pressure, the reasoning of § 2 applied to an infinitesimal fall of pressure gives us

$$\mu C_p = \rho \quad (15)$$

Without numerical data on the quantities involved in equation

(8) it is not obvious what the sign of μ or ρ will be, to say nothing of magnitude.

But there is another expression for ρ , which may easily be shown to be equivalent to equation (8); it is *

$$\rho = -\frac{\partial}{\partial p} (pv)_\theta - \left(\frac{\partial \epsilon}{\partial p} \right)_\theta \quad (16)$$

This equation, while useless for computation, is quite clear physically. Of the terms on the right side, the first is the external work done by the fluid, and the second is the increase of its internal energy, both per unit *fall* of pressure at constant temperature and measured for unit mass.

The second term may also, though rather loosely, be called the internal work, and it includes the work done against the cohesive forces, self-attraction, or internal pressure during the expansion of the fluid with falling pressure. Equation (16) is therefore in the simple form

$$(\text{heat put in}) = (\text{work given out}) + (\text{increase of internal energy}).$$

Suppose, to take an extreme case, that the air reaching the valve has been cooled all the way to its normal boiling point by the cold derived from boiling off some of the liquid air which surrounds the lower turns of the worm. When it is released through the valve, its increase of volume is greater than if it started under the same pressure at a higher temperature, since its initial density is greater at the low temperature. Hence both the external and the internal work might be expected to be greater and the absorption of heat greater than if the expansion started at a higher temperature. We could not tell from equations (8) or (16) just the numerical value of ρ or μ without data on the properties of air down to its boiling point, and these are lacking. We are certain, however, by the general reasoning of §§ 2 and 3, that the greater cold due to starting the expansion at a very low temperature produces just enough more liquid in the jet to offset the liquid boiled away in cooling the air down to this very low temperature before it arrives at the valve.

* See note at the end of this paper.

As for the process that goes on actually inside the nozzle, valve, or porous plug, complicated as it is by kinetic energy of eddy currents which has not yet been dissipated, it can not be followed in detail. Pressure and temperature cease to have any meaning under such turbulent conditions. Whether the jet of air from the valve is to be looked upon as liquid evaporating or gas condensing is somewhat a matter of words and can not in practice be decided from the appearance of the jet. The advantage of applying thermodynamics to such problems as those discussed in this paper is that it frequently, as in the present case, enables us to pass over intermediate details and arrive, notwithstanding, at results which are as trustworthy as the two laws of thermodynamics and the experimental data we have to use.

The thermodynamic theory of liquefiers in which the expansion valve is replaced by a motor is somewhat less simple than that of the Hampson liquefier, because even if the expansion in the motor be assumed to be isentropic, the interchanging process which goes on in the regenerator is essentially irreversible. The allowance to be made for the increase of entropy involved is somewhat uncertain, and in obtaining numerical values for x one is driven to the use of successive approximations, so that the computations are more laborious. Sufficient data are available in the case of air to permit of an approximate solution under practicable initial conditions.

If a turbine is to be used as the motor, its design is subject to the same general principles as that of a steam turbine. With an initial pressure of 40 atmospheres and an initial temperature of 16°C ., the exit velocity of the mixture of liquid and gaseous air from a Laval nozzle would be considerably less than in the steam turbine, and the kinetic energy per cubic centimeter not very different from that in the Laval steam turbine. It may therefore be practicable to use a simple, single stage, impulse turbine as a substitute for the expansion valve of the Hampson liquefier.

The theory of this type of liquefier will be given in a subsequent paper, when it is hoped that some experimental results may also be communicated.

NOTE ON THE EXPRESSIONS FOR ρ

Equation (16) or

$$\rho = - \left(\frac{\partial}{\partial p} (pv) \right)_\theta - \left(\frac{\partial \epsilon}{\partial p} \right)_\theta \quad (16)$$

is much the clearest mathematical expression of the Joule-Thomson effect ρ . For it is merely a statement that if unit mass of the fluid expands so that its pressure falls by unity, the quantity of heat that must be supplied, if the temperature is to be kept from falling, is equal to the external work done by the fluid plus the simultaneous increase in its internal energy; and this, again, is merely a statement that the first law or principle of the conservation of energy is applicable to the process. The only peculiarity of the equation is that the external work done by the fluid in an infinitesimal expansion does not take the form $p\delta v$ which it has for a reversible expansion against a pressure equal to that of the gas. Instead, we have for the irreversible expansion through a porous plug, where the fluid enters the plug at one pressure and leaves it at a different pressure without doing any external work while in the plug, a quantity of work $\delta(pv)$ given out by the fluid.

To show that equation (16) is equivalent to equation (8) we may proceed as follows: If W represents work done on any system, Q heat given to it, and ϵ its internal energy, the first law states that in any infinitesimal change of state

$$\delta\epsilon = \delta Q + \delta W \quad (a)$$

The second law states that if the change is reversible

$$\frac{\delta Q}{\theta} = \delta\eta \quad (b)$$

where η , the entropy, is determined solely by the instantaneous state of the system and not by its past history, and where θ is the temperature on Kelvin's absolute thermodynamic scale. Since in a reversible expansion the work done on the fluid is $-p\delta v$, equations (a) and (b) give, as the combined expression of the two laws of thermodynamics when applied to reversible infini-

tesimal changes of state of a mass of fluid subject to no outside forces except an uniform pressure p , the equation

$$\delta\epsilon = \theta\delta\eta - p\delta v \quad (c)$$

upon which may be founded the whole of the thermodynamic study of fluids in equilibrium.

For an isothermal change in which the increase of pressure is δp , equation (c) becomes

$$\left(\frac{\partial\epsilon}{\partial p}\right)_\theta = \theta\left(\frac{\partial\eta}{\partial p}\right)_\theta - p\left(\frac{\partial v}{\partial p}\right)_\theta \quad (d)$$

By subtracting $\delta(\theta\eta - pv)$ from both sides of equation (c) we get

$$\delta(\epsilon - \theta\eta + pv) = -\eta\delta\theta + v\delta p \quad (e)$$

and since ϵ , η , and v are all completely determined by θ and p , the first member of (e) is a perfect differential and the equation is in the form

$$\delta\zeta = \left(\frac{\partial\zeta}{\partial\theta}\right)_p \delta\theta + \left(\frac{\partial\zeta}{\partial p}\right)_\theta \delta p$$

where

$$\zeta = \epsilon - \theta\eta + pv; \quad \left(\frac{\partial\zeta}{\partial\theta}\right)_p = -\eta; \quad \left(\frac{\partial\zeta}{\partial p}\right)_\theta = v$$

But since the order of differentiation with regard to two independent variables is immaterial, it follows that

$$\left[\frac{\partial}{\partial p}\left(\frac{\partial\zeta}{\partial\theta}\right)_p\right]_\theta = \left[\frac{\partial}{\partial\theta}\left(\frac{\partial\zeta}{\partial p}\right)_\theta\right]_p$$

or

$$-\left(\frac{\partial\eta}{\partial p}\right)_\theta = \left(\frac{\partial v}{\partial\theta}\right)_p \quad (f)$$

Substituting from (f) in (d), we have

$$\left(\frac{\partial\epsilon}{\partial p}\right)_\theta = -\theta\left(\frac{\partial v}{\partial\theta}\right)_p - p\left(\frac{\partial v}{\partial p}\right)_\theta \quad (g)$$

and substituting from (g) in (16), we have

$$\rho = -\frac{\partial}{\partial p}(pv)_\theta + \theta \left(\frac{\partial v}{\partial \theta} \right)_p + p \left(\frac{\partial v}{\partial p} \right)_\theta$$

or since

$$\frac{\partial}{\partial p}(pv)_\theta = p \left(\frac{\partial v}{\partial p} \right)_\theta + v$$

we have finally

$$\rho = \theta \left(\frac{\partial v}{\partial \theta} \right)_p - v \quad (8)$$

This is equation (8), which has thus been obtained by a simple series of operations from equation (16). It expresses the value of the Joule-Thomson effect in terms of directly measurable quantities—the specific volume, temperature and rate of thermal expansion at constant pressure—and is therefore convenient for computing purposes. Equation (16), on the other hand, has the advantage that its correctness is obvious to anyone familiar with the law of conservation of energy, as soon as the nature of the process to which it refers, namely, expansion through a fine-grained porous plug, is understood. For qualitative reasoning, equation (16) is therefore frequently preferable to equation (8).

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CONTENTS

	Page
Platinum Resistance Thermometry at High Temperatures	
. <i>C. W. Waidner and G. K. Burgess</i>	149
The Daylight Efficiency of Artificial Illuminants	
. <i>Herbert E. Ives</i>	231
Coupled Circuits in which the Secondary has Distributed Inductance and Capacity	
. <i>Louis Cohen</i>	247
Effect of Phase Harmonics upon Acoustic Quality. <i>M. G. Lloyd and P. G. Agnew</i>	255
White Light from the Mercury Arc and its Complementary . <i>Herbert E. Ives</i>	265
The Regulation of Potential Transformers and Magnetizing Current	
. <i>M. G. Lloyd and P. G. Agnew</i>	273
The Determination of the Constants of Instrument Transformers	
. <i>P. G. Agnew and T. T. Fitch</i>	281



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PLATINUM RESISTANCE THERMOMETRY AT HIGH TEMPERATURES.

By C. W. Waidner and G. K. Burgess.

CONTENTS.

	Page.
I. INTRODUCTION.....	150
II. THERMOMETERS AND METHODS OF USE.....	152
Methods of measurement; construction of thermometers; heating by measuring current; calibration of thermometers; constants of the thermometers.	
III. MEASUREMENT AND REPRODUCIBILITY OF FREEZING AND MELTING POINTS.....	158
Electric furnaces; furnace manipulation; crucibles; surface oxidation; depth of immersion; undercooling; the observations; freezing points of Sn, Cd, Pb, Zn, Sb, Ag, Cu, Al, Ag ₃ -Cu ₂ and Cu-Cu ₂ O eutectics; metals from different sources.	
IV. THE SCALE OF THE PLATINUM THERMOMETER.....	175
Corrections for thermometers of impure platinum; precision of measurement with platinum thermometer; modification of Callendar's formula; other calibration formulæ; other methods of calibration at high temperatures.	
V. CHANGES IN CONSTANTS OF THERMOMETERS AT HIGH TEMPERATURES..	178
Changes with discontinuous heating; changes during a freezing point determination.	
VI. COMPARISON OF RESISTANCE AND THERMOELECTRIC SCALES.....	182
VII. THE PALLADIUM THERMOMETER.....	183
VIII. THE BOILING POINT OF SULPHUR.....	184
Variation of S. B. P. with pressure; the S. B. P. tubes; gas and electric heating; superheating; radiation shields; temperature distribution within S. B. P. tube; temperature distribution within radiation shield; on the uncertainty in the accepted value of the S. B. P.	
IX. GENERAL CONCLUSIONS.....	193
X. APPENDIX: TABLES.....	198
XI. BIBLIOGRAPHY.....	223

I. INTRODUCTION.

The present investigation was undertaken to determine the advantages and limitations of platinum resistance thermometers at high temperatures, especially with reference to reproducibility of scale, method of calibration, formulæ best expressing the relation between resistance and temperature, effect of impurities in the platinum wire, changes due to high temperatures, and the most satisfactory method of construction and use. The reproducibility of high temperatures by the melting or freezing points of metals obtained from different sources and furnished as chemically pure was also studied.

We realize that a very considerable amount of most valuable work has been done by several investigators in the domain of platinum thermometry of precision at high temperatures, yet it seemed to us that with the facilities at our disposal there was opportunity to study some of the questions above referred to with still greater exactness, especially in view of the publication of several recent researches on the gas thermometer at high temperatures, which afford, through the temperatures of the freezing points of pure metals, an indirect but very certain comparison of the gas and platinum scales.

The platinum resistance thermometer was first seriously proposed and used for the measurement of high temperatures by Siemens,* who in 1871 submitted several such thermometers to the British association for test. The results of these tests^a showed large and progressive changes in resistance after each exposure to high temperatures. The further investigation of the platinum thermometer by Callendar,¹² fifteen years later, showed that if the thermometer was so constructed that the platinum wire was protected from contamination by the supporting frame, the changes in resistance after some hours' exposure to high temperatures (red heat), were very small (a few parts in 10,000).

Callendar¹² has shown that if we define the *platinum temperature*, pt , by the relation

* References will be found in the Bibliography, Sec. XI, p. 223.

$$pt = 100 \frac{R - R_0}{R_{100} - R_0} \quad (1)$$

where R = observed resistance at the temperature, t° ,

R_0 = " " " 0°C. ,

R_{100} = " " " 100°C. ,

the relation between the platinum temperature, pt , and the temperature, t , on the scale of the gas thermometer is represented by the relation

$$t - pt = \delta \left\{ \frac{t}{100} - 1 \right\} \frac{t}{100} \quad (2)$$

where δ is a constant for any given sample of platinum, having a value of about 1.50 for pure platinum and a higher value for impure platinum.

The *fundamental interval*, FI , of a thermometer, is defined by the relation

$$FI = R_{100} - R_0 \quad (3)$$

and the *fundamental coefficient*, c , by the relation

$$c = FI / 100 R_0 \quad (4)$$

Three temperatures only are necessary for the standardization of a platinum thermometer. Two of these temperatures generally chosen are the melting point of ice and the temperature of saturated steam at known pressure. The third point recommended by Callendar and Griffiths,¹⁸ and generally used on account of the numerous careful determinations that have been made of it, is the boiling point of sulphur, observed under carefully specified conditions.

The work of numerous investigators has established the validity of the above relation (Eq. 2) and has shown that the platinum thermometer, standardized by observations in ice, steam, and sulphur vapor, will serve to reproduce temperatures on the gas scale within the range -80° to $+1100^\circ \text{C}$ almost, if not quite, within the limits of accuracy of the gas thermometer. The scale defined by the constant δ , determined by such a calibration, begins to depart appreciably from the scale of the Hydrogen or

Helium thermometer at temperatures below -100° , and for the measurement of temperatures in the interval 0° to -200° , it is best to choose as calibration temperatures melting ice, solid CO_2 , and ether, and the boiling point of liquid oxygen. If still lower temperatures are to be measured, it is necessary to use the temperatures of boiling or solidification of hydrogen, and if the highest attainable accuracy is desired it will probably be necessary to use more than three calibration temperatures, and an equation of a higher degree connecting the resistance and the temperature. For the calibration of calorimetric resistance thermometers intended for the measurement of small temperature changes, the transition point of sodium sulphate ($32^{\circ}384$) may be used with advantage as the third calibration temperature.¹¹⁷

II. THERMOMETERS AND METHODS OF USE

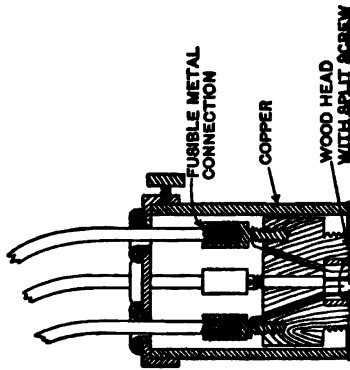
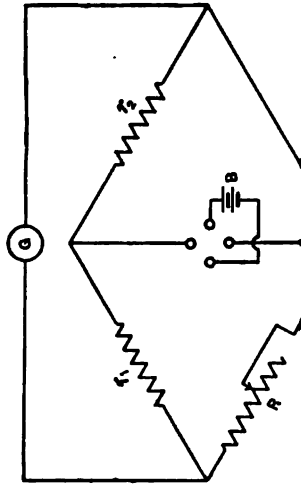
In the present investigation 11 resistance thermometers were used, made of wire of varying degrees of purity and of various sizes from 0.1 to 0.6 mm diameter, and with resistances, R_0 , ranging from 0.11 ohm to 21.5 ohms. Two of these thermometers were made of pure palladium wire 0.3 mm diameter, while the other 9 were of platinum.

Three types of thermometer were used, which may be briefly designated as

- a. Compensated lead type.
- b. Potential terminal type.
- c. Combined compensated lead and potential terminal type.

These three types are illustrated diagrammatically in Plates I and II. In the compensated lead type, due to Callendar,¹⁸ two fairly large platinum leads (preferably over 0.6 mm diameter), as nearly identical as possible with the leads to the coil, run down the stem of the thermometer, and are connected at their lower ends. When these leads are put into the arm of the Wheatstone bridge adjacent to the arm in which the thermometer coil is inserted the thermometer is compensated for variations in temperature of the leads to the coil, due to variable depth of immersion. To compensate for the effects of heat conduction from the coil along the heavier platinum leads, the ends of the compensating leads are

PLATE I



joined by a short length of fine platinum wire of the same size as the wire of which the coil is made. In thermometers of the potential terminal type two leads are fused to each end of the coil. In this type of thermometer both the current and potential leads may obviously be made of much smaller wire than for the compensating lead type. The method of using this thermometer, which consists in comparing the potential difference at the terminals of the coil with the potential difference at the terminals of a known resistance carrying the same current, will be understood from the diagram. In thermometers of the combined compensated lead and potential terminal type the current leads should be made of fairly large wire to secure better compensation. If one of the potential leads is omitted we have a three-lead compensated thermometer of the Siemens type.⁵

Methods of Measurement.—The compensated lead thermometers (type *a*, and type *c* when so used) were used (see Plate II) with a specially designed Wheatstone bridge,²² having mercury contact plugs and manganin resistance coils ranging from 0.01 to 50 ohms. The coils were immersed in a motor-stirred and thermostatically controlled oil bath. The finer steps were obtained on three shunt dials giving steps of 0.001, 0.0001, and 0.00001 ohm, respectively. The 1000-ohm ratio coils were used throughout this work. The ratio coils could be quickly reversed by interchanging two mercury contact links, thus eliminating the effect of inequality of the ratio coils. The bridge was used with a Griffiths' oil immersed key,²⁴ which kept the galvanometer circuit always closed except at the instant of making the battery circuit, thus eliminating the effects of small thermoelectric currents in the circuits. As a further precaution measurements were made with the battery and galvanometer circuits alternately reversed, as well as with the ratio arms interchanged, in the following sequence:

r/n		r/n
1. Battery +, Galv. +		3. Battery -, Galv. +
2. " - " +		4. " + " +
7. " - " -		5. " + " -
8. " + " -		6. " - " -

The connections are shown diagrammatically in Plate II.

This bridge has been calibrated several times a year since its construction in 1903 and at intervals of about eight weeks during the progress of this investigation. The small changes, amounting to a few parts in 100,000, are apparently seasonal and are undoubtedly due to the effects of moisture.

The method of using the potential terminal thermometers is shown in Plate II. The potential difference at the terminals of the thermometer coil was compared with the potential difference at the terminals of the resistance R by means of a special Leeds and Northrup type K potentiometer. In some of the later work a Diesselhorst potentiometer, constructed by Otto Wolff, of Berlin, was also used. The resistance R , which was taken from a mercury contact resistance box of manganin coils in a stirred and thermostatically controlled oil bath, could be adjusted, to the nearest 0.01 ohm, to equality with the resistance of the thermometer coil, so that nearly equal potential differences were compared.

The low resistance thermometers ($R_0 = 0.11$ ohm) were used as potential terminal instruments. In some of the earlier tests these thermometers were calibrated in connection with a special slide-wire double Kelvin bridge submitted with the thermometers by the makers, Messrs. Leeds and Northrup, who have applied this method of resistance thermometry to industrial work.

The order of accuracy of the resistance measurements throughout the work was 1 or 2 parts in 100,000.

Construction of Thermometers.—The thermometers, with the exception of Nos. 479 and 480, from the Cambridge Scientific Instrument Company, and the two low-resistance thermometers ($R_0 = 0.11$ ohm) 9837 and 9838, from Leeds and Northrup, were very kindly constructed for this investigation by Mr. H. C. Dickinson and Mr. J. J. Crowe of this Bureau. The essential details of construction are shown in Plate I. The coil is wound on a notched mica frame. The several leads are insulated from one another by mica washers. The platinum leads terminate in the head of the thermometer in small copper cups containing a fusible alloy by means of which connection is made to stranded leads (carefully adjusted to equality for the compensated lead thermometers)

which serve to connect the thermometer to the measuring apparatus. The head of the thermometer is made of wood thoroughly dried and boiled in paraffine. This head terminates at the bottom in a split screw which serves to grip the porcelain stem when the surrounding double walled copper head is screwed on to the wooden head. The thick metal head insures a quite uniform temperature distribution throughout the inclosure and thus reduces to a minimum the thermal electromotive forces within the head of the thermometer. Undue heating of the head of the thermometer at high temperatures, with consequent increase in these emf's., can be avoided by passing a current of air through the space between the walls.

The containing tubes were of Berlin porcelain glazed on the outside only, except for thermometers 9837 and 9838, which were inclosed in quartz.

A modified form of mica frame was used in the construction of thermometers 1787 F and 1787 G, with a view to diminishing the possible effects of strains in the platinum wire of the coil due to thickening of the mica after repeated exposures to high temperatures. The construction of the mica frame for these thermometers will be understood from the illustration. The wire was threaded through holes in the mica, and the two independent mica strips, b_1 , b_2 , not being in contact with the mica strip a , gave a flexible mounting for the wire.

The wire of which the thermometer coils were constructed was of varying degrees of purity, with values of δ ranging from 1.50 to 1.80. In Table I will be found a summary of the constants of the thermometers.

Heating by Measuring Current.—The measuring current through the coils of the thermometers was from 0.004 to 0.010 ampere for the different thermometers, except for the low-resistance thermometers. For any given thermometer the same measuring current was always used. Measurements made with other currents showed that the resulting values of pt were practically identical with the pt corresponding to infinitely small current. Two different currents were used with the low resistance thermometers, viz., 0.1 and 0.5 ampere. While with the larger cur-

rent the excess of temperature of the coil above its surroundings was more than 1° in the observed value of R_0 , the value of FI found was practically identical for both these currents.

TABLE 1.
Range of Constants of Platinum Thermometers.

No. of Thermometer	R_0	FI	c	d	Method of Measurement	Meas. current, amp.	Diam. of wire, mm.
1787 A	21.3476 21.0617	4.4067 4.4203	.00206426 210065	1.571 1.569	Poten- tiometer.	.004	0.1
1787 C	3.48779 3.48174	1.34298 1.34114	.00385052 385192	1.504 1.504	Poten- tiometer.	.010	.15
1787 F	2.85355 2.85567	1.11228 1.11312	.00389787 389793	1.503 1.503	Bridge, 3 leads.	.006	.2
1787 F	2.84782 2.84987	1.11069 1.11119	.00390014 389909	1.503 1.503	Poten- tiometer.	.010	.2
479	4.27342 4.28823	1.64585 1.64513	.00385136 383638	1.516 1.508	Bridge, 4 leads.	.006	.2
478	5.15665 5.16566	1.99745 2.00000	.00387354 387172	1.507 1.507	Poten- tiometer.	.007	.2
480	2.62081	1.00503	.00383480	1.551	Bridge, 4 leads.	.006	.2
1787 E	2.92482 2.92226	0.50511 .50540	.00172630 172948	1.803 1.803	Poten- tiometer.	.005	.3
9837	0.11275 .11307	.04324 .04325	.00383350 38251	1.54 1.54	Poten- tiometer.	.50	.6
9838	.10951 .10954	.04192 .04184	.0038280 39196	1.56 1.56	Poten- tiometer.	.50	.6

Range of Constants of Palladium Thermometers.

1787 G	1.87254 1.86284	0.62761 .62696	0.00335163 33590	2.890	Bridge, 3 leads.	.006	.3
1787 G	1.87345 1.86802	.62809 .62890	.00335258 336667	2.890	Poten- tiometer.	.015	.3
Pd II	1.30897	.43591	.00333017	2.947	Poten- tiometer.	.01	.35

The effect of using different measuring currents (with thermometer 1787 C.) ranging from 0.0025 to 0.100 ampere, is shown in Table 2*. The current normally used with this thermometer

* See Appendix, p. 198.

throughout the investigation was 0.01 ampere. It will be seen that a measuring current almost, if not quite, five times as great might have been used without appreciably affecting the results.

For a given small excess in temperature of the platinum coil above the temperature of its surroundings, the energy radiated in steam is 3.7 and in sulphur 52 times that radiated at 0° C, assuming that the radiation from platinum is proportional to the fifth power of the absolute temperature. For constant measuring current the energy supplied to the coil at the S. B. P. is only 2.6 times the energy supplied at 0° C. Hence, since it is observed that the excess of temperature of the coil at the S. B. P. is only about 25 per cent less than the excess of temperature at 0° C, it follows that only a relatively small portion of the energy supplied to the coil by the measuring current is lost by radiation, by far the greater part of the loss being due to convection and conduction.

Calibration of Thermometers.—The thermometers were calibrated by the Callendar-Griffiths method by observations of the resistance in melting ice, steam, and the vapor of boiling sulphur.

The resistance in ice, R_0 , was determined in a finely divided mixture of pure ice saturated with distilled water. The resistance in steam was determined in the usual laboratory form of Regnault hypsometer. The temperature of the steam in this type of hypsometer was compared with the temperature of the steam in the International Bureau type of hypsometer, designed by Chappuis, by taking measurements with the same thermometer in the two hypsometers alternately. The observed temperature of the steam in the Regnault agreed with that in the Chappuis hypsometer to within 0°.01, being about 0°.003 higher for slow rate of boiling and about 0°.006 higher for fairly rapid rate of boiling. The observations in steam were accompanied by simultaneous observations of the atmospheric pressure with a Fuess standard barometer, accurate to 0.02 mm. The temperature of the steam was then obtained from the observed barometric height reduced to 0°, latitude 45°, sea level, and from Broch's steam tables.

The observations in sulphur, together with simultaneous barometer observations, were carried out with the apparatus and

in the manner fully described under the head of "The boiling point of sulphur," p. 184.

The S. B. P. was taken as $444^{\circ}.70$ at standard atmospheric pressure.

Each thermometer was recalibrated at frequent intervals throughout the progress of the work. Several consecutive freezing (or melting) point determinations were usually made, requiring several hours, during which time the temperature of the crucible of metal varied only a few degrees above and below the melting point. Before and after each group of such determinations the thermometer was recalibrated. For the observations at the higher temperatures, where the changes in the constants of the thermometer become appreciable, the resulting freezing points are given in Tables* 8-10, as deduced from both the calibration before and the calibration after the observations in the metal.

Constants of the Thermometers.—In Table 1 (p. 156) are indicated, for each thermometer, the values of its characteristic constants, R_0 , FI , c , and δ , as observed at the beginning of this investigation and at its close, and also the method of measurement used with each instrument. The changes in the constants due to use of the thermometers at high temperatures are discussed subsequently (p. 178).

III. MEASUREMENT AND REPRODUCIBILITY OF FREEZING AND MELTING POINTS.

The determinations with thermometers of pure platinum of the freezing and melting points of the several metals used in this investigation are given below. The results of measurements, by means of thermocouples calibrated at the temperatures of freezing of zinc, antimony, and copper, made by one of the authors,† over a period of several years, on the reproducibility of these fixed points, as given by metals from different makers, are also included. The thermoelectric determinations at the lower temperatures are less precise than those with resistance thermometers.

* All tables except Nos. 1, 11, 12, 15, and 25, are in the appendix.

† Burgess.

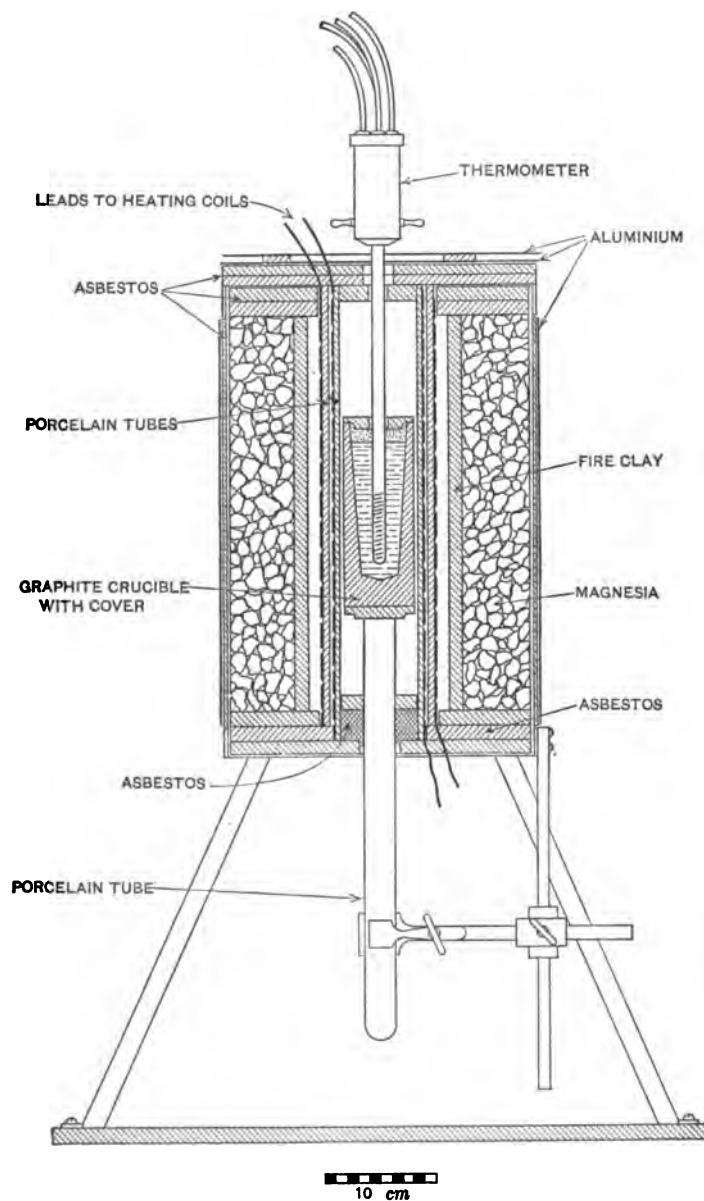
With the exception of the silver from the United States Mint in Philadelphia, the copper from the Baltimore Copper Works, and the aluminium from the Pittsburgh Reduction Works, all the materials were purchased without specifying other than the usual purest product of the several chemical manufacturers. All the metals were protected from oxidation and the measurements were carried out as explained below. A very sharp freezing point, i. e., a constant temperature extending over some minutes, can be obtained with all the metals used except aluminium, if they are pure and of sufficient bulk. This fact eliminates all ambiguity in the definition of the temperature of a freezing point.

Electric Furnaces.—The freezing and melting points were taken in well-insulated resistance furnaces, mounted vertically, with two separate heating coils of platinum ribbon, 0.01 mm thick and 2 cm wide, wound on porcelain tubes, with the turns so spaced as to allow the establishment of a uniform temperature over a considerable length of furnace by using a separate rheostat control in each circuit. The arrangement of a furnace containing a crucible of metal with a thermometer in place is shown in Plate III.

Furnace Manipulation.—With the above set-up, it is possible to hold a metal at its freezing point for an indefinite time. In practice it was the custom, when taking a freezing point, to so adjust the currents in the heating circuits as to obtain a freeze extending over ten to fifteen minutes, during which interval, for a pure metal, the temperature would usually remain constant to a few hundredths of a degree. At the higher temperatures it was deemed desirable to prevent overheating of the thermometer head in order to reduce to a minimum the thermal emf's. This was effected by passing an air current through the double-walled head of the thermometer. When this was done the head could be grasped by the hand for stirring without discomfort. In order to avoid breaking the porcelain containing tube, it was preheated in a gas furnace before introducing it into the metal.

Crucibles.—Two kinds of crucible were used, of Dixon graphite and of Acheson graphite, the latter being turned from rods. The crucibles were about 13.5 cm inside depth, 5 cm inside diameter at the top, and 3.5 cm at the bottom, holding therefore about

PLATE III



CRUCIBLE RESISTANCE FURNACE

180 cc of metal, or about 1.5 kilograms. A chemical analysis of the Acheson graphite, cut from several crucibles, was made by Dr. W. F. Hillebrand of this bureau. The total ash was found to be 0.3 per cent, one-half of which was Fe_2O_3 (equivalent to 0.11 per cent Fe). No differences in the observed values of any of the freezing points could be traced to the effects of impurities introduced from the crucibles.

Surface Oxidation and sublimation of the metals were practically eliminated, even for antimony, by putting a layer of powdered graphite and a graphite crucible cover over the surface.

Depth of Immersion.—Measurements were usually taken with the thermometer immersed to within 1 cm of the bottom of the crucible. The effect of varying depths of immersion was carefully studied by taking several consecutive freezes in each metal, everything else remaining the same. No difference greater than a few hundredths of a degree in the freezing points could be detected when the thermometer was raised 2 cm from its normal position in zinc, antimony, silver, and copper. Raising a thermometer 4 cm in zinc gave a freezing point too low by $1^\circ.3$.

Undercooling.—To avoid any considerable undercooling, as well as to insure a more certain uniformity of temperature over the length of the thermometer coil, especially when working with such metals as antimony, which, when left quiet, naturally undercools considerably and which has also a low heat conductivity, it was the practice to stir the liquid metal as long as possible by means of the thermometer itself.

Most, if not all, of these metals develop some undercooling, which is in general the more marked the higher the temperature to which the metal is brought above its freezing point, or the more rapid the rate of cooling, unless the liquid metal is stirred. When undercooling occurs, the temperature to which the metal rises is not appreciably different from the freezing point as determined when undercooling is practically eliminated by stirring.

The Observations.—In tables 3 to 10 in the appendix are given the necessary data relating to all the freezing and melting point determinations with thermometers of *pure platinum* [$\delta = 1.50$ to 1.51], of *impure platinum* [Therm. No. 1787A ($\delta = 1.57 \pm$) and No. 1787E ($\delta = 1.80$)] and of *palladium* [Therm. No. 1787G ($\delta =$

2.88±)]. No observations are omitted except a single day's work on copper, when a leak was found in the bridge circuits.

It will be seen that the thermometers of impure platinum and the palladium thermometer invariably give a lower temperature, t , than those of pure platinum for temperatures below the S. B. P. and a higher temperature above the S. B. P.

Tin.—Eleven determinations of the freezing point were made on three samples of tin: Eimer & Amend C. P. rods of 1905 in a Dixon graphite crucible, Baker & Adamson C. P. rods of 1905 in a Dixon crucible, and "Kahlbaum" tin of 1909 in an Acheson graphite crucible. The essential details of these determinations are given in Tables 3, and 12 (p. 174), from which it would appear that on the scale of the resistance thermometer of pure platinum the freezing point of "Kahlbaum" tin is $231^{\circ}.92 \pm 0.02$, and that of each of the other two samples is about $0^{\circ}.05$ lower. Undercooling, which is considerable when the metal cools without stirring from a relatively high temperature, could not be completely eliminated by stirring for any of these samples.

Measurements made with *Pt-Rh* and *Pt-Ir* thermocouples in 1903-1905, assuming "Kahlbaum" tin to freeze at $231^{\circ}.92$, gave the following results:

F. P. of Tin.—Thermoelectric Determinations.

Source	Number of Couples Used	Number of Dets.	Freezing Point
"Kahlbaum"	4	6	$231^{\circ}.92$
Eimer & Amend (Metallic, Henderson Bros.).....	4	8	$231^{\circ}.99$
Eimer & Amend C. P. sticks	4	4	$231^{\circ}.85$
Baker and Adamson C. P. sticks	2	2	$231^{\circ}.97$

In these earlier experiments no undercooling was observed for the Baker & Adamson tin.

The freezing point of tin is therefore readily reproducible to within $0^{\circ}.1$ with material purchased from various chemical firms. Even "Metallic" tin, i. e., tin of commercial purity, seems to have about the same freezing point as the pure metal. This may be due to the fact that the effect of some impurities on the freezing

point of tin is first to lower this temperature very slightly, and then to raise it with increasing percentage of impurity. This would explain why the commercial tin appears to have the highest freezing point of all the samples. Its freezing point curve is less flat than for the purer tin, as should be the case when impurities are present.

Cadmium.—The samples of cadmium used were "Kahlbaum," Baker & Adamson, and J. T. Baker,* all purchased in 1909. As shown in Tables 4 and 12, the freezing point of the "Kahlbaum" sample is $321^{\circ}.01 \pm 0.04$; of the Baker & Adamson metal, $320^{\circ}.39$, and of J. T. Baker's, $320^{\circ}.44$.

The undercooling of cadmium before freezing can be nearly eliminated by stirring.

Lead.—Three samples of this metal were used, "Kahlbaum," J. T. Baker's "Lead Granulated, Test Lead," stated to contain no silver; and Baker & Adamson's "Test Lead, free from Ag," all of 1909. As shown in Tables 5 and 12, the mean value of the freezing point of lead is $327^{\circ}.43$, the determinations for a single sample agreeing to $0^{\circ}.03$. The "Kahlbaum" lead is about $0^{\circ}.3$ lower than Baker's and $0^{\circ}.2$ lower than Baker & Adamson's. Lead undercools slightly even with stirring.

Determinations with four thermocouples, in 1903 and 1905, of the freezing point of Eimer & Amend C. P. lead gave from $327^{\circ}.3$ to $327^{\circ}.5$. Pig lead had a less sharp freezing point some $1^{\circ}.5$ low, and lead pipe was about 2° low. The data for these commercial samples of lead are given for the reason that lead hardening and annealing baths are used in the industries, and the question has often arisen whether this grade of lead can be used as a control point for industrial pyrometers.

Zinc.—More determinations of the freezing point were made with zinc than with any other metal, and the results obtained with the platinum thermometers, as shown in Tables 6 and 12, indicate a remarkable degree of concordance, both for determinations by means of different thermometers of the freezing point of the same material and for determinations with material from different sources. Thus, samples of "Kahlbaum" zinc bought in 1903, 1907,

* Marked "Cadmium Metal" with an indicated analysis showing iron 0.004 per cent and arsenic, trace.

and 1908 give a freezing point of $419^{\circ}.37 \pm 0.02$, one of these being in Acheson and the others in Dixon graphite crucibles. One of these samples had been in use in the same crucible since 1903 for calibrating thermocouples.

Determinations of the freezing points were also made on the following samples:

Eimer & Amend "C. P. in sticks," two crucibles of 1903 and 1905 mixed, and Baker & Adamson "C. P. in sticks." The former,* which had been repeatedly used, gave a lower value, $419^{\circ}.25$, and the latter a slightly higher value, $419^{\circ}.42$, than the Kahlbaum zinc.

Measurements made with thermocouples in 1905 gave the following results for the freezing point taken in a gas furnace:

F. P. of Zinc—Thermoelectric Determinations.

Source	Number of Couples Used	Number of Dets.	Freezing Point
"Kahlbaum".....	8	10	$419^{\circ}.3$
Eimer & Amend "C. P. in sticks".....	8	12	$419^{\circ}.0$
Eimer & Amend "Metallic".....	4	6	$416^{\circ}.8$
Baker & Adamson "C. P. in sticks".....	4	8	$419^{\circ}.1$

Other thermoelectric determinations made in 1907 gave equally concordant results.

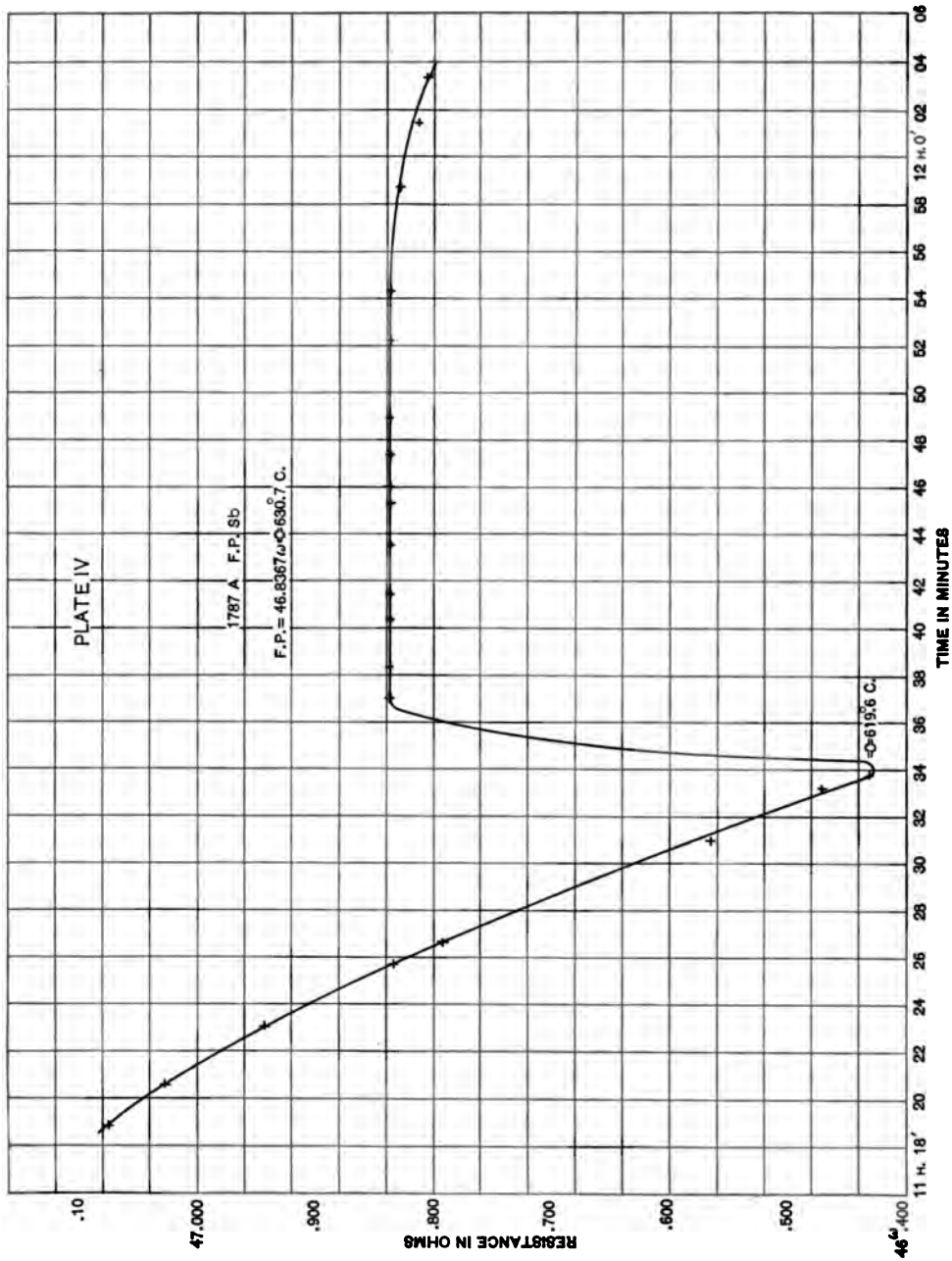
The undercooling of zinc is almost inappreciable, being only a few hundredths of a degree unless the metal has been considerably overheated. It is therefore unnecessary to stir when taking a freeze.

A considerable number of determinations of the melting point of zinc, made with the resistance thermometers, gave results agreeing to $0^{\circ}.1$ with the freezing point, even for fairly rapid heating.

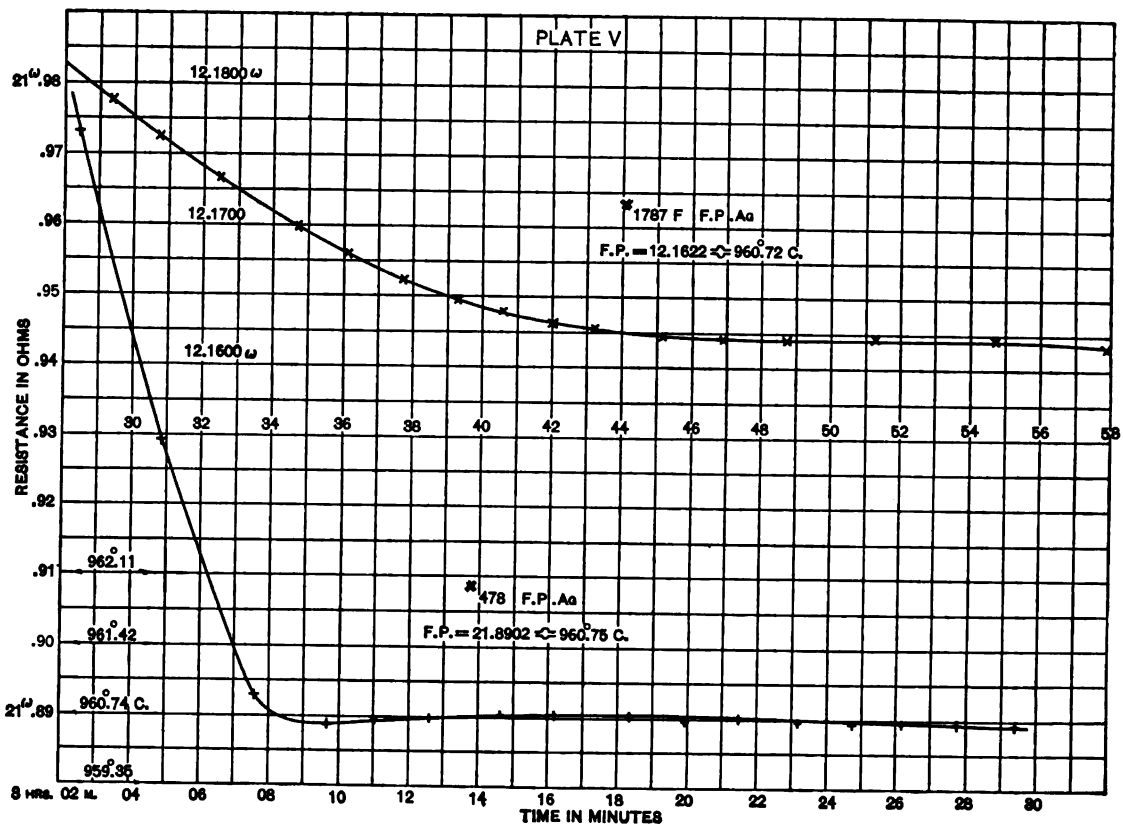
Antimony.—"Kahlbaum" antimony† has been used as a fixed point for a number of years with satisfaction, and this material,

* Dr. E. T. Allen finds, for two samples from different lots of Eimer & Amend zinc "C. P. in sticks," total impurities 0.063 and 0.049 per cent, with lead as the principle component, it being 0.051 and 0.041; see *Phys. Rev.* **19**, p. 177; 1904; *Am. Jl. Sci.* **26**, p. 454; 1908.

† An analysis of this material has been published; Fritz Henz, *Inaugural Dissertation*, Zurich, 1903 (Pub. in Leipzig), Fe=0.012, Cu=0.004, Pb=0.003 per cent.



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purchased at different times over a period of six years, appears to be a very uniform product as determined by the constancy of the freezing point. Antimony from other sources, however, so far as our experience goes* is not to be relied upon as being of sufficient purity to warrant its use for defining a fixed temperature by means of its freezing point.

Antimony oxidizes very rapidly in the air and somewhat even in the presence of graphite, but this oxidation does not appear to affect the freezing point appreciably. This metal normally undercools very considerably, even to the extent of 30° or more, but whether this is nearly eliminated or not, by stirring, practically the same temperature of freezing is obtained. In Plate IV is given a freezing point curve for antimony when the metal is stirred with the thermometer. The melting point agrees with the freezing point to within $0^{\circ}.1$, and they approach each other as the slowness of freeze and melt are increased.

Seven determinations of the freezing point of three samples of "Kahlbaum" antimony with two resistance thermometers gave $630^{\circ}.71 \pm 0.13$. (See tables 7 and 12.)

Thermoelectric determinations in gas and electric furnaces of the freezing point of antimony from various sources gave the following results, assuming Sb "Kahlbaum" = $630^{\circ}.7$:

F. P. of Antimony—Thermoelectric Determinations.

Source	Date of Purchase	Number of Couples	Number of Dets.	Temperature
"Kahlbaum".....	1903-1909	6	8	$630^{\circ}.7$
Eimer & Amend C. P.	1903	4	6	$621^{\circ}.0$
Eimer & Amend "Metallic".....	1903	2	6	$619^{\circ}.1$
J. T. Baker † "Antimony metal".....	1909	2	4	$626^{\circ}.2$
Baker & Adamson.....	1909	2	4	$626^{\circ}.0$
Merck "Highest Purity".....	1907	1	2	$624^{\circ}.7$

* See also Day and Allen, *Phys. Rev.* **19**, p. 177; 1904.

† Analysis on label: Fe=0.01, Cu=0.005, As=0.002.

Pb=0.002, Zn=0.000, Sn=0.000 %.

It is evident that for this metal the determination of the freezing point is a very delicate test of the purity of the sample.

Silver.—Only one sample of silver was used, purchased in the form of drops from the Philadelphia mint, as the purest silver obtainable. Its analysis was given as 99.995 fine. An analysis of another sample of this grade of silver has also been made by Dr. E. F. Allen,* who found only 0.0032 per cent impurities, one-third of which was iron.

Eight determinations with three resistance thermometers gave a freezing point of $960^{\circ}.88 \pm 0.16$. See Tables 8 and 12, which also show how constant the indications of these thermometers remain even at such a high temperature, as is seen from a comparison of the calibrations made before and after the freezing point determinations.

The silver point is very sharply defined, as might be expected from the high conductivity and purity of this metal, and although there is some slight undercooling if the freezing point is approached from a high temperature this disappears entirely with stirring or when the rate of cooling before freezing is small. Sample freezing point determinations with and without undercooling are shown in Plate V.

Copper.—As an upper limit for the exact determination of temperatures by the usual type of platinum resistance thermometer, the freezing point of copper was chosen, and an examination of the data in Tables 9 and 12 shows that the precision attainable at this temperature is already considerably less than at the silver F. P., though still sufficiently good for work within about 1° . On account of the changes in the constants of the platinum thermometer, which occur at this high temperature, it is better to depend on the thermocouple for temperature measurements in this region, since the advantages of very great sensitiveness of the resistance instrument are masked by these changes.

Copper from several sources was used. The only product furnished with an analysis was that obtained from the Baltimore Copper Smelting and Rolling Company, which company very

* Am. Jl. Sci. (4), 26, p. 456; 1908.

kindly furnished us, in 1908, with a specially prepared electrolytic sample in the form of cathode plates.*

This copper had a freezing point of $1083^{\circ}.0 \pm 0.5$ from seven determinations with three resistance thermometers.

Kahlbaum electrolytic copper in a Dixon crucible gave $1081^{\circ}.4$; a Dixon crucible filled with copper of uncertain origin, but which had been in use for several years, $1082^{\circ}.7$; and Eimer & Amend's "Copper,† C. P. drops, cooled in hydrogen" of 1909, $1083^{\circ}.9$ in Acheson graphite for two determinations with the same thermometer.

It should be noticed that very slight changes in either the fundamental or the difference coefficient will cause considerable differences in the computed temperatures at the copper point. Thus a change in δ from 1.50 to 1.51 is equivalent to about 1° temperature difference at 1083° .

When the metal is protected from oxidation, the copper freezing point is very sharp, the undercooling being at most only $0^{\circ}.2$ to $0^{\circ}.3$ (see Plate VI), and often inappreciable.

The copper freezing point has been used with entire satisfaction as a fixed point in calibrating pyrometers at the Bureau of Standards since 1903. This metal is so readily obtained sufficiently pure, at small cost, that its freezing point is particularly adapted for use as a standardizing temperature.

Even commercially pure copper in the form of bars or wire usually has a freezing point within a degree of that of the specially prepared material.

A great many determinations of this temperature were made by means of thermocouples from 1903 to 1909, on material from various sources, both in gas and electric furnaces. Among the kinds of copper used were copper wire, copper in bars, Baker &

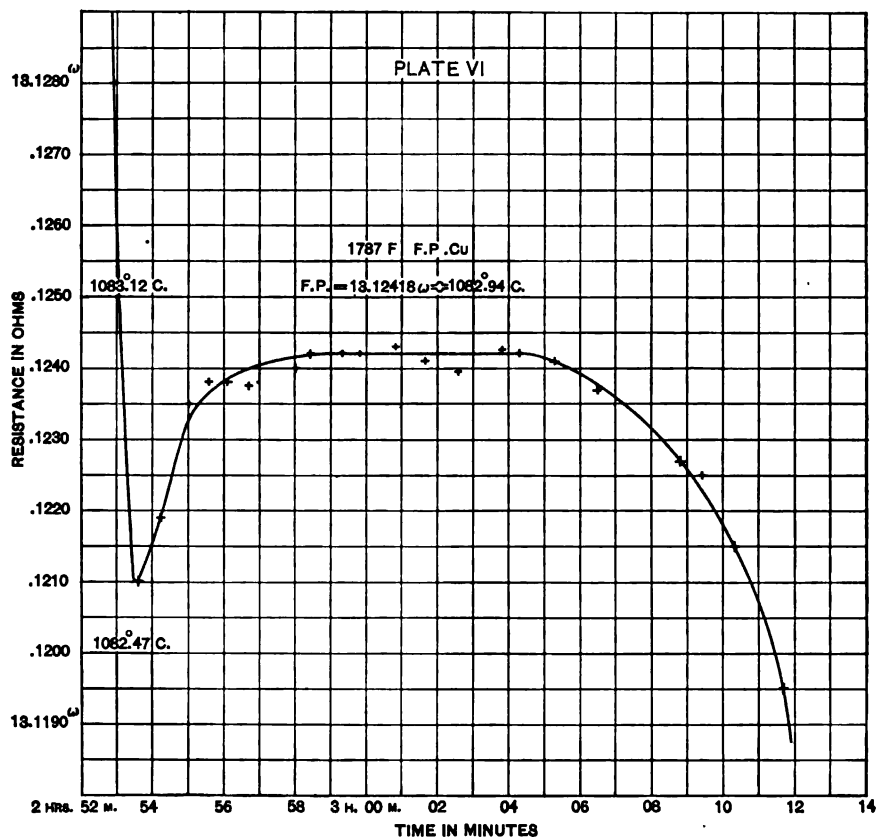
*Mr. Wm. M. Pierce reports the analysis of this copper as follows.

Ag, Au, As, Sb, Pb, none; Fe, S, trace; Cu 99.995 per cent by direct electrolytic test.

"In making the test for Ag, As, Sb, Fe, S, and Pb, large quantities of the samples were used: six assay tons for Ag, 300 grams for As and Sb, and 200 grams for Fe, S, and Pb."

†An analysis of this grade of copper is given by Day and Allen, *Phys. Rev.* **10**, p. 180, 1904, showing total impurities = 0.040 per cent, of which Fe = 0.011, Zn = 0.010, Ag = 0.018, and Pb = 0.001.

Adamson's "copper, C. P. reduced by hydrogen," and Kahlbaum's electrolytic granulated. No certain difference could be detected in the freezing points of any of these materials, but when using the powdered copper reduced by hydrogen great care has to be taken not to oxidize the metal. When determinations are made in a gas furnace the metal tends to oxidize partially, even in graphite,



unless a reducing flame is used. The presence of the soluble oxide, Cu_2O , lowers the copper freezing point until, at saturation, the copper-cuprous oxide eutectic point is reached at about $1,063^\circ$. In a gas furnace, however, unless special precautions are taken, neither the freezing point of pure copper nor that of the eutectic of $\text{Cu-Cu}_2\text{O}$, is obtained alone, but an intermediate temperature, which is usually nearer the Cu F.P. The eutectic point can always be detected except for very small percentages of the oxide.

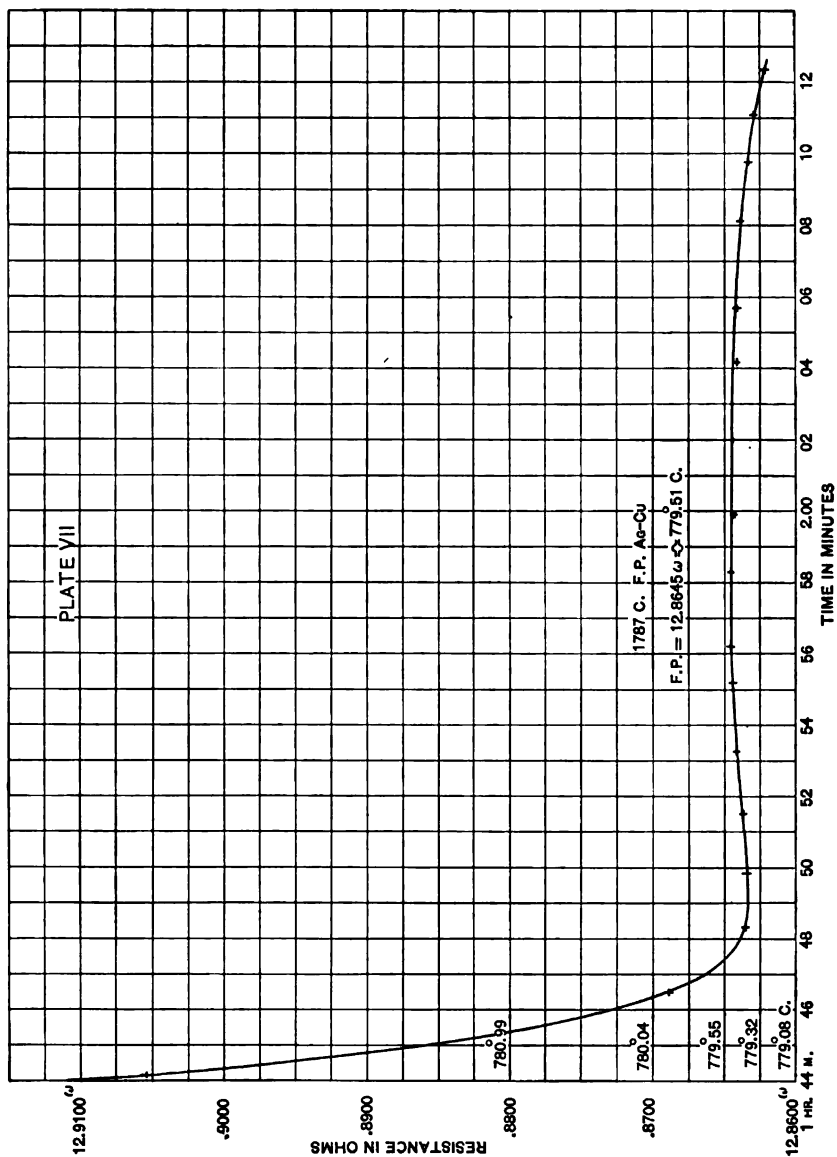
Silver-Copper Eutectic.—With the exception of the “Kahlbaum” antimony point at $630^{\circ}.7$ there is no satisfactory fixed point between zinc and silver or copper, and it is especially desirable to have such a temperature located in the neighborhood of 800° . We have found that the Ag-Cu eutectic alloy (Ag, Cu,) meets this requirement in a very satisfactory way.

A preliminary study of the conditions necessary for the constancy of this eutectic temperature was made with a thermocouple and gave the following results, using as components silver from the Philadelphia mint, and copper, “C. P. drops cooled in hydrogen,” from Eimer & Amend:

Silver-Copper Eutectic—Thermoelectric Determinations.

Per cent Ag-Cu	Composition in Grams Ag-Cu	Range of Transformation Temperature	Alloy, cooled from	Eutectic Temperature	Remarks
50-50	150-150	$1^{\circ}.5$	828°	777°.50	
50-50	150-150	.3	812	777°.89	
50-50	150-150	$1^{\circ}.1$	821	778°.72	
50-50	150-150	.2	906	778°.23	{ Slight undercool.
50-50	150-150	.5	977	777°.89	
50-50	150-150	.4	857	776°.33	Stirred.
50-50	150-150	.4	929	776°.43	Stirred.
60-40	255-150	.8	945	773°.07	Stirred.
60-40	255-150	.7	818	774°.99	
71-29	355-145	.0	825	779°.49	{ Slight but long undercool.

These results show that the presence of an excess of copper is disadvantageous in two ways: first, the eutectic temperature is not sharply marked, as shown by the column headed “range of transformation temperature”; and second, the value of the eutectic temperature observed depends upon the degree of mixture of the excess component with the eutectic, as is shown by comparing the freezes taken with and without stirring. Although the lowest temperature is obtained when stirring the noneutectic mixture, yet this appears to be considerably below the true eutectic temperature, as obtained when the alloy of the eutectic composition



is used. The reason for this retarding of the separation of the eutectic by the admixture of copper is uncertain, but it may be due to the greater conductivity of the copper.

For the measurements with the resistance thermometers, an Acheson crucible was filled with 1200 grams of the alloy 71Ag-29Cu; measurements were also made with the eutectic composition 72Ag-28Cu, and with 73Ag-27Cu. The observations are shown in Table 10 and a summary is given in Table 12. The best value to assign to the eutectic temperature from these observations is $779^{\circ}.20 \pm 0.15$.

This temperature is as sharply marked as the freezing point of most of the pure metals; a sample freezing curve is given in Plate VII. It will be noticed that the undercooling is slightly different from that of a pure metal. This undercooling is of the order of $0^{\circ}.1$; but instead of the temperature rising rapidly to its maximum, as in the case of a metal, the rise is very slow, taking almost half of the time of duration of a freeze. This slight undercooling could not be avoided by stirring.

Copper-cuprous Oxide Eutectic.—Some measurements were made with thermocouples in order to determine the temperature and reproducibility of the Cu-Cu₂O eutectic point in both gas and electric furnaces. In agreement with the work of Heyn,* we find that the cooling curve for copper in the presence of its oxide when the mixture has been heated in a gas furnace does not, in general, correspond to the composition of the components originally put into the crucible, on account of the varying reducing and oxidizing action of the atmosphere in such a furnace. Unless the atmosphere is strongly reducing, however, the eutectic point is sharp to within 1° at 1063° . When the alloy of the eutectic composition (Cu₂O = 3.5 per cent) is introduced into an electric furnace of the type used in these experiments (Plate III), the eutectic temperature is as sharply defined as the freezing point of pure copper and is normally preceded by an undercooling of 2 or 3° . From six observations we find this eutectic temperature to be $1063^{\circ}.2 \pm 0.3$ on the temperature scale of this investigation (on which F.P. of Cu = $1083^{\circ}.0$).

* C. Heyn, Kupfer und Sauerstoff, Mitt. Königl. Tech. Versuchsanstalten 18, p. 315; 1900.

Aluminium.—No determinations of the freezing point of aluminium were made with the platinum-resistance thermometer. Two series of measurements were carried out, however, with thermocouples, one in 1905 and the other in 1909, on the same material but with separate samples. The Pittsburgh Reduction Company in 1903 furnished us with a specially prepared sample of exceptional purity. The analysis of this sample was given as: Al, 99.78; Si, 0.12; Fe, 0.10; Cu, trace. The other samples were in the form of bars from Eimer & Amend and from Kahlbaum.

On the temperature scale used in this investigation the following values were found for the freezing points of these aluminium samples:

F. P. of Aluminium—Thermoelectric determinations.

Source	Al F. P.
Pittsburgh Reduction Co.....	658.0
Kahlbaum, bars.....	654.5
Eimer & Amend, bars.....	656.0

The observations of 1905 and 1909 agree to within 1°, which is about the uncertainty in successive observations with the same thermocouple. The freezing point of aluminium is not sharp, even for the purest of these samples, so that this metal is the least adapted of all we have tried for defining a fixed temperature. Small quantities of an impurity, such as iron or silicon, lower the aluminium freezing point very considerably.

Metals from Different Sources.—The data on the freezing points of the metals from various sources may be grouped together so as to show the reproducibility of the fixed temperature that each defines. This has been done in Table 11, in which are indicated, for each sample, the source of the material, a brief description, the date of purchase, and the freezing point. In the last column is given the "reproducibility factor" for each metal, which we have taken as the average deviation of a single observation in degrees C from the mean F. P., including only those samples of C. P. grade.

TABLE 11.

Reproducibility of Fixed Points by Means of Metals from Different Sources.

Metal	Source	Description	Date of Purchase	Freezing Point °C.	Reproducibility Factor
Tin	Kahlbaum	"Kahlbaum"	1903, 1905, 1909	231°.92	0°.06
	Elmer & Amend	"C. P. in sticks"	1905	231°.85	
	Henderson Bros.	"Metallic" in bars	1905	*(231°.99)	
	Baker & Adamson	"C. P. in sticks"	1905	231°.97	
	Baker & Adamson	"C. P. in sticks"	1909	231°.85	
Cadmium	J. T. Baker Co.	"Mossy metal"	1909	231°.84	0.26
	J. T. Baker Co.	"Cadmium metal"	1909	320°.44	
	Kahlbaum	"Kahlbaum"	1909	321°.01	
	Baker & Adamson	"Cadmium"	1909	320°.39	
	Baker & Adamson	"Test lead free from Ag"	1909	327°.47	
Lead	J. T. Baker Co.	"Lead granulated, test lead"	1909	327°.58	0.10
	Kahlbaum	"Kahlbaum"	1909	327°.26	
	Elmer & Amend	"C. P. lead"	1905	327°.40	
		Pig lead	1905	(326°.0)	
		Lead pipe	1905	(325°.5)	
Zinc	Elmer & Amend	"C. P. in sticks"	1903-1905	419°.25	0.06
	Elmer & Amend	"Metallic"	1903	(416°.8)	
	Kahlbaum	"Kahlbaum"	1903, 1907, 1908	419°.37	
	Baker & Adamson	"C. P. in sticks"	1903	419°.42	
	Kahlbaum	"Kahlbaum"	1903-1909	630°.71	
Antimony	Elmer & Amend	"C. P."	1903	621°.0	2.3
	Elmer & Amend	"Metallic"	1903	(619°.1)	
	J. T. Baker Co.	"Antimony metal"	1909	626°.2	
	Baker & Adamson	"Antimony"	1909	626°.0	
	Merck	"Highest purity"	1909	624°.7	
Aluminium	Pittsburgh Reduction Co.	99.78 aluminium	1903	658°.0	1.2
	Kahlbaum	In bars	1905	654°.5	
	Elmer & Amend	In bars	1905	656°.0	
	Elmer & Amend	"C. P. drops, cooled in hydrogen."	1909		
	Kahlbaum	Electrolytic, granulated	1907		
Copper	Baker & Adamson	"C. P. reduced by hydrogen"	1905	1083°.0	†1.0
	Baltimore Copper Works.	99.995 copper	1908		
		In bars			
		Wire			

* Numbers in parentheses are not included in computing the "Reproducibility Factor."

† This uncertainty is due more to oxidation than to impurities.

IV. THE SCALE OF THE PLATINUM THERMOMETER.

The scale defined by the platinum thermometer, calibrated in ice, steam, and sulphur, may be best expressed in terms of the freezing points of the pure metals. In Table 12 are grouped the results of the determinations of a series of such freezing points as obtained with platinum of several grades of purity.

The temperatures determined with thermometers of pure platinum, for which the fundamental coefficient is 0.0039, and the difference coefficient 1.50, are evidently in very close agreement with temperatures on the gas scale. When, however, thermometers are made of impure platinum having $\delta = 1.57$ or 1.80 , as shown in the table, and the Callendar method of reduction is used, the agreement of the two temperature scales, platinum resistance and gas, is seen to be far from satisfactory.

The precision at the various fixed points is also shown in Table 12, where the column headed "Range," when a single sample is used, gives a measure of this precision in terms of the difference between the highest and the lowest determinations, omitting none.

An idea of the reproducibility of these fixed points in terms of samples from several sources is also given by this same column, "range," for those cases in which several samples are used, as indicated in the column headed "number of samples."

The freezing points determined by Heycock and Neville²⁷ with the resistance thermometer are also included in Table 12. They are computed for S. B. P. = $444^{\circ}.70$ instead of for S. B. P. = $444^{\circ}.53$.

Correction for Thermometers of Impure Platinum.—From the measurements with thermometers made of wires with various difference coefficients it is possible to find, graphically, the corrections necessary to reduce the indications of any thermometer of impure platinum to the scale of the thermometer of pure platinum.

Such a series of corrections is given in Table 13 for values of δ varying from 1.525 to 1.900, and for temperatures from 200° to 1100° . The corrections are in degrees C and reduce the apparent temperatures, calculated by the Callendar formula from readings at the ice, steam, and sulphur points, to the scale of the pure platinum thermometer of $\delta = 1.505$.

Precision of Measurement with Platinum Thermometer.—The range of values of all the F. P. determinations taken with a single thermometer in a given crucible of metal will serve as a measure of the precision of measurement attainable with that thermometer at the temperature in question.

In Table 14 is given the range of temperatures at a series of fixed points obtained with thermometers 1787C. and 1787A., the former of pure platinum and the latter of platinum for which $\delta = 1.57$. It will be seen that the greatest range observed at any temperature up to the copper point is $0^{\circ}.4$, and the average range is only $0^{\circ}.11$, or practically within the reproducibility of many of these metals from new samples. The thermometers of impure platinum give as good precision as those of pure platinum; but it is to be remembered that the constants of the impure metal undergo the greater changes with heating, necessitating more frequent recalibration of such thermometers.

Modification of Callendar's Formula.—For thermometers of impure platinum, used at high temperatures, Callendar's formula may be modified so as to give results agreeing very closely with temperatures on the scale of the thermometer of pure platinum or on the gas scale. If a fourth calibration point is taken, such as the copper or silver freezing point, the difference coefficient may be written in the form, $\delta = a + bt$. Platinum temperatures, pt , are then computed in the usual way; but the true temperatures are now obtained by using the interpolated value of δ for that temperature, which need be known only approximately.

Using the silver freezing point as the fourth calibration temperature to determine a and b , we have calculated several of the freezing points of the metals by this method for thermometers of impure platinum and of palladium, as shown in Table 15, where are also grouped for comparison the temperatures on the scale of pure platinum and for the thermometers of impure platinum, as reduced by the ordinary Callendar method of calibration (*i. e.*, at ice, steam, and S. B. P.).

It is evident from an inspection of the table that for the impure platinum wires the calibration at four points by the modified Callendar method reproduces the temperature scale of pure platinum in the range 0 to 1100° more closely than the latter is known in terms of the gas thermometer scale.

TABLE 15.

Temperature Scale for Impure Platinum and Palladium. Modification of Callendar Formula.

$$[\theta = a + bt \text{ (from S. B. P. and Ag F. P.)}]$$

Metal	F. P. on scale of Therm. of Pure Pt $\theta = 1.503$	Thermometers of Impure Pt				Pd Therm.	
		No. 1787A		No. 1787E		No. 1787G	
		$\theta = 1.570$ from S. B. P. calibration	$\theta = a + bt$ from $a = 1.606$ $b = -0.0053$	$\theta = 1.803$ from S. B. P. calibration	$\theta = a + bt$ from $a = 1.900$ $b = -0.00219$	$\theta = 2.890$ from S. B. P. calibration	$\theta = a + bt$ from $a = 3.020$ $b = -0.00327$
Sn.....	231°.90	231°.82	231°.89			231°.62	231°.80
Cd.....	321°.01	320°.95	321°.05			320°.52	320°.74
Sb.....	630°.7	631°.2	630°.5	632°.6	630°.8	633°.0	629°.2
Ag-Cu.....	779°.2	781°.2	779°.1	784°.6	779°.3	787°.6	777°.1
Ag.....	960°.9	966°.2	(961°.0)	975°.3	(961°.0)	992°.5	(961°.0)
Cu.....	1083°.0	1091°.9	1083°.3	1106°.2	1082°.3	1152	1091

For the palladium thermometer the modified Callendar formula is not sufficient to give true temperatures, although up to the silver point the greatest divergence is only about 2° .

Other Calibration Formulæ.—It was thought worth while to see if other calibration formulæ would represent the resistance-temperature relation for platinum and palladium. Dickson⁸⁸ suggested the formula for platinum:

$$(R + a)^2 = p(t + b)$$

where a , p , and b are constants, and applied it to the observations of several observers. This formula possesses the apparent theoretical advantage over Callendar's of not requiring a maximum for the resistance of platinum. We find that, for thermometers of platinum or of palladium, calibrated at 0° , 100° , and S. B. P., Dickson's formula does not give satisfactory values of the freezing points. For example, thermometer 1787C., of pure platinum, calibrated in this way gives $232^\circ.13$ for tin, and $1051^\circ.3$ for copper. For thermometer 1787G, of palladium, the freezing point of copper would be 1005° , and of the Ag-Cu eutectic, $754^\circ.4$.

The adaptability of other types of formula has been discussed at length by Callendar.¹²

Other Methods of Calibration at High Temperatures.—The temperature scale defined by the resistance thermometer (of pure platinum), as used in this investigation and as almost universally used, is based on calibration in ice, steam, and sulphur. Higher temperatures are then determined by an extrapolation of the resistance-temperature relation thus found. If the thermometer is intended for use at temperatures much above S. B. P., it could be standardized at other known temperatures, for example, ice, S. B. P. or F. P. of Zn, and the F. P. of Ag or Cu, in which case extrapolation would be avoided. Instead of representing the parabolic resistance-temperature relation by the Callendar equation, it would be represented by the equivalent equation:

$$R_t = R_0(1 + at - \beta t^2)$$

This method of standardization was not used in the present investigation for the reason that the usual method (ice, steam, and sulphur) leads to values for the freezing points of Sb, Ag, and Cu, which are in such excellent agreement with the best gas thermometer determinations of these freezing points that the differences are almost, if not quite, within the accuracy of measurements attained up to the present time in gas thermometry.

V. CHANGES IN CONSTANTS OF THERMOMETERS AT HIGH TEMPERATURES.

Changes with Discontinuous Heating.—If the calibration data of each thermometer are arranged chronologically, the changes produced in the thermometric constants R_0 , FI , c , and δ , due to the various temperatures to which the instrument has been subjected, become immediately apparent. Tables 16 to 23 show these changes for the thermometers used in this investigation.

A comparison of the behavior of thermometers made of different grades of platinum and subjected to similar heat treatment shows that the constants of pure platinum undergo less change than those of impure platinum, although it is difficult to eliminate the part played by the mica frame in causing variations in these constants. It may be noted that, generally, the pure platinum shows a tendency to become less pure, as is seen by the decrease in c for thermometers 1787F, 478, 479, although for 1787C the

value of c remained practically constant, while for thermometers made of less pure platinum the value of c increased, as if the platinum were being purified by the loss by evaporation of impurities such as iridium, due to successive heatings. In support of this view it is to be noticed that the effect of continued annealing at 1100° is the same for all of the thermometers, namely, to increase the value of c .

For the thermometers of pure platinum there is a slight rise in zero (R_0) amounting only to a few tenths of a degree, while for those of impure platinum and of palladium the zero falls, and, in general, by relatively much greater amounts. For example, with 1787A, for which $\delta=1.57$, R_0 decreased by about $6^{\circ}.5$, and with 1787C, for which $\delta=1.50$, R_0 increased by less than $0^{\circ}.15$, for about the same heat treatment.

An examination of the behavior of the thermometers made of platinum of different grades shows the advantage of employing the purest metal for work of precision. It will also be seen that the changes in thermometric constants for all types are relatively small at temperatures up to the melting point of silver, but that they are often considerable at the copper point.

Thermometers 1787F and 1787G were used by both the Bridge and Potentiometer methods, and in comparing results by these two methods, Tables 17 and 22, it should be noted that the values of R_0 and FI differ, due to the different resistance units of the measuring apparatus, and also because the thermometer is not exactly the same instrument when used by the two methods.

The changes produced in δ by heating, after the thermometer is well annealed, are extremely small, and no certain conclusions can be drawn regarding the slight variations observed, as they appear for the most part to be within the limit of experimental error.

Changes during a F. P. Determination.—In determining freezing points at the higher temperatures it is necessary to determine, by calibrating the thermometer both before and after the freezing-point measurement, the changes occurring while the freezing point was being taken. These changes become important, and allowance must be made for them in work to $0^{\circ}.1$ above 700° , and in some cases at lower temperatures. This

effect increases materially with the temperature, and is the main factor in reducing the precision attainable at high temperatures. An examination of the experiments on Ag, Cu, and Ag-Cu (Tables 8, 9, and 10) shows that changes amounting to several tenths of a degree at these temperatures may arise from this cause. These changes are indicated in condensed form in Table 24, for values of both pt and t . In general, the temperature assigned to a freezing point was the mean value obtained from the calibrations before and after the freeze.

Some measurements made on two Leeds and Northrup resistance thermometers, Nos. 9837 and 9838, the coils of which were made of much larger platinum wire (diam. = 0.6 mm), show a much greater constancy at high temperatures for thermometers made of thick wire. These coils were not mounted on mica frames but were freely suspended and were thus not subject to strains resulting from thickening of the mica frame (see Plate I). The resistance in ice was quite low, about 0.11 ohm, but a much larger measuring current could, of course, be used than with the fine wire thermometers, so that the sensitiveness was not reduced in proportion to the resistance. These two thermometers were submitted to the same heat treatment. After a very thorough annealing it was found that an exposure for five hours at 1200°, three and one-half hours at 1250°, and one hour at 1300°, in an electric furnace, caused a total increase of R_0 and a change in the FI corresponding to only a few tenths of a degree (see Table 23). The error at 1200° at the close of the experiments, due to using the calibration found at the beginning of the experiments was of the order of 3°, the two thermometers showing approximately the same changes.

These thermometers were mounted in fairly clear quartz tubes, and it may be worthy of mention that while this material withstood with only slight crystallization the exposure to temperatures of 1200° or 1300° for some ten or fifteen hours, it deteriorated very rapidly after that, and developed numerous cracks and spots where transformations in the structure of the material were evident.

TABLE 25
Comparison of Scales of Platinum Thermometer and Thermocouples.

Metals	Temp. from Res. Therms.*	Temperature from Thermocouples $E=a+bT+cT^2$									
		P_1 Heraeus 1903 Pt, 90 Pt-10 Rh		P_1 Heraeus 1908 Pt, 90 Pt-10 Rh		S_1 Johnson Matthey 1904 Pt, 90 Pt-10 Rh		S_2 Heraeus 1908 Pt, 90 Pt-10 Rh		W_1 Pellin Pt, 90 Pt-10 Ir	
		I	II	I	II	I	II	I	II	I	II
Zn.....	419°.37	†419°.37	419°.37	419°.37	419°.37	419°.37	419°.37	419°.37	419°.37	419°.37	419°.37
Sb.....	630°.7	630°.7	630°.9	630°.7	631°.0	630°.7	630°.4	630°.7	631°.3	630°.7	630°.7
Ag-Cu..	779°.2	779°.1	779°.2	778°.9	779°.2	779°.6	779°.2	778°.6	779°.2	779°.2	779°.2
Ag.....	961°.0	960°.7	960°.8	960°.8	960°.9	961°.4	961°.2	960°.6	961°.0	960°.9	960°.9
Cu.....	1083°.0	1083°.0	1083°.0	1083°.0	1083°.0	1083°.0	1083°.0	1083°.0	1083°.0	1083°.0	1083°.0

* Scale defined by resistance thermometer of pure platinum calibrated in terms of the Callendar equation from observations in ice, steam, and sulphur (444°.70).

† Numbers in heavy type taken as calibration points for thermocouples, i. e., for computing constants a , b , and c of the equation $E=a+bT+cT^2$. The remaining two temperatures are then the freezing points on the thermoelectric scale defined by this equation.

‡ For convenience in computation, the formula $T=a+bE+cE^2$ is sometimes used as the calibration equation for a thermocouple. It should be noted that this formula does not exactly agree with the above formula, even within the range of calibration temperatures, and differs very much for higher extrapolated temperatures.

VI. COMPARISON OF RESISTANCE AND THERMOELECTRIC SCALES.

To compare these two temperature scales a number of thermocouples were calibrated at the freezing points of the metals previously used with the resistance thermometers. The results are given in Table 25. The calibration equations of the couples, $E = a + bt + ct^2$, were computed from the observed electromotive forces, E , at three temperatures, indicated by heavy type in the table. The observations were reduced in two ways. In the first column under each thermocouple are given the computed values of the freezing point of the silver-copper eutectic alloy, and of silver, when the freezing points of zinc, antimony, and copper are taken as calibration temperatures. These are the fixed points often employed for the standardization of thermocouples. In the second column under each thermocouple are given the computed values of the freezing point of antimony and of silver, when the freezing points of zinc, Ag-Cu eutectic, and copper are taken as calibration temperatures.

The observations show that the thermocouples calibrated at three temperatures are in excellent agreement at the intermediate temperatures, the average deviation from the mean of their indications being about $0^{\circ}.2$, and the maximum deviation $0^{\circ}.5$. Furthermore, the temperature scale defined by these couples calibrated at three temperatures (as given by the scale of the platinum thermometer) is in agreement at the intermediate temperatures with the scale of the platinum thermometer, as defined by the Callendar equation, to within $0^{\circ}.3$.

The method of comparing these temperature scales here adopted, viz, the determination of the freezing points of a number of metals, is capable of great precision and has some decided advantages over the usual method of intercomparison even in a most carefully compensated electric furnace. Harker,⁸⁸ using the latter method, finds the gas, thermoelectric, and resistance scales to agree to within 2° on the average, in the range 450° to 1000° .

VII. THE PALLADIUM THERMOMETER.

It was thought of sufficient interest to include measurements made with a thermometer of pure palladium, as this would furnish evidence as to the generality of the Callendar formula for pure metals.

The constants of such a thermometer, 1787G, calibrated in the same way as the platinum thermometer, are given in Table 1; the individual observations on the freezing points of the metals in the several tables 3 to 10; the behavior of the constants R_0 , FI and c in Table 22; and a comparison of the results obtained with the platinum and palladium thermometers at the several freezing points in Tables 26 and 27.

It is evident that the palladium thermometer by no means satisfies the Callendar equation at the higher temperatures. It follows that this formula is not general in its applicability to metallic resistance thermometers, even for metals of the platinum group, in spite of the fact that for thermometers of pure platinum it may be used for extrapolation to at least 1100° and give results in excellent agreement with the gas scale.

Between 0° and nearly 600° , however, the Callendar equation gives, for the palladium thermometer, temperatures agreeing with those found by the platinum thermometer to within 1° , and for the greater part of this range to within 0.5° .

The Callendar formula, modified by making $\delta = a + bt$, gives better values for temperatures obtained with the palladium thermometer than the simple Callendar formula (see Table 15). Dickson's equation (p. 177) does not satisfy the observations with palladium at all.

An independent equation for the temperature-resistance relation of palladium to 1100° appears to require five constants, so that it was considered impracticable to indicate such an equation.

The behavior of palladium as regards changes in its constants, due to heating (to 1100°) is about the same as that of platinum wire having $c = 0.0021$ and $\delta = 1.57$, see Tables 20 and 22.

Another thermometer of palladium, Pd II, was made with metal from another source; but its calibration constants, Table 1, as well as the thermoelectric behavior of the wire of which it was made, indicated a less pure material, and no further measurements were made with it.

VIII. THE BOILING POINT OF SULPHUR.

Regnault⁴ made a series of observations on the relation between pressure and temperature of the vapor of sulphur for pressures ranging from 250 to 3000 mm. As only two of these observations were in the neighborhood of atmospheric pressure, various values for the S. B. P. have been deduced from these observations by different experimenters who have endeavored to utilize Regnault's results.* As these determinations were carried out in a thick-walled iron-tube boiling-point apparatus heated in a furnace, the observed temperatures were too high, due to superheating of the sulphur vapor, a fact which was recognized by Regnault.

After an extensive series of experiments with an iron-tube S. B. P. apparatus, Callendar and Griffiths¹⁵ were finally led to adopt a glass-tube apparatus. The sulphur was boiled in the outer tube of the well known Victor Meyer vapor-density apparatus, about 4 cm diameter, 40 cm long, and having a spherical enlargement at the lower end. The tube was protected against heat loss by a suitable asbestos covering extending from the center of the bulb to within about 4 cm of the upper end. To prevent superheating of the sulphur vapor, the heating by the Bunsen flame was limited to the lower half of the spherical bulb containing the sulphur, the depth of which was such that it somewhat more than filled the spherical enlargement. The supply of heat was regulated so that the line of condensation of the sulphur vapor remained fixed a short distance (about 2 cm) above the asbestos covering. A series of instructive experiments carried out by these observers showed that in order to find the temperature of the vapor, or at least to attain a definite and reproducible temper-

* By the boiling point of sulphur, often abbreviated to S. B. P., will be understood the temperature on the scale of the gas thermometer of the saturated vapor of sulphur at standard atmospheric pressure (i. e., 760 mm of mercury at 0° and at latitude 45°, sea level), as measured in a definite form of boiling-point apparatus and under carefully specified conditions.

Using the two observations nearest atmospheric pressure, Callendar and Griffiths¹⁵ deduce from Regnault's data 447°.48 for S. B. P. Weinhold⁷ deduces from Regnault's empirical formula 448°.4. From a recomputation of these observations Ramsay¹⁰ gives 448°.34. Taking the four observations of Regnault nearest to atmospheric pressure (467.45 mm to 1308.54 mm), Chappuis and Harker⁶¹ deduce from these data 447°.51 as the S. B. P.

ature, it is not sufficient merely to insert a thermometer in the vapor. They found that when the platinum thermometer was immersed in the sulphur vapor, unprotected by any radiation screens it read about $1^{\circ}.2$ lower than when thoroughly shielded, owing to cooler liquid sulphur condensed in the upper parts of the tube running down over the bulb of the thermometer, and to loss of heat by radiation to the cooler walls of the tube. The effect of adding the umbrella (see Plate VIII) above the coil of the thermometer was to raise the observed temperature about $0^{\circ}.3$, and the effect of adding a glass-tube radiation screen and two perforated asbestos disks below was to produce a further rise of about $0^{\circ}.5$. Adding a second coaxial glass-tube radiation screen produced an additional rise of about $0^{\circ}.2$. Replacing the inner glass tube with a brass tube gave a further rise in temperature. In the final form of radiation screen used by these observers, shown in Plate VIII, the inner glass tube is covered with platinum foil. With this apparatus in its final form, these observers found for the *S. B. P.* $444^{\circ}.53$ at standard atmospheric pressure, on the scale of the constant pressure air thermometer, the platinum thermometers used in this determination having been previously compared with the constant pressure air thermometer.

A brief note²⁶ on the results of a determination of the *S. B. P.* made at the Physikalisch-Technische Reichsanstalt, by means of several mercurial thermometers that had been compared with a constant volume porcelain bulb air thermometer, gave for the *S. B. P.* $444^{\circ}.5$. The temperature scale of the air thermometer on which this work is based was shown to be in satisfactory agreement with the temperature scale of the air thermometer of Wiebe and Böttcher.

Chappuis and Harker²¹ determined the boiling point of sulphur by means of two platinum thermometers in an apparatus similar in all essential details to that used by Callendar and Griffiths, using as a radiation screen a perforated asbestos cone, similar to that used by Heycock and Neville,²⁷ as shown in Plate VIII. These thermometers had been previously compared with a constant volume nitrogen thermometer, having a bulb of Berlin porcelain. This work gave for the *S. B. P.* $445^{\circ}.2$. In this work the expan-

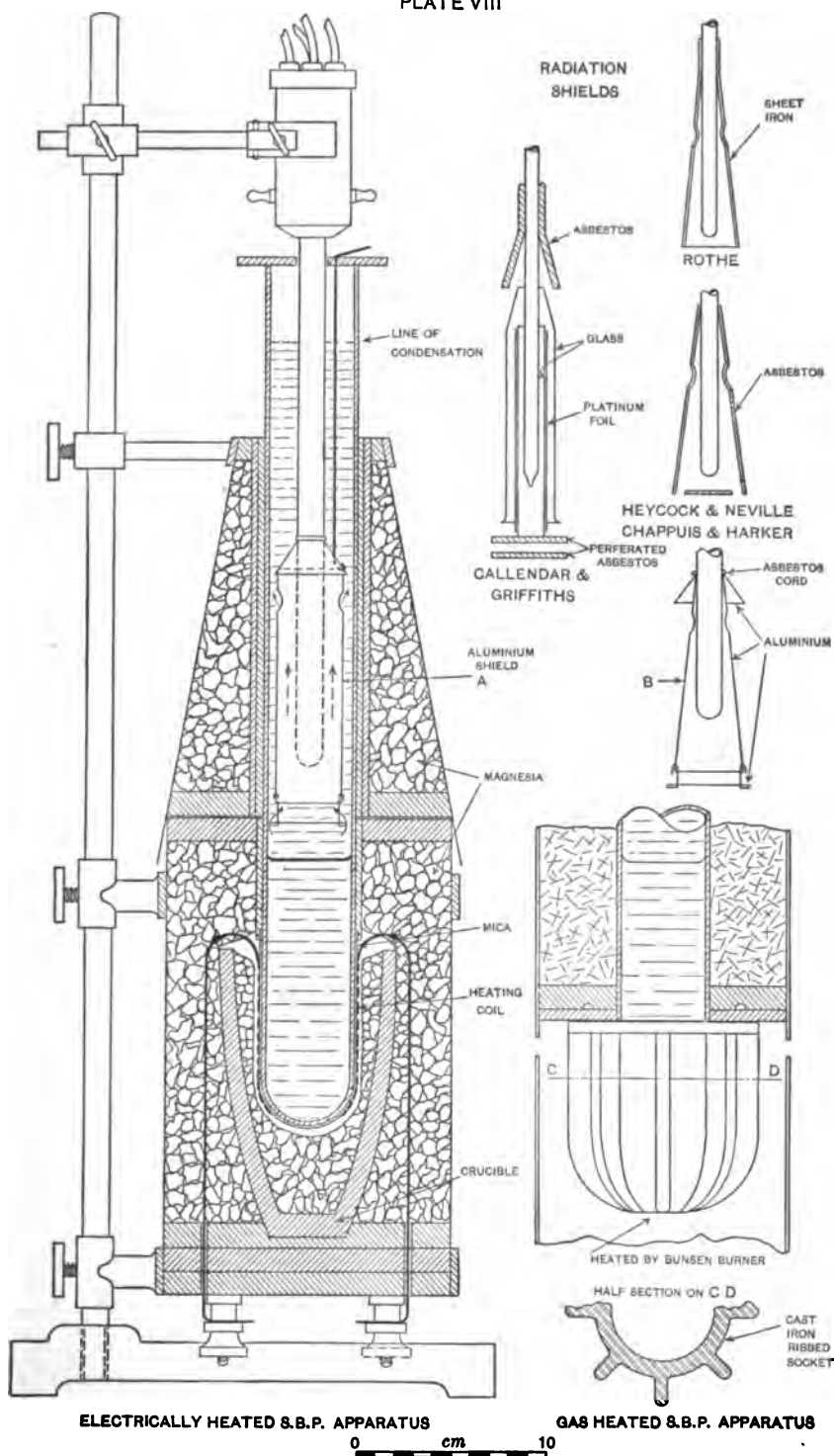
sion of the porcelain was determined only for the interval 0° to 100° , and the value at higher temperatures was determined by extrapolation. Subsequently, after Holborn and Day⁵³ had determined the expansion of Berlin porcelain at high temperatures, Chappuis⁷⁰ recomputed the results previously found in collaboration with Harker, and was led to the value *S. B. P.* = $444^{\circ}.7$ on the scale of the constant volume nitrogen thermometer.

Holborn⁵³ compared several thermometers of pure platinum wire with a constant volume nitrogen thermometer in the interval 300° to 500° and found that the results of observation were in agreement with the Callendar equation to within $0^{\circ}.1$, the value of δ being 1.489. While Holborn did not use these thermometers in an actual determination of the *S. B. P.*, it is evident that they would give $444^{\circ}.55$ for the *S. B. P.*, since platinum of the same degree of purity as that used in these thermometers has a $\delta = 1.50$, when this constant is determined by calibration in ice, steam, and sulphur, and when the boiling point of sulphur is taken as $444^{\circ}.70$.

Rothe⁸⁰ determined the boiling point of sulphur in an electrically heated *S. B. P.* apparatus similar to that described and used by the authors in the present work, using for this purpose four gas-filled mercurial thermometers, the relation of the scales defined by these thermometers to the standard gas scale of the Physikalisch-Technische Reichsanstalt being known. The correction for emergent stem of the thermometer, which amounted to $1^{\circ}.5$ to 3° , was determined by means of a Mahlke fadenthermometer. A conical radiation screen of thin sheet iron, as shown in Plate VIII, was used. Rothe found for the *S. B. P.* $444^{\circ}.7$ on the constant volume scale of the air thermometer. With two thermocouples that were calibrated on the scale of the Holborn and Day constant volume (*Pt-Ir* bulb) nitrogen thermometer, the standard high temperature scale of the Reichsanstalt, but which was not primarily intended for the highest attainable accuracy at the lower range of temperatures, Rothe found for the *S. B. P.* $445^{\circ}.0$.

Variation of *S. B. P.* with Pressure.—The following relations for the variation in temperature of the *S. B. P.* with pressure have been used:

PLATE VIII



$$t = t_{760} + 0.082 (p - 760) \dots\dots\dots (1)$$

$$t = t_{760} + 0.088 (p - 760) \dots\dots\dots (2)$$

$$t = t_{760} + 0.090 (p - 760) + 0.0002 (p - 760)^2 \dots\dots\dots (3)$$

$$t = t_{760} + 0.0912(p - 760) - 0.000042 (p - 760)^2 \dots\dots\dots (4)$$

$$t = t_{760} + 0.0904(p - 760) - 0.0000519 (p - 760)^2 \dots\dots\dots (5)$$

Equations (1) and (2) were deduced from Regnault's observations, the former by Griffiths from observations including a wide range of pressures, and the latter by Chappuis and Harker from the observations nearest atmospheric pressure. Equation (3) was computed by Chree⁵⁷ from a large number of observations of the steam points of platinum thermometers carried out at different times at Kew Observatory, the total range of barometric pressure being, however, only about 30 mm. Equations (4) and (5), the former given by Holborn and Henning¹³⁰ and the latter by the Harker and Sexton¹²⁷, are the results of experiments especially planned with a view to the determination of the temperature-pressure relation, and are, therefore, to be preferred. These equations, which are in very close agreement, were determined from experiments over a range of 100 mm above and below standard atmospheric pressure.

S. B. P. Tubes.—The construction of the electrically heated S. B. P. apparatus in which most of our observations were made is shown in sectional drawing in Plate VIII. Two forms of gas-heated tubes were also tried, one a Victor Meyer tube similar to that used by Callendar and Griffiths, in which the heating was done by a Bunsen flame applied directly to the lower part of the spherical enlargement at the bottom of the tube, the other in which the lower end of the glass tube was inclosed by the iron casting shown in Plate VIII, the heat being supplied to this iron jacket by a Bunsen flame.

The order of accuracy with which the S. B. P., observed in this way, serves to fix a definite and reproducible temperature, and the precautions that are necessary, will be seen from the following experiments which are briefly described:

Gas and Electric Heating.—A series of measurements made alternately with these three S. B. P. tubes,* using the same ther-

* We have had tubes which would last for a long time; at other times the cracking of the tubes was very annoying, and required frequent rebuilding of the apparatus due to the sulphur coming into contact with the heating coil. Heating coils of nickel, constantan, etc., were used at different times, but a heating coil of thin platinum ribbon was finally adopted. For these reasons the gas-heated type would probably prove more satisfactory for ordinary laboratory use.

mometer and the same radiation shields, has shown that the tubes serve to define practically the same temperature, the S. B. P. observed in the two gas-heated tubes being $0^{\circ}.03$ and $0^{\circ}.04$ higher, respectively, than in the electrically heated tube.

Superheating.—To avoid superheating of the sulphur vapor, the depth of liquid sulphur should be such that its upper surface is some distance (several cms at least) above the upper end of the electric heating coil or that portion of the glass tube in contact with the heating surface (flame or hot gases). For example, with an electric heating coil 7 cm high and with a depth of 8 cm of sulphur, there were still evidences of superheating for a distance of some 7 cm above the surface of the liquid, the superheating being about $0^{\circ}.1$, $0^{\circ}.05$, and $0^{\circ}.01$, respectively, at distances of 4.5, 5.5, and 6.5 cm above the surface. With a depth of only 3.5 to 4 cm of liquid sulphur in the tube, the observed superheating at 6.5, 10, 17, and 30 cm distance from the liquid surface (in a tube 55 cm long) was 20° , $8^{\circ}.3$, $2^{\circ}.5$, and $1^{\circ}.1$, respectively. With a depth of 12 cm of sulphur the maximum variation in temperature, when taken as hereafter described, for the interval 4 cm above the surface to 20 cm above the surface was only about $0^{\circ}.02$ or $0^{\circ}.03$. When naphthalene is boiled in the electrically heated tube, the observed boiling point (with thermometer No. 479) for 8 and 16 cm depth of liquid in the tube was identical (viz, $218^{\circ}.01$), while when the depth of liquid was only 4 cm the observed boiling point was $0^{\circ}.31$ higher, showing the effect of superheating due to the contact of the vapor with the portion of the wall heated by the coil above the surface of the liquid.

By lowering the thermometer, with its attached radiation shields, into the liquid sulphur, it was found that the temperature of the liquid was some $5^{\circ}.4$ higher than the temperature observed in the vapor.

Radiation Shields.—We have repeated most of the experiments of Callendar and Griffiths on the effects of shielding the thermometer in the sulphur vapor and have made many additional experiments in order to determine the conditions to be specified in order to reproduce the S. B. P. with the highest accuracy. The two types of radiation shields used in this work are shown in Plate

VIII. The temperatures observed with these two shields were very nearly identical, *A* giving a temperature very slightly lower than *B* (about $0^{\circ}.02$). With the lower disk of *A* removed, which gives the type of shield used by Rothe, the observed temperature was very slightly lower ($0^{\circ}.015$). A shield exactly similar to that used by Callendar and Griffiths was also tried. It gave a temperature about $0^{\circ}.03$ or $0^{\circ}.04$ higher than the shield *B*. The effect of adding another aluminium cylinder around *B* was to give a temperature about $0^{\circ}.03$ higher than the single cylindrical shield *B*. If all radiation shields are dispensed with the thermometer reads about $1^{\circ}.3$ low, and if the small cone alone is used above the coil, the thermometer reads about 1° low.

Temperature Distribution within S. B. P. Tube.—To determine whether such circumstances as distance of thermometer coil from liquid surface, from line of condensation, etc., has any effect on the observed S. B. P., the thermometer, together with its shield, was moved up and down along the axis of the S. B. P. tube and observations were taken with the coil of the thermometer at various distances from the liquid surface (from about 7 cm from center of coil to surface of sulphur to a position where the top of the radiation shield is just below the condensation line on the S. B. P. tube). With the thermometer in shield *B*, mounted so that the lower end of the porcelain tube was in the plane of the base of the surrounding aluminium cylinder, the temperature remained constant (within $0^{\circ}.01$) until the top of the shield was about 1 cm below the line of condensation of the sulphur, in which position it read $0^{\circ}.02$ low, the observed temperature falling off very rapidly upon further raising the thermometer. A similar experiment was also tried with the thermometer coil mounted about 4.5 cm farther up in the shield with the same result. Notwithstanding the excellent agreement ($0^{\circ}.01$ or $0^{\circ}.02$) in the observed temperatures when the coil of the thermometer was placed in these two different positions within the radiation shield, thermocouples (of *Pt*, *Pt-Ir*, and *Pt*, *Pt-Rh*), used to explore the distribution of temperature within that portion of a porcelain tube (similar to the thermometer tube) under the radiation shield, showed very appreciable point-to-point variations within this region, amounting, as will be seen later, to as much as $0^{\circ}.5$.

While the conical shield *A* gives a temperature only about $0^{\circ}.02$ lower than the cylindrical shield *B*, it must be placed farther down in the sulphur vapor, well below the line of condensation. With the conical shield *A* there is a marked drop in temperature when the thermometer, with its shield, is raised, which is noticeable when the top of the shield is still some 4 or 5 cm below the condensation line. By putting an additional small umbrella above the shield *A* this effect disappears, and the thermometer, with its shield, can be raised so that the top of the shield is within 1 cm of the condensation line without producing a change in the observed temperature exceeding $0^{\circ}.01$.

Temperature Distribution within Radiation Shield.—As has been stated, the observed resistances of a platinum thermometer when mounted in the two different positions within the shield, as already described, were almost identical. Notwithstanding this fact, when the temperature distribution inside a porcelain tube, similar to the thermometer tube, was explored with a thermocouple, it gave evidence of very appreciable point-to-point variations of temperature of the order of $0^{\circ}.4$ or $0^{\circ}.5$. If these are true variations in temperature, one would expect to find a different resistance for the coil of the platinum thermometer when mounted in different positions within the shield, especially as the thermometer set-up was sensitive to well within $0^{\circ}.01$. It is possible that a part of the variation observed with the couples is apparent rather than real, due, perhaps, to the heterogeneity of the wires of the couple, which would give slightly different electromotive forces with variable depth of immersion in the heated region. Heræus couples of *Pt*, *Pt-Rh*, and of *Pt*, *Pt-Ir*, however, both gave about the same temperature-distribution curve. Further experiments are contemplated with a special resistance thermometer, having a very short coil of a few millimeters of fine (Wollaston) platinum wire.

On the Uncertainty in the Accepted Value of the S. B. P.—On account of the satisfactory agreement of the determinations of the sulphur boiling point by different observers using different gas thermometers it has been very generally assumed that this temperature was known to $0^{\circ}.1$ or $0^{\circ}.2$. A consideration of some recent work with the gas thermometer, however, throws some

doubt on this conclusion and would seem to show that the uncertainty in the present accepted value of the S. B. P. ($444^{\circ}.5$ to $444^{\circ}.7$) may be as much as 1° .

In the present investigation the value assumed for the S. B. P.—viz, $444^{\circ}.70$ —gives $419^{\circ}.37$ for the F. P. of zinc, as determined by the platinum resistance thermometer. Day and Clement¹³⁵ have recently published the preliminary results of a series of elaborate and painstaking measurements with a constant volume nitrogen thermometer and find for the F. P. of zinc $418^{\circ}.5$. If this temperature were taken as one of our known calibration temperatures for the platinum thermometer, the other two temperatures being that of ice and of steam, it would give for the S. B. P. $443^{\circ}.7$ —that is, 1° lower than the generally accepted value. The desirability of further careful work with the gas thermometer is therefore evident. In this connection it is of interest to note the value of the S. B. P. recently found by Eumorfopoulos,¹³⁶ using the Callendar form of compensated constant pressure air thermometer, viz, $443^{\circ}.6$. The uncertainty in the coefficients of expansion used for the glass bulb of his air thermometer is such, however, that this value of the S. B. P. may be in error by more than 1° , as is recognized by both Messrs. Eumorfopoulos and Callendar.¹³⁴

If the true temperature of the vapor of sulphur, boiling at standard atmospheric pressure and measured under the conditions specified by the several investigators who have determined this point, should ultimately be found to be $443^{\circ}.7$ instead of $444^{\circ}.7$, as we have taken it, it would result in changing the value of δ from 1.505 to 1.447 and in lowering the resulting values of the freezing points of Sn, Cd, Pb, Zn, Sb, Ag-Cu, Ag, Cu, by $0^{\circ}.19$, $0^{\circ}.44$, $0^{\circ}.47$, $0^{\circ}.87$, $2^{\circ}.3$, $3^{\circ}.9$, $6^{\circ}.5$, $8^{\circ}.8$, respectively.

The Callendar method of calibration in ice, steam, and sulphur would not then be applicable for extrapolation to high temperatures, as it would give for the F. P. of copper $1,074^{\circ}$ instead of $1,083^{\circ}$ and the experimental evidence available makes it reasonably certain that the latter value can hardly be in error by more than 2° or 3° .

While the agreement of the scale of the platinum thermometer with the Holborn-Day gas scale is remarkable when we consider

that the former scale is dependent on an extrapolation from the boiling point of sulphur, yet on account of the purely empirical nature of the Callendar equation this agreement can not be used as an argument in favor either of the correctness of the present accepted values of the freezing points of the metals or to decide between these two values of the S. B. P., although, taken in connection with the other corroborative experimental evidence previously referred to, the higher value of the S. B. P. ($444^{\circ}.70$) and the resulting temperature scale as used in this investigation appear the more probable.

S. B. P. on Thermodynamic Scale.—Berthelot¹⁸⁷ and Buckingham¹⁸⁸ have recently discussed the probable differences between the temperature scales of the constant volume and the constant pressure gas thermometer and the absolute thermodynamic scale. The S. B. P. found by Callendar and Griffiths ($444^{\circ}.53$ on the constant pressure air scale) and by Harker and Chappuis ($444^{\circ}.70$ on the constant volume nitrogen scale), when reduced in accordance with the conclusions of the above authors, is $444^{\circ}.9$ on the thermodynamic Centigrade scale. In some recent work with the platinum resistance thermometer, Holborn and Henning¹⁸⁹ have used the value $445^{\circ}.0$.

IX. GENERAL CONCLUSIONS.

a. Temperature Scales.

1. Temperatures defined by the resistance thermometer of pure platinum calibrated by Callendar's formulæ at 0° , 100° , and 444.70° , the boiling point of sulphur, agree with temperatures on the generally accepted gas scale in the range 0 to 1100° C, to within the degree of reproducibility of the latter.

Freezing Point of:	Sn	Cd	Zn	Sb	Ag ₂ -Cu ₂	Ag	Cu
Thermometers of Pure Pt..... $\left\{ \begin{array}{l} c=0.0039 \\ d=1.50 \end{array} \right\}$	231°.90	321°.01	419°.37	630°.71	779°.20	960°.88	1083°.0
Thermometer of Impure Pt 1787A.. $\left\{ \begin{array}{l} c=0.0021 \\ d=1.57 \end{array} \right\}$	231°.82	320°.95	419°.32	631°.25	780°.86	966°.21	1091°.9
Thermometer of Impure Pt 1787B.. $\left\{ \begin{array}{l} c=0.0017 \\ d=1.80 \end{array} \right\}$				632°.65	784°.60	975°.22	1106°.2

2. Thermometers made of impure platinum, and calibrated in the same way, do not define the same temperature scale as that of pure platinum, the divergence increasing with the impurity of the metal, as shown in the preceding table.

3. If thermometers of impure platinum are calibrated at a fourth temperature, such as the silver freezing point, and the difference coefficient of the Callendar formula is written $\delta = a + bt$, then the scale so defined by these thermometers is brought into very close agreement with that of the thermometer of pure platinum, as follows:

Freezing Point of:	Sn	Cd	Sb	Ag-Cu ₂	Ag	Cu
Thermometers of Pure Pt..... ($\delta=1.50$)	231°.90	321°.01	630°.7	779°.2	960°.9	1083°.0
Thermometer of Impure Pt 1787A.....	231°.89	321°.05	630°.5	779°.1	(961°.0)	1083°.3
Thermometer of Impure Pt 1787E.....			630°.8	779°.3	(961°.0)	1082°.3

For Thermometer 1787A, $\delta = 1.608 - 0.00853 t$.

For Thermometer 1787E, $\delta = 3.020 - 0.00327 t$.

4. The Dickson formula $(R + a)^2 = p(t + b)$ is not applicable to either platinum or palladium thermometers at high temperatures.

5. The palladium thermometer, calibrated by the Callendar method, gives temperatures agreeing to 1° with the scale of the platinum thermometer up to 600°. It requires a fourth degree equation to express its resistance-temperature relation from 0 to 1100°.

6. The following table of freezing points represents the scale of the resistance thermometer of pure platinum as determined from this investigation, using only the purest metals:

Metal	Sn	Cd	Pb	Zn	Sb	Al	Ag-Cu ₂	Ag	Cu	Cu-Cu ₂ O
Freezing Point.....	231°.92	321°.01	327°.43	419°.37	630°.71	658°.0	779°.20	960°.9	1083°.0	1063°.2
Average deviation for purest metal.....	.02	.02	.10	.02	.30	.5	.25	.20	.5	.3
Reproducibility factor.....	.06	.26	.10	.06	2.3	1.2			1.0	

The average deviation of a single observation is a measure of the precision attained when two or more thermometers were used in a single metal sample. The reproducibility factor is the average deviation of the metals, designated as C. P. or better, from their mean F. P.

7. Thermocouples of Pt, 90 Pt-10 Rh and Pt, 90 Pt-10 Ir, calibrated at three temperatures in terms of the formula $E = a + bt + ct^2$, define a temperature scale in the interval 200° to 1100°, which is in agreement with the platinum resistance scale at intermediate temperatures to within 0°.3.

b. Behavior of Platinum Thermometers.

8. When used at high temperatures the constants of platinum thermometers undergo gradual changes, which necessitate frequent recalibration in work of high precision. The use of thermocouples for work above the F. P. of silver is therefore much less laborious.

9. These changes in thermometric constants are greatly reduced but not completely eliminated, by annealing the thermometers for at least several hours, at a temperature higher than that at which they are subsequently to be used.

10. For thermometers of pure platinum, the changes in their constants after the thermometers have been annealed, are very much less than for those of impure platinum; thus, R_0 changes only by a few tenths of a degree for pure platinum but by several degrees for impure.

11. These changes are least for pure platinum wires of large diameter and suspended free from strains. For such thermometers, made of wire 0.6 mm in diameter, R_0 changed by only a few tenths of a degree after repeated and continued heatings at 1200° to 1300°.

12. For impure platinum wire the effect of high temperatures is to decrease R_0 and to increase the fundamental coefficient, c , that is, the effect is as if the wire became purer, possibly because of the evaporation of impurities, for example, iridium. If the platinum is pure, the slight changes indicate a contamination of the wire and the effect of strains, as is evidenced by decrease in c and the increase in R_0 . The total change observed is a resultant of the effects of strains, of annealing, and of contamination and purification. In the use to which these thermometers were subjected in this investigation, the changes in the value of δ were almost within the limits of experimental error. For impure platinum there is evidence of a slight decrease in this constant.

c. Calibration Temperatures.

13. The freezing points of the pure metals are the most satisfactory calibration temperatures, being superior to boiling points in constancy, ease of manipulation, and reproducibility. Unless

a very slow rate of heating is used, melting points are less satisfactorily fixed and apparently slightly higher than freezing points.

14. The temperature of the separation of the silver-copper eutectic is an entirely satisfactory fixed point at $779^{\circ}.2$, when the alloy is of the eutectic composition, or about 72Ag-28Cu. The eutectic point of copper-cuprous oxide is also a satisfactory calibration temperature at $1063^{\circ}.2$, when proper experimental precautions are taken to maintain the eutectic composition.

d. The Boiling Point of Sulphur.

15. The temperature defined by the vapor of sulphur boiling at known pressure and under carefully specified conditions is a most satisfactory fixed point, reproducible to a few hundredths of a degree. These conditions are briefly:

(a) Sulphur to be boiled in a glass tube 4 to 5 cm diameter and of sufficient length to insure ample depth of immersion in the vapor (tube about 40 cm satisfactory);

(b) Sufficient depth of liquid sulphur to avoid superheating;

A criterion for the above conditions is absence of change in the observed resistance when the thermometer with its attached radiation shield is moved some centimeters up and down the tube;

(c) Surrounding coil of thermometer with a suitable radiation shield to avoid the cooling effects of the condensed sulphur running down the stem of the thermometer and of radiation to the cooler parts of the apparatus.

16. Both gas and electrically heated S. B. P. tubes were used; the former gave a temperature $0^{\circ}.03$ or $0^{\circ}.04$ higher than the latter.

17. The determinations of the S. B. P. by different observers is discussed at length. The determinations of these observers are in satisfactory agreement, the values found ranging, with one exception, from $444^{\circ}.5$ to $444^{\circ}.7$. The bearing of some recent work with the gas thermometer on the value of the S. B. P. is considered, which shows that the above value may be too high. The evidence, however, is not sufficient as yet to warrant any change in the generally accepted and widely used value of the S. B. P.

We take great pleasure in acknowledging the assistance rendered by Messrs. J. J. Crowe and H. A. Ginsburg, both in the experimental work and in the computations involved in this investigation.

WASHINGTON, June 26, 1909.

X. APPENDIX.*

TABLE 2.

Heating Effect of Measuring Current.

[Thermometer 1787C.]

Amperes	R_0	R_{100}	$R_{44.33}$	F. I.	pt_{∞}	δ
2.50×10^{-3}	3.48160	4.82275	9.13220	1.34115	421°.33	1.503
5.00×10^{-3}	3.48164	4.82277	9.13220	1.34113	421°.33	1.503
10.00×10^{-3}	3.48174	4.82287	9.13213	1.34113	421°.31	1.505
25.0×10^{-3}	3.48286	4.82401	9.13308	1.34115	421°.30	1.505
50.0×10^{-3}	3.48705	4.82819	9.13608	1.34114	421°.21	1.511
100.0×10^{-3}	3.50373	4.84546	9.14832	1.34173	420°.69	1.545

Amperes	Heating of Pt Coil above Surrounding Temperature		
	ΔT_0	ΔT_{100}	ΔT_{∞}
2.50×10^{-3}	$0^{\circ}.001 \pm$	$0^{\circ}.001 \pm$	
5.00×10^{-3}	.004	.002	
10.00×10^{-3}	.011	.010	
25.0×10^{-3}	.095	.095	$0^{\circ}.067$
50.0×10^{-3}	.41	.41	.29
100.0×10^{-3}	1.65	1.69	1.20

R at F. P. of Silver		$pt_{\Delta_{\infty}}$	δ	$t_{\Delta_{\infty}}$
0.100 amp.	0.0100 amp.			
14.7079		835°.06	1.545	963°.64
	14.6995	836°.43	1.504	960°.83
	14.7009	836°.54	1.504	960°.98
14.7104		835°.24	1.545	963°.89

* The following tables are in the text: 1 on p. 156, 11 on p. 173, 12 on p. 174, 15 on p. 177, 25 on p. 181.

TABLE 3.
Freezing Point of Tin.

Date	Therm. No.	R _o	FI	θ	R _{sa}	p _{sa}	t	Remarks
Feb. 5, '09	1787C., P. T.	3.47977	1.33997	1.507	6.52466	227°.24	231°.85	F. P., E. & A., '05, D. C.
May 13, '09	"	{ 3.48164	1.34105	1.505	{ 6.52906	{ 227°.24	{ 231°.84	" Baker, '09, "
		{ 3.48177	1.34105	1.505		{ 227°.23	{ 231°.83	"
Mar. 17, '09	1787F., W. B.	2.85456	1.11265	1.500	5.38326	227°.27	231°.85	" B. & A., "
"	"	2.85456	1.11265	1.500	5.38420	227°.35	231°.94	" (K), " A. C.
"	"	2.85456	1.11265	1.500	5.38397	227°.33	231°.92	" " " "
Mar. 18, '09	" P. T.	2.84881	1.11084	1.508	5.37370	227°.30	231°.91	" " " "
"	"	2.84881	1.11084	1.508	5.37360	227°.29	231°.90	" " " "
Jan. 30, '09	1787A., P. T.	21.3347	4.4077	1.572	31.3388	226°.97	231°.77	" B. & A., '05, D. C.
"	"	21.3347	4.4077	1.572	31.3427	227°.06	231°.87	M. P., " " "
Feb. 5, '09	"	21.3347	4.4077	1.572	31.3402	227°.00	231°.80	F. P., E. & A., " "
Mar. 18, '09	"	21.0734	4.4213	1.565	31.1129	227°.07	231°.85	" (K), '09, A. C.
Mar. 17, '09	1787G., W. B. Palladium.	1.87759	0.63158	2.890	3.28485	222°.81	231°.62	" " " "

F. P. = Freezing point. M. P. = Melting point. D. C. = Dixon graphite crucible. A. C. = Acheson graphite crucible. E. & A. = Eimer and Amend.
 B. & A. = Baker and Adamson (K) = "Kahlbaum." P. T. and W. B. indicate method of measurement: Potentiometer, or Wheatstone Bridge.
 Where two values are given in brackets, the upper refers to the calibration before the F. P. or M. P. determination and the resulting value of the
 F. P. or M. P., and the lower to the calibration after and the corresponding value of the F. P. or M. P. computed therefrom.

TABLE 4.
Freezing Point of Cadmium.

Date	Therm. No.	R _c	FI	δ	R _{cd}	P _{cd}	t	Remarks
Feb. 27, '09	1787C., P. T.	3.47967	1.34094	1.505	7.64123	310°.35	321°.03	F. P., (K), '09, A. C.
"	"	3.47967	1.34094	1.505	7.64015	310°.27	320°.94	M. P., " " "
May 12, '09	"	{ 3.48164	1.34105	1.505	{ 7.63632	{ 309°.81	320°.44	{ F. P., Baker, '09, D. C.
"	"	{ 3.48177	1.34105	1.505	{ 7.63580	{ 309°.80	320°.43	{ " B. & A., '09, " "
"	"	{ 3.48164	1.34105	1.505	{ 7.63563	{ 309°.77	320°.40	{ " " " "
"	"	{ 3.48177	1.34105	1.505	{ 7.63563	{ 309°.76	320°.39	{ " " " "
Mar. 16, '09	1787F., W. B.	2.85456	1.11265	1.501	6.30820	310°.40	321°.05	" (K), " " "
Feb. 27, '09	1787A., P. T.	21.1732	4.4165	1.568	34.8576	309°.85	320°.97	" " " "
"	"	21.1732	4.4165	1.568	34.8559	309°.81	320°.93	M. P., " " "
Mar. 16, '09	1787G., W. B. Palladium.	1.87759	0.63158	2.890	3.77290	300°.09	320°.52	F. P., " " "

TABLE 5.
Freezing Point of Lead.

Date	Therm. No.	R ₀	FI	δ	R ₉₀	ρ_{90}	t	Remarks
Mar. 25, '09	1787F., W. B.	2.85455	1.11265	1.500	6.37505	316°.41	327°.59	F. P., Baker, Ag free, test.
"	"	2.85455	1.11265	1.500	6.37501	316°.40	327°.58	F. P., Baker, Ag free, test.
Apr. 22, '09	1787C., P. T.	3.48164	1.34105	1.505	7.72003	316°.05	327°.24	F. P., $\text{\textcircled{K}}$, '09.
"	"	3.48164	1.34105	1.505	7.72043	316°.08	327°.27	" " "
May 13, '09	"	{ 3.48164	{ 1.34105	{ 1.505	{ 7.72278	{ 316°.26	{ 327°.47	{ M. P., B. & A. test, Ag free, D. C.
"	"	{ 3.48177	{ 1.34105	{ 1.505	{ 7.72278	{ 316°.25	{ 327°.46	{ " " "
"	"	{ 3.48164	{ 1.34105	{ 1.505	{ 7.72278	{ 316°.26	{ 327°.47	{ F. P., B. & A. test, Ag free, D. C.
"	"	{ 3.48177	{ 1.34105	{ 1.505	{ 7.72278	{ 316°.25	{ 327°.46	{ " " "

TABLE 6.
Freezing Point of Zinc.

Date	Therm. No.	R ₀	F I	δ	R _{2a}	P _{2a}	t	Remarks
Nov. 19, '08	1787A., P. T.	21.3476	4.4057	1.571	38.9003	398°.32	419°.36	Fairly rapid freeze.
20, '08	"	21.3476	4.4057	1.571	38.8972	398°.25	419°.28	" "
Dec. 15, '08	"	21.3361	4.4071	1.571	38.8887	398°.28	419°.31	Slow freeze.
"	"	21.3361	4.4071	1.571	38.8880	398°.26	419°.29	" "
"	"	21.3361	4.4071	1.571	38.8880	398°.26	419°.29	" "
"	"	21.3361	4.4071	1.571	38.8886	398°.28	419°.31	" "
Dec. 5, '08	480 W. B.	2.62081	1.00503	1.551	6.62603	398°.52	419°.28	Rapid freeze.
"	"	2.62081	1.00503	1.551	6.62482	398°.40	(419°.16)	
Dec. 8, '08	479 W. B.	4.27342	1.64585	1.516	10.84077	399°.02	419°.32	
"	"	4.27340	1.64585	1.516	10.83975	398°.97	419°.27	Bottom of therm. 5 cm from bottom of crucible.
"	"	4.27338	1.64585	1.516	10.84036	399°.00	419°.30	
Dec. 9, '08	"	4.27323	1.64585	1.516	10.84046	399°.02	419°.32	
"	"	4.27323	1.64585	1.516	10.82075	397°.82	(417°.99)	Bottom of therm. 5 cm from bottom of crucible.
"	"	4.27323	1.64585	1.516	10.83951	398°.96	(419°.26)	3 cm from bottom.
Jan. 4, '09	1787C., P. T.	3.48779	1.34298	1.506	8.84889	399°.19	419°.36	All above dets. $\odot$ Zn in A. C.
"	"	3.48779	1.34298	1.506	8.84902	399°.20	419°.37	

Jan. 21, '09	"	3.47971	1.33989	1.506	8.82960	399 .28	419 .46	B. & A., C. P. Zn in D. C.
22, '09	1787A., P. T.	21.3396	4.4076	1.571	38.8982	398 .37	419 .41	B. & A., C. P. Zn in D. C.
"	"	21.3396	4.4076	1.571	38.8975	398 .36	419 .40	B. & A., C. P. Zn in D. C.
Jan. 23, '09	"	21.3396	4.4076	1.571	38.8964	398 .33	419 .37	Ⓚ Zn in D. C. In use since 1903 for testing couples.
"	1787C., P. T.	3.47971	1.33989	1.506	8.82896	399 .23	419 .40	
"	"	3.47971	1.33989	1.506	8.82896	399 .23	419 .40	
Jan. 25, '09	1787A., P. T.	21.3395	4.4076	1.571	38.8942	399 .28	419 .32	Ⓚ Zn in D. C. In use since 1907.
"	1787C., "	3.47964	1.33989	1.506	8.82841	399 .19	419 .36	
"	"	3.47964	1.33989	1.506	8.82843	399 .19	419 .36	
Jan. 26, '09	"	3.47970	1.33991	1.506	8.82713	399 .09	419 .25	E. & A., C. P. Zn in D. C. Mixture of 1903 and 1905 samples.
"	1787A., P. T.	21.3395	4.4076	1.571	38.8915	398 .22	419 .25	

TABLE 7.
Freezing Point of Antimony.

Date	Therm. No.	R _o	FI	δ	R _{ab}	P _{ab}	t	Remarks
Nov. 21, '08	1787A., P. T	21.3384	4.4067	1.571	46.8340	578°.56	631°.24	(K) 1908, in A. C.
Jan. 26, '09	"	21.3384	4.4073	1.572	46.8334	578.47	631.17	" " "
Jan. 28, '09	"	21.3360	4.4077	1.572	46.8367	578.55	631.26	(K) 1905, " D. C.
Jan. 29, '09	"	21.3348	4.4077	1.572	46.8384	578.61	631.34	(K) 1907, " "
Nov. 21, '08	1787C., P. T	3.4831	1.34163	1.506	11.2688	580.32	630.73	(K) 1908, in A. C.
"	"	3.4831	1.34163	1.506	11.2664	580.14	630.52	" " "
Jan. 27, '09	"	3.4797	1.33994	1.507	11.2567	580.40	630.87	" " "
Jan. 28, '09	"	3.4798	1.33994	1.507	11.2573	580.44	630.92	(K) 1905, " "
Jan. 29, '09	"	3.4798	1.33994	1.507	11.2566	580.39	630.86	(K) 1907, " "
Jan. 27, '09	479 W. B	4.2736	1.64580	1.508	13.8182	579.94	630.35	(K) 1908, " "
Feb. 1, '09	"	4.2735	1.64573	1.508	13.8175	579.93	630.34	" " "
Feb. 3, '09	"	4.2735	1.64573	1.512	13.8208	580.13	630.74	(K) 1907, " "
Feb. 1, '09	1787 G. W. B	1.8750	0.62861	2.890	5.2405	535.38	632.82	(K) 1908, " "
"	" P. T	1.8725	0.62761	2.890	5.2332	535.48	632.99	(K) 1908, " "
Feb. 3, '09	" W. B	1.8741	0.62817	2.890	5.2386	535.60	633.16	(K) 1907, " "
Apr. 16, '09	1787E., P. T	2.92415	0.50480	1.802	5.81135	571.95	632.68	(K) 1907, " D. C.
"	"	2.92382	0.50511	1.807	5.81178	571.66	632.53	No cover.
"	"	2.92415	0.50480	1.802	5.81178	572.03	632.78	" " "
"	"	2.92382	0.50511	1.807	5.81178	571.75	632.64	" " "

Melting Point of Antimony.

Nov. 21, '08	1787A., P. T	21.3384	4.4067	1.571	46.8261	578°.38	631°.02	Ⓚ 1908, in A. C.
Jan. 28, '09	"	21.3354	4.4077	1.572	46.8286	578.38	631.07	Ⓚ 1905, " D. C.
Jan. 29, '09	"	21.3347	4.4077	1.572	46.8293	578.41	631.10	Ⓚ 1907, " "
Nov. 21, '08	1787C., P. T	3.4831	1.34163	1.506	11.2691	580.34	630.75	Ⓚ 1908, " A. C.
"	"	3.4831	1.34163	1.506	11.2707	580.46	630.90	" " " "
Jan. 27, '09	"	3.4797	1.33994	1.507	11.2556	580.32	630.78	" " " "
Jan. 29, '09	"	3.4798	1.33997	1.507	11.2556	580.30	630.75	Ⓚ 1907, " D. C.
Feb. 3, '09	1787 G. W. B.	1.8741	0.62817	2.890	5.2379	535.49	633.00	" " " "
Feb. 1, '09	" P. T	1.8750	0.62861	2.890	5.2389	535.13	632.45	Ⓚ 1908, " A. C.

TABLE 8.
Freezing Point of Silver.

Date	No. of Therm.	R. _o	FI	β	R _{A₂}	P _{A₂}	t	t mean	Remarks
Mar. 11, '09	478 P. T	{ 5.16514 5.16618	2.00011 1.99957	1.507 1.503	{ 21.8904 21.8904	{ 836°.22 836°.39	960°.88 960°.65	{ 960°.76 960°.76	A. C.
"	" "	{ 5.16514 5.16618	2.00011 1.99957	1.507 1.503	{ 21.8898 21.8898	{ 836°.19 836°.36	960°.84 960°.60	{ 960°.72 960°.72	A. C.
June 11, '09	1787C., P. T	3.48174	1.34114	1.504	14.7009	836°.54	960°.98		"
"	" "	3.48174	1.34114	1.504	14.6995	836°.43	960°.83		"
Mar. 8, '09	1787 F. W. B	{ 2.85459 2.85456	1.11274 1.11265	1.503 1.501	{ 12.16260 12.16260	{ 836°.49 836°.56	960°.79 960°.66	{ 960°.72 960°.72	"
"	" P. T	{ 2.84890 2.84881	1.11077 1.11084	1.504 1.508	{ 12.14004 12.14004	{ 836°.46 836°.41	960°.86 961°.25	{ 961°.06 961°.06	"
"	" P. T	{ 2.84890 2.84881	1.11077 1.11084	1.504 1.508	{ 12.1401 12.1401	{ 836°.46 836°.42	960°.86 961°.26	{ 961°.06 961°.06	" M. P.
"	" P. T	{ 2.84890 2.84881	1.11077 1.11084	1.504 1.508	{ 12.13953 12.13953	{ 836°.41 836°.37	960°.79 961°.20	{ 961°.00 961°.00	"
"	" W. B	{ 2.85459 2.85456	1.11274 1.11265	1.503 1.501	{ 12.16214 12.16214	{ 836°.45 836°.52	960°.73 960°.61	{ 960°.67 960°.67	"
"	479 W. B	4.28823	1.64513	1.513	18.0387	835°.83	961°.02		" Broke after F. P.

Mar. 11, '09	1787A., P. T	{ 21.0862 21.0734 }	4.4203 4.4213	1.568 1.565	{ 57.9980 57.9982 }	{ 835.05 835.15 }	{ 966.31 966.10 }	{ 966.20 966.22 }	" Impure Platinum.
"	" "	{ 21.0862 21.0734 }	4.4203 4.4213	1.568 1.565	{ 57.9980 57.9982 }	{ 835.06 835.16 }	{ 966.32 966.11 }	{ 966.22 966.22 }	" "
Apr. 17, '09	1787E., P. T	{ 2.92382 2.92276 }	0.50511 0.50517	1.804 1.803	{ 7.07231 7.07200 }	{ 821.30 821.41 }	{ 975.30 975.35 }	{ 975.32 975.24 }	" "
"	" "	{ 2.92382 2.92276 }	0.50511 0.50517	1.804 1.803	{ 7.07231 7.07200 }	{ 821.24 821.36 }	{ 975.22 975.27 }	{ 975.24 975.27 }	" "
Mar. 15, '09	1787 G. W. B	{ 1.87869 1.87759 }	0.63124 0.63158	2.890 2.890	{ 6.52849 6.52861 }	{ 736.61 736.39 }	{ 992.74 992.27 }	{ 992.50 992.54 }	" Palladium.
"	" "	{ 1.87869 1.87759 }	0.63124 0.63158	2.890 2.890	{ 6.52849 6.52861 }	{ 736.63 736.41 }	{ 992.78 992.29 }	{ 992.54 992.29 }	" "
"	" P. T	{ 1.87498 1.87397 }	0.63093 0.63084	2.883 2.883	{ 6.52460 6.52460 }	{ 736.95 737.21 }	{ 992.14 992.68 }	{ 992.41 992.68 }	" "

TABLE 9.—Freezing Point of Copper.

Date	Therm. No.	R.	F. I.	δ	R _{Cu}	p _{Cu}	t	t mean	Remarks
Feb. 19, '09	1787C., P. T	{ 3.47967 3.47904	{ 1.34058 1.34076	{ 1.508 1.508	{ 15.8439 15.8488	{ 922°.30 922°.23	{ 1082°.77 1082°.69	{ 1082°.7 ₃ 1083°.2 ₆	Old Pot., D. C.
Feb. 19, '09	"	{ 3.47967 3.47904	{ 1.34058 1.34076	{ 1.508 1.508	{ 15.8488 15.8538	{ 922°.67 922°.59	{ 1083°.31 1083°.21	{ 1083°.2 ₆ 1083°.3 ₀	{ M. P. Old Pot., D. C., quite rapid.
Feb. 20, '09	"	{ 3.47904 3.47988	{ 1.34076 1.34094	{ 1.506 1.506	{ 15.8538 15.8558	{ 922°.97 922°.78	{ 1083°.43 1083°.17	{ 1083°.3 ₀ 1083°.5 ₀	K. elect., D. C.
"	"	{ 3.47904 3.47988	{ 1.34076 1.34094	{ 1.506 1.506	{ 15.8558 15.8560	{ 923°.11 922°.93	{ 1083°.63 1083°.38	{ 1083°.5 ₀ 1083°.5 ₂	{ M. P., K. elect., D. C., fast melt.
Mar. 1, '09	"	{ 3.47947 3.48041	{ 1.34094 1.34080	{ 1.505 1.508	{ 15.8560 15.8565	{ 922°.97 922°.93	{ 1083°.27 1083°.77	{ 1083°.5 ₂ 1083°.7 ₆	B. C. W., A. C.
Mar. 5, '09	"	{ 3.48041 3.48128	{ 1.34080 1.34086	{ 1.508 Not redet.	{ 15.8565 15.8570	{ 923°.04 922°.97	{ 1083°.85 1083°.68	{ 1083°.7 ₆ 1083°.8 ₁	B. C. W., A. C.
"	"	{ 3.48128 3.48128	{ 1.34086 1.34086	{ Not redet. Not redet.	{ 15.8570 15.8593	{ 922°.07 923°.14	{ 1083°.89 1083°.99	{ 1083°.8 ₁ 1083°.9 ₆	{ B. C. W., A. C. { E. A., & Drops, '09, A. C. with cover.
Mar. 6, '09	"	{ 3.48128 3.48156	{ 1.34086 1.34104	{ Not redet. 1.509	{ 15.8593 15.8591	{ 923°.14 923°.12	{ 1083°.99 1083°.97	{ 1083°.9 ₆ 1083°.9 ₄	{ E. A., & Drops, '09, A. C. with cover.
"	"	{ 3.48156 3.48156	{ 1.34104 1.34104	{ 1.509 1.508	{ 15.8591 23.6006	{ 922°.98 922°.14	{ 1083°.91 1081°.47	{ 1083°.9 ₄ 1081°.5 ₆	K. elect., D. C.
Feb. 24, '09	478 P. T	{ 5.16070 5.16248	{ 1.99969 2.00013	{ 1.501 1.505	{ 23.6006 23.6011	{ 922°.14 921°.85	{ 1081°.47 1081°.65	{ 1081°.5 ₆ 1081°.5 ₆	M. P., K. elect., D. C.
"	"	{ 5.16070 5.16248	{ 1.99969 2.00013	{ 1.501 1.505	{ 23.6011 23.6120	{ 922°.17 921°.87	{ 1081°.50 1081°.67	{ 1081°.5 ₆ 1082°.46	B. C. W., A. C.
Mar. 1, '09	"	{ 5.16248 5.16514	{ 2.00013 2.00011	{ 1.505 1.507	{ 23.6120 23.6120	{ 922°.42 922°.29	{ 1082°.46 1082°.58	{ 1082°.5 ₂ 1082°.5 ₂	B. C. W., A. C.

Mar. 1, '09	478 P. T	{ 5.16248 5.16514	2.00013 2.00011	1.505 1.507	{ 23.6021 921.92	{ 1081.74 1081.88	{ 1081.8 ₁ 1081.88	{ M. P. very rapid, A. C., B. C. W.
Feb. 24, '09	479 W. B	{ 4.27916 4.28715	1.64497 1.64500	1.505 1.514	{ 19.4283 920.94	{ 1080.33 1080.44	{ 1080.7 ₀ 1080.99	{ K. elect., D. C.
"	"	{ 4.27916 4.28715	1.64497 1.64500	1.505 1.514	{ 19.4343 921.30	{ 1080.85 1081.53	{ 1081.2 ₀ 1081.53	{ M. P., K. elect., D. C.
Mar. 2, '09	"	{ 4.28715 4.28823	1.64500 1.64513	1.514 1.513	{ 19.4484 921.66	{ 1082.76 1082.40	{ 1082.5 ₀ 1082.40	{ B. C. W., D. C.
Mar. 2, '09	1787E., W. B	{ 2.85344 2.85459	1.11239 1.11274	1.494 1.509	{ 13.1241 922.91	{ 1082.06 1083.81	{ 1082.9 ₀ 1083.81	{ B. C. W., D. C.
"	1787E., P. T	{ 2.84782 2.84890	1.11069 1.11077	1.504 1.504	{ 13.0929 ₀ 922.41	{ 1082.32 1082.08	{ 1082.2 ₀ 1082.08	{ B. C. W., D. C.
Feb. 20, '09	1787A., P. T	{ 21.2475 21.1734	4.4132 4.4165	1.575 1.568	{ 61.9037 921.24	{ 1091.77 1092.12	{ 1091.9 ₀ 1092.12	{ K. elect., D. C.
"	"	{ 21.2475 21.1734	4.4132 4.4165	1.575 1.568	{ 61.8990 921.90	{ 1091.28 1091.63	{ 1091.4 ₀ 1091.63	{ M. P., K. elect., D. C.
Mar. 1, '09	"	{ 21.1730 21.1418	4.4165 4.4179	1.568 1.570	{ 61.8642 921.35	{ 1090.83 1091.75	{ 1091.2 ₀ 1091.75	{ B. C. W., A. C.
Mar. 6, '09	"	{ 21.1418 21.0862	4.4179 4.4203	1.570 1.568	{ 61.8733 921.96	{ 1092.04 1092.85	{ 1092.4 ₀ 1092.85	{ E. & A., Drops, A. C.
"	"	{ 21.1418 21.0862	4.4179 4.4203	1.570 1.568	{ 61.8593 922.72	{ 1091.60 1092.40	{ 1092.0 ₀ 1092.40	{ E. & A., Drops, A. C.
Apr. 20, '09	1787E., P. T	{ 21.0862 2.92226	4.4203 0.50540	1.568 1.803	{ 7.49870 905.5	{ 1106 1106	{ 1106 1106	{ B. C. W., A. C.
Feb. 18, '09	{ Palladium 1787G., W. B	{ 1.86284 1.86284	.62696 .62696	2.878 2.878	{ 6.89936 803.32	{ 1151 1151	{ 1151 1151	{ Old Pot., D. C.
"	"	{ 1.86284 1.86802	.62696 .62890	2.878 2.884	{ 6.89978 803.4	{ 1151 1153	{ 1151 1153	{ M. P. Old Pot., D. C.
"	" P. T	{ 1.86802 1.86802	.62890 .62890	2.884 2.884	{ 6.91594 802.7	{ 1153 1153	{ 1153 1153	{ Old Pot., D. C.

TABLE 10.—Freezing Point of Silver-Copper Alloy.

Date	Therm. No.	R ₀	F. I.	δ	R _{Ag-Cu}	μ_{Ag-Cu}	t	t mean	Remarks
71 Ag-29 Cu.									
Mar. 26, '09	1787C., P. T	{ 3.48156	1.34104	1.509	{ 12.8645	{ 699.68	{ 779°.64	{ 779°.52	F. P.
"	"	{ 3.48153	1.34099	1.505	{ 12.8645	{ 699.70	{ 779.39		
		{ 3.48156	1.34104	1.509	{ 12.8646	{ 699.68	{ 779.64	{ 779.52	"
		{ 3.48153	1.34099	1.505	{ 12.8646	{ 699.71	{ 779.40		
Mar. 27, '09	1787F., P. T	{ 2.84881	1.11084	1.508	{ 10.6244	{ 699.97	{ 779.94	{ 779.60	"
		{ 2.84974	1.11121	1.504	{ 10.6244	{ 699.65	{ 779.26		
"	1787A., P. T	{ 21.0734	4.4213	1.565	{ 51.9268	{ 697.84	{ 781.10	{ 781.24	"
		{ 21.0620	4.4239	1.572	{ 51.9268	{ 697.68	{ 781.38		
"	"	{ 21.0734	4.4213	1.565	{ 51.9251	{ 697.80	{ 781.05	{ 781.18	"
		{ 21.0620	4.4239	1.572	{ 51.9251	{ 697.64	{ 781.32		
"	"	{ 21.0734	4.4213	1.565	{ 51.9286	{ 697.88	{ 781.15	{ 781.29	M. P. Fast melt.
		{ 21.0620	4.4239	1.572	{ 51.9286	{ 697.72	{ 781.43		
72 Ag-28 Cu (Eutectic).									
Mar. 31, '09	1787C., P. T	{ 3.48153	1.34099	1.505	{ 12.8623	{ 699.54	{ 779°.18	{ 779°.16	F. P.
		{ 3.48164	1.34105	1.505	{ 12.8623	{ 699.50	{ 779.13		
Mar. 29, '09	1787F., W. B	{ 2.85542	1.11295	1.504	{ 10.6425	{ 699.68	{ 779.28	{ 779.20	"
		{ 2.85567	1.11312	1.504	{ 10.6425	{ 699.55	{ 779.13		
"	" P. T	{ 2.84974	1.11121	1.508	{ 10.6249	{ 699.70	{ 779.60	{ 779.46	"
		{ 2.84987	1.11119	1.504	{ 10.6249	{ 699.70	{ 779.32		

Mar. 29, '09	1787F., W. B.	{ 2.85542 2.85567 }	1.11295	1.504	{ 10.6434 699.76 }	{ 779.40 699.63 }	{ 779.32 779.24 }	"
"	" P. T.	{ 2.84974 2.84987 }	1.11121 1.11119	1.508 1.504	{ 10.6249 699.70 }	{ 779.60 779.32 }	{ 779.46 779.32 }	M. P.
Mar. 31, '09	478, P. T.	{ 5.16618 5.16566 }	1.99957 2.00000	1.503 1.505	{ 19.15225 699.33 }	{ 778.93 778.91 }	{ 778.92 778.91 }	F. P.
"	"	{ 5.16618 5.16566 }	1.99957 2.00000	1.503 1.505	{ 19.15424 699.43 }	{ 779.06 779.04 }	{ 779.05 779.04 }	"

73 Ag-27 Cu.								
Apr. 1, '09	1787C., P. T.	{ 3.48153 3.48164 }	1.34099 1.34105	1.505 1.505	{ 12.8620 699.48 }	{ 779.15 779.10 }	{ 779.12 779.10 }	F. P.
"	"	{ 3.48153 3.48164 }	1.34099 1.34105	1.505 1.505	{ 12.8651 699.71 }	{ 779.45 779.40 }	{ 779.42 779.40 }	M. P. Fast melt.
"	1787A., P. T.	{ 21.0620 21.0617 }	4.4239 4.4243	1.572 1.570	{ 51.9222 697.52 }	{ 781.25 781.03 }	{ 781.14 781.03 }	F. P.
Apr. 19, '09	1787E., P. T.	{ 2.92276 2.92226 }	0.50517 .50540	1.803 1.803	{ 6.39844 687.81 }	{ 784.96 784.67 }	{ 784.82 784.67 }	M. P.
"	"	{ 2.92276 2.92226 }	.50517 .50540	1.803 1.803	{ 6.39792 687.92 }	{ 784.82 784.53 }	{ 784.68 784.53 }	F. P.
"	"	{ 2.92276 2.92226 }	.50517 .50540	1.803 1.803	{ 6.39741 687.60 }	{ 784.69 784.39 }	{ 784.54 784.39 }	"
Mar. 27, '09	1787G., P. T.	{ 1.87397 1.87312 }	.63084 .63101	2.883 2.878	{ 5.85843 631.58 }	{ 787.85 787.33 }	{ 787.59 787.33 }	"

Note.—All determinations in Table 10 in Acheson graphite crucible.

TABLE 13.—Temperature Corrections for Platinum of Different δ .

[Thermometer calibrated by Callendar method, ice, steam, and S. B. P.]

Temp. °C	Correction in °C for values of δ given below							
	1.525	1.550	1.575	1.600	1.650	1.700	1.800	1.900
200°	+0.02	+0.05	+ 0.08	+ 0.10	+ 0.14	+ 0.16	+ 0.20	+ 0.21
300°	+ .02	+ .05	+ .08	+ .11	+ .19	+ .27	+ .45	+ .55
400°	.00	.00	+ .01	+ .03	+ .08	+ .14	+ .29	+ .37
500°	— .02	— .05	— .09	— .11	— .18	— .24	— .39	— .57
600°	— .09	— .18	— .30	— .40	— .62	— .88	— 1.42	— 1.96
700°	— .33	— .70	— 1.03	— 1.32	— 1.78	— 2.2	— 2.9	— 3.5
800°	— .90	— 1.65	— 2.24	— 2.7	— 3.6	— 4.4	— 5.8	— 7.1
900°	— 1.90	— 3.1	— 4.0	— 4.9	— 6.5	— 8.1	— 10.8	— 13.5
1000°	— 3.3	— 5.2	— 6.8	— 8.2	— 10.7	— 13.1	— 17.1	— 20.8
1100°	— 5.5	— 8.1	— 10.3	— 12.2	— 15.7	— 18.7	— 24.3	— 29.1

TABLE 14.—Precision with Thermometer 1787C.

Metal	Freezing Point	Number of Dets.	Range
Cd, Baker & Adamson.....	320° .39	2	0° .01
Pb, Baker & Adamson.....	327 .47	2	.01
Zn, "Kahlbaum".....	419 .36	4	.01
Sb, "Kahlbaum," 1908.....	630 .71	3	.35
Sb, "Kahlbaum," M. P.....	630 .81	3	.15
71 Ag-29 Cu.....	779 .52	2	.00
Ag, Mint.....	960 .90	2	.10
Cu, B. C. W.....	1083 .30	3	.29
Cu, E. & A. drops.....	1083 .95	2	.02
Cu, Kahlb. Electrolytic.....	1083 .40	2	.10

Precision with Thermometer 1787A.

Sn, Baker & Adamson.....	231° .89	2	0° .10
Cd, "Kahlbaum".....	321 .05	2	.04
Zn, Baker & Adamson.....	419 .40	2	.01
Sb, "Kahlbaum".....	630 .50	2	.07
71 Ag-29 Cu.....	779 .40	2	.04
Ag, Mint.....	961 .00	2	.02
Cu, E. & A. drops.....	1083 .3	2	.40

TABLE 16.—Changes of Zero, F. I. and c of 1787C. ($\delta = 1.503$.)

Date	Obs.	R.	ΔR_0 in ° C	F. I.	Per cent change in F. I.	c	Per cent change in c	History of Thermometer 1787C
Nov. 21, 1908.		3.4831		1.34163		0.00385183		After Sb. 2 times (previously annealed)
Jan. 4, 1909.		3.48779		1.34298		.00385052	0	" Zn 2 "
21.....	1	3.47971	0	1.33989	0		+.001 ₅	" " 1 " (mica frame repaired)
23.....	2	71	.000	89	.000		+.001 ₅	" " 1 "
25.....	3	64	-.005	89	.000		+.003 ₆	" " 2 "
26.....	4	70	-.001	91	+.001 ₅		+.003 ₄	" " 1 "
27.....	5	70	-.001	94	+.003 ₄		+.005 ₄	" Sb 1 "
28.....	6	80	+.007	94	+.003 ₄		+.002 ₆	" " 1 "
29.....	7	80	+.007	94	+.003 ₄		+.002 ₆	" " 1 "
Feb. 6.....	8	84	+.010					" Sn 1 "
8.....	9	68	-.002	1.34058	+.051		+.054	" Annealing 2 hrs. at 1100° +
20.....	10	04	-.050	76	+.065		+.086	" Cu 1 time
24.....	11	88	+.013	94	+.078		+.074	" " 2 "
Mar. 1.....	12	47	-.018					" Cd 2 "
2.....	13	3.48041	+.052	80	+.068		+.049	" Cu 1 "
6.....	14	3.48128	+.117	86	+.072		+.029	" " 2 "
12.....	15	56	+.138	104	+.085		+.034	" " 2 "
21.....	16	53	+.136	999	+.082		+.031	" Ag-Cu. 2 "
26.....	17	64	+.144	105	+.087		+.033	" " 2 "
Apr. 1.....	18	64	+.144	105	+.087		+.033	" " 1 "
June 7.....	19	74	+.152	114	+.093		+.036	" Cd, 3 times; Pb, 2 times.

TABLE 17.
Changes of Zero, F. I. and c of 1787F. ($\delta=1.503$.)

Date	Obs.	R.	ΔR , in ° C.	F. I.	Per cent change in F. I.	c	Per cent change in c	History of Thermometer 1787 F
ON WHEATSTONE BRIDGE.								
1909.								
Feb. 10.....	1	2.85355	0	1.11228	0	0.00389787	0	After annealing 2 hrs. at 1100°
25.....	2	344	-.010	249	+.019	876	+.023	" S 2 times
Mar. 3.....	3	459	+.093	274	+.041	807	+.005	" Cu 2 "
9.....	4	456	+.091	265	+.033	780	-.002	" Ag 4 "
29.....	5	542	+.168	295	+.060	904	+.030	" Cd 1; Sn 4, Pb 2, Ag-Cu 1
30.....	6	567	+.190	312	+.076	793	+.0016	" Ag-Cu 2 times
ON POTENTIOMETER.								
Feb. 9.....	1'	2.84782	0	1.11069	0	0.00390014	0	After annealing 2 hrs. at 1100°
25.....	2'	776	-.005					" S 2 times
Mar. 3.....	3'	890	+.097	077	+.007	.00389894	-.031	" Cu 2 "
16.....	4'	881	+.089	087	+.016	942	-.018	" Ag 4 "
29.....	5'	974	+.173	121	+.047	933	-.021	" Cd 1, Sn 4, Pb 2, Ag-Cu 1
30.....	6'	987	+.184	119	+.045	909	-.027	" Ag-Cu 2 times

TABLE 18.
Changes of Zero, F. I. and c. of 478. ($\partial = 1.507$.)

Date	Obs.	R.	$\Delta R.$ in ° C	F. I.	Per cent change in F. I.	c	Per cent change in c	History of Thermometer, 478
1909.								
Feb. 5.....	1	5.15665	0	1.99745	0	0.00387354	0	After annealing 2 hrs. at 1100°.
23.....	2	5.16070	+ .202	969	+ .112	464	+ .028	" 5 mins. in Cu. Coil straightened.
25.....	3	248	.292	2.00013	.134	436	+ .022	" Cu 1 time.
Mar. 6.....	4	514	.424	11	.133	232	— .032	" " 1 "
12.....	5	618	.476	1.99957	.106	050	— .078	" Ag 2 times.
31.....	6	566	.450	2.00000	.128	172	— .047	" Ag-Cu 2 "

TABLE 19.
Changes of Zero, F. I. and c. of 479. ($\delta = 1.512$.)

Date	Obs.	R.	ΔR , in ° C	F. I.	Per cent change in F. I.	c	Per cent change in c	History of Thermometer, 479
Dec. 8, 1908....	1	4.27342	0	1.64585	0	0.00385136	0	Old thermometer reannealed. After Zn 4 times.
9.....	2	23	-.012	85	.000	154	+.005	
Jan. 26, 1909....	3	83	+.025	85	.000	100	-.009	
Feb. 1.....	4	45	+.002					
1.....	5	53	+.007	72	-.008	096	-.010	" " 3 "
2.....	6	04	-.023	55	-.018	101	-.009	" Sb 2 "
3.....	7	50	+.005					" " 1 time.
4.....	8	56	+.008					" " 1 "
8.....	9	4.26905	-.271	04	-.049			" " 1 "
17.....	10	812	-.321	46	-.024	341	+.053	" 2 hrs. at 1100°+.
23.....	11	4.27916	+.348	497	-.053	525	+.101	" S.
25.....	12	4.28715	+.832	500	-.052	4414	-.187	" Cu 2 times.
Mar. 3.....	13	823	+.898	13	-.044	3705	-.372	" Cu 1 time.
						638	-.389	" Ag 1 "

TABLE 20.—Changes of Zero, F. I. and c. of 1787A. ($\beta=1.570$.)

Date	Obs.	R. ₀	ΔR_0 in ° C	F. I.	Per cent change in F. I.	c	Per cent change in c	History of Thermometer, 1787 A
Nov. 19, 1908....	1	21.3476	0	4.4067	0	0.00206426	0	First annealed at 1100°.
21.....	2	384	-.21	67	.000	515	+.043	After Zn 1 time and Sb 1 time.
Dec. 15.....	3	61	-.26	71	+.009	556	.063	" Zn 4 times.
Jan. 22, 1909....	4	96	-.18	76	.020	546	.058	" Zn 3 "
25.....	5	95	-.18	76	.020	547	.059	" Zn 2 "
26.....	6	84	-.21	73	.014	543	.057	" Sb 1 time.
27.....	7	72	-.24	77	.023	573	.071	" Sb 1 "
28.....	8	48	-.29	77	.023	597	.083	" Sb 2 times.
29.....	9	47	-.29	78	.025	602	.085	" Sb 2 "
Feb. 6.....	10	41	-.31					" Sn 1, and S.
6.....	11	21.2475	-2.27	4.4132	.148	.00207704	.619	" Annealing 2 hrs. at 1100°+.
24.....	12	.1734	-3.94	.41648	.222	208586	1.046	" Cu 1 time.
Mar. 1.....	13	.1730	-3.95					" Cd 2 times.
2.....	14	.1418	-4.66	.41786	.254	208963	1.229	" Cu 1 time.
9.....	15	.0827	-5.99	.42539	.424	209906	1.686	" Cu 2 times.
11.....	16	.0862	-5.91	03	.309	872	1.669	" Several hrs. in S.
12.....	17	.0734	-6.20	128	.331	804	1.636	" Ag 2 times.
27.....	18	.0620	-6.46	39	.390	210042	1.752	" Ag-Cu 3 times.
Apr. 1.....	19	.0617	-6.46	43	.399	065	1.763	" " 1 time.

TABLE 21.
Changes of Zero, F. I. and c. of 1787E. ($\delta = 1.803$.)

Date	Obs.	R.	$\Delta R.$ in ° C	F. I.	Per cent change in F. I.	c	Per cent change in c	History of Thermometer, 1787 E
1909.								
Apr. 15.....	1	2.92415	0	0.50480	0	0.00172630	0	After annealing 1.5 hrs. at 1100°.
17.....	2	382	-.006	511	+.06	757	+.08	" Sb and Ag 2 times each.
19.....	3	226	-.039	540	+.19	948	+.19	" Ag-Cu 2 times.

TABLE 22.—Changes of Zero, F. I. and c of 1787G. ($\theta = 2.890$.)

Date	Obs.	R.	$\Delta R.$ in ° C	F. I.	Per cent change in F. I.	c	Per cent change in c	History of Thermometer, 1787 G
ON WHEATSTONE BRIDGE.								
Feb. 1, 1909.....	1	1.87254	0	0.627606	0	0.00335163	0	After annealing at 1100°.
2.....	2	398	+ .23	.62817	+ .089	206	+ .013	" Sb 2 times.
3.....	3	400	+ .23					" Sb 1 time.
3.....	4	414	+ .26					" 2 hrs. at 1100°-1150°.
8.....	5	6268	-1.57	.62696	-.104	6590	+ .426	Thermometer repaired.
17.....	6	284	-1.55					Annealed 5 min. in gas furnace.
Mar. 15.....	7	1.87869	0	.63124	0	0.00335098	+ .241	After Ag 2 times.
16.....	8	759	- .174	158	+ .054	6378	+ .362	" Cd and Sn 1 time each.
22.....	9	756	- .179					
ON POTENTIOMETER.								
Jan. 27.....	1'	1.87345	0	0.62809	0	0.00335258	0	After annealing.
Feb. 1.....	2'	504	+ .25	861	+ .083	252	-.005	" Sb 2 times.
5.....	3'	6802	- .86	890	+ .129	6667	+ .420	" Annealing 2 hrs. at 1100°-1150°.
Mar. 12.....	4'	1.87498	0	0.63098	0	.00336499	+ .370	Thermometer repaired.
16.....	5'	397	- .17	084	-.022	633	+ .400	Annealed 5 min. in gas furnace.
30.....	6'	312	- .30	101	+ .005	876	+ .482	After Ag 2 times. " Ag-Cu 3 "

TABLE 23.
Changes of Zero, F. I., and c. of Nos. 9837 and 9838.

Heat Treatment.	9837. $\delta=1.54_6$			9838. $\delta=1.55_7$		
	R _o	F I	c	R _o	F I	c
5 hrs. at 1100°	0.11275	0.04324	0.0038350	0.10951	0.04192	0.0038280
5 " " 1200°	78	32	411	48	200	363
1.5 " " 1250°	83	26	340	50	190	265
1 " " 1300°	88	50	536	56	220	518
2 " " 1250°	307	25	251	54	184	196

TABLE 24.
Changes in Thermometers at High Temperatures.

	Thermometers in copper. ¹				
	1787 C	1787 F	1787 A	478	479
$p_{t_1}-p_{t_2}$ { Mean	+0°.12	+0°.28	-0°.78	+0°.21	+0°.38
Range	+ .03 to + .19	+ .17 to + .39	-.41 to -1.00	+ .12 to + .30	+ .14 to + .50
t_1-t_2 { Mean	+0.07	-0.75	-0.65	-.15	-.33
Range	+ .26 to -.50	+ .24 to -1.75 ⁴	-.35 to -.92	-.12 to -.18	+ .36 to -.68
					See 9.
					"
					1787 G ⁴

Thermometers in silver.¹

pt_1-pt_{10} { Mean.....	0° 00	-0° 10	-0° 17	See 8.	+0° 06
Range.....	+0° 05' to -0° 07'	-0° 10 to -0° 10	-0° 17 to -0° 17	"	+0° 22 to -0° 26
t_1-t_{11} { Mean.....	-0° 19	+0° 21	+0° 24	"	+0° 14
Range.....	+0° 13' to -0° 41'	+0° 21 to +0° 21	+0° 23 to +0° 24	"	+0° 49 to -0° 54

Thermometers in Ag-Cu Alloy.²

pt_1-pt_{10} { Mean.....	+0° 01	+0° 12	+0° 12	Not used.	Not used.
Range.....	+0° 04 to -0° 03	+0° 32 to 0° 00	+0° 16 to +0° 06	"	"
t_1-t_{11} { Mean.....	+0° 13	+0° 04	-0° 16	"	"
Range.....	+0° 25 to +0° 05	+0° 28 to -0° 68	+0° 18 to -0° 28	"	"

¹ Thermometers had been previously annealed in electric furnace at 1050-1100° for several hours.² Thermometers had been previously used in copper.³ Thermometers had been previously used in copper and silver.⁴ Palladium thermometer.⁵ This large difference is due to a too low value of δ (1.494) observed before and a too high value of δ (1.509) observed after the two determinations in molten copper.⁶ Thermometer 1787 F used as potential terminal thermometer.⁷ Thermometer 1787 F used as Wheatstone Bridge three-lead compensated thermometer.⁸ Thermometer 479 used once. No calibration after, as thermometer found broken on cooling.⁹ Thermometer 1787 G broke; could not be recalibrated after 3 determinations in copper of Feb. 18, 1909.¹⁰ pt_1 =platinum temperature computed from calibration before determination of freezing point.¹¹ pt_2 =platinum temperature computed from calibration after determination of freezing point.¹² t_1 =true temperature computed from calibration before determination of freezing point.¹³ t_2 =true temperature computed from calibration after determination of freezing point.

TABLE 26.

Comparison of Platinum and Palladium Thermometers at Freezing Point Temperatures.

[Callendar's Formula. Calibration Data: Ice, Steam, Sulphur (444°.70).]

Platinum $\begin{cases} \delta = 1.505 \\ c = 0.00385 \text{ to } 389 \end{cases}$ Palladium $\begin{cases} \delta = 2.890 \\ c = 0.00336 \end{cases}$

Metal.	Temperature by Platinum Therm.	Temperature by Palladium Therm.	Δ Pd-Pt	Range, ° C.	Number of—	
					Obs.	Samples
Sn.	231°.90	231°.62	+ 0°.28		1	1
Cd.	321 .01	320 .52	— .49		1	1
Pb.	327 .58	326 .90	— .68		1	1
Sb.	630 .71	632 .99	+ 2 .28	0.34	3	2 K
Ag.-Cu.	779 .20	787 .59	+ 8 .39		1	1
Ag.	960 .88	992 .49	+31 .61	.13	3	1
Cu.	1,083 .0	1,152	+69	2	2	1

TABLE 27.

Corrections to the Palladium Thermometer.

[Palladium Temperatures computed by Callendar's Formula. Calibration Data, Ice, Steam, Sulphur (444°.70).]

Temperatures centigrade (Platinum Therm.)	Temperatures centigrade (Palladium Therm.)	Correction to temperatures with Palladium Therm.
200°	199°.80	+0°.20
300	299 .56	+ .44
400	399 .70	+ .30
500	500 .40	— .40
600	601 .5	— 1.5
700	704 .7	— 4.7
800	810 .5	—10.5
900	921 .0	—21.0
1,000	1,040 .8	—40.8
1,100	(1,175)	(—75)

XI. BIBLIOGRAPHY.

The following is not intended to be a complete bibliography of electrical resistance thermometry, but it is hoped that all the more important original contributions to the subject are included, as well as a sufficient number of technical papers treating of the applications of the resistance thermometer. Many duplicate references, mainly from the engineering press and reviewing journals, are omitted. Most of the English papers and many of the others will be found abstracted in the *London Electrician*.

Some references are given which deal with those properties of platinum which are of interest in connection with a study of its resistance at high temperatures.

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DAYLIGHT EFFICIENCY OF ARTIFICIAL ILLUMINANTS.

By Herbert E. Ives.

Among students of illumination it is generally agreed that the best light for human use is daylight. Therefore the nearer an artificial light approaches daylight in color and distribution, the higher it should rank. There has been developed, however, no simple means of measuring an artificial light source in such a manner that a numerical value can be given to its approach in quality to daylight. In other words, there has been no basis for estimating the "daylight efficiency."

In so far as color is concerned, progress has recently been made along two lines which should make possible and profitable an attempt to supply this deficiency. On the side of physical research extended series of spectrophotometric measurements of daylight have been made, notably by E. L. Nichols. On the technical side there have been developed new illuminants of high efficiency, and of color considerably nearer daylight than those previously known.

The present investigation was prompted by the question whether it might not be possible, with some of the more efficient illuminants, to imitate daylight by the use of properly colored globes or screens, without lowering the resultant efficiency below that of some of the older sources. Knowing the spectral relation of a source to daylight, it is a simple problem to determine the spectral absorption necessary to reduce the light to daylight color. Knowing the intensity equivalent of the different colors of the spectrum, it is possible to determine the loss in intensity due to the use of the colored absorbing screen.

The problem up to this point is an entirely practical one. After having worked out some numerical values in the way described

below, it occurred to the writer that the white light obtained in this manner might be used as a convenient measure of the quality of an artificial light. The ratio $\frac{\text{intensity of available white light}}{\text{total intensity of source}}$ offers itself as a measure of the illuminant's efficiency with regard to color.

In the following paper are given the results of applying this idea to a number of the more prominent artificial illuminants. Two methods are developed; the first method, suggested above, by a discussion of absorbing screens, applicable to sources with a continuous spectrum, and giving the actual available nonselective white light; the second method, by means of the primary color sensations, applicable to both selective and nonselective sources, and giving the amount of white sensation, in general only partly or not at all available.

FIRST METHOD.

White Light Efficiency as Obtained by the Discussion of Absorbing Screens.—If we assume the intensity of the daylight spectrum to be unity at all points, the spectral intensity of any other light, as determined by the spectrophotometer, may be plotted on such a scale that the lowest point of the resultant curve is also unity. The colored screen which in combination with the source will transmit light of daylight color is the one whose percentage transmission at each point of the spectrum is the reciprocal of the ordinate of the illuminant's spectral curve. The first step in the artificial production of daylight by this method is therefore the determination of the spectral relationship of artificial sources to daylight. For this purpose is available the work of E. L. Nichols on average daylight in terms of the acetylene flame. Average daylight as determined by Nichols has the following intensities throughout the spectrum as compared with acetylene, equality being assumed, according to the usual practice, at $.59\mu$.¹

$\frac{.725}{.450}$	$\frac{.620}{.865}$	$\frac{.590}{1.00}$	$\frac{.530}{1.34}$	$\frac{.460}{2.07}$	$\frac{.420}{2.59}$	$\frac{.390}{2.63}$ μ
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Other light sources may be compared in the spectrophotometer with acetylene, and hence with daylight.

¹Transactions of the Illuminating Engineering Society, May, 1908.

Measurements of several illuminants, gas, glow lamps, etc., are given in the paper referred to. The electric glow lamps are not, however, specified according to watts per candle, the only exact mode of specification where relative color values are to be considered, and the Welsbach mantles are not classified according to cerium content, which determines the color. The writer, therefore, thought it worth while, for the purposes of this paper, to make spectrophotometer measurements, against acetylene, of carbon, tungsten, and other glow lamps at the exact watts per candle at which each is rated for commercial comparison, and of a Welsbach mantle whose cerium content is known.

The carbon and tungsten lamps were measured for watts per mean horizontal candle, and were 4 and 3.1 for the carbon, 1.25 for the tungsten. The mean spherical values were obtained by the use of reduction factors. The metallized and tungsten lamps were measured directly for mean spherical candlepower on an integrating photometer. The Welsbach mantle contained $\frac{3}{4}$ per cent cerium, being the mantle recommended for residential illumination by the Welsbach Company.

The measurements were made with a Lummer-Brodhun spectrophotometer; a tungsten lamp at constant voltage was used as a comparison source, and the substitution method was used. Special care was taken to exclude the effect of scattered light at the ends of the spectrum, by the use of colored glasses between the eye and eye slit. Measurements were carried to $.67\mu$ in the red and to $.42\mu$ in the blue.

The results of these measurements are given in Tables I and II; Table I in terms of acetylene, Table II in terms of average daylight, assuming the daylight acetylene ratios determined by Nichols.

It will be observed that if we plot as curves the data of Table II, the lowest point of all the curves is in the blue. In order, therefore, to determine the appropriate absorbing screens, these curves must be drawn with a point in the extreme blue made equal to unity. The choice of this point is largely arbitrary. The shorter the wave length chosen the smaller the percentage transmission of the absorbing screen for the other portions of the spectrum. A point should therefore be chosen of as long a wave length as possible

without too great a loss of blue light. In the present paper the "equality point" has been taken as $.42\mu$. This is one of the points measured in Nichols's daylight investigation, and is the

TABLE I.

Spectrophotometer Measurements of Various Artificial Lights Against Acetylene.

	Source.	.42	.45	.50	.55	.59	.62	.65	.67 μ
1	Glow lamp, carbon, 4.85 watts per mean spherical cp.....	.445	.550	.700	.875	1.00	1.10	1.19	1.26
2	Glow lamp, carbon, 3.75 watts per mean spherical cp.....	.495	.595	.735	.885	1.00	1.09	1.17	1.23
3	Glow lamp, metallized, 3.1 watts per mean spherical cp.....	.560	.655	.780	.905	1.00	1.065	1.13	1.17
4	Glow lamp, tantalum, 2.6 watts per mean spherical cp.....	.595	.680	.810	.920	1.00	1.055	1.12	1.15
5	Glow lamp, tungsten, 1.56 watts per mean spherical cp.....	.785	.815	.875	.945	1.00	1.03	1.09	1.11
6	Acetylene.....	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
7	Welsbach mantle, $\frac{1}{2}$ per cent cerium.....	1.28	1.40	1.42	1.22	1.00	.86	.71	.64
		.4358 μ		.5460 μ		.5780 μ			
8	Mercury arc (Nichols).....	16.81		3.75		1.00			

TABLE II.

Light Sources Compared with Average Daylight.

	Source.	.42	.45	.50	.55	.59	.62	.65	.67 μ
1	Glow lamp, carbon, 4.85 watts per mean spherical cp.....	.171	.250	.438	.718	1.00	1.27	1.65	1.97
2	Glow lamp, carbon, 3.75 watts per mean spherical cp.....	.191	.265	.460	.730	1.00	1.25	1.63	1.93
3	Glow lamp, metallized, 3.1 watts per mean spherical cp.....	.216	.295	.485	.750	1.00	1.23	1.56	1.83
4	Glow lamp, tantalum, 2.6 watts per mean spherical cp.....	.231	.310	.504	.759	1.00	1.22	1.54	1.80
5	Glow lamp, tungsten, 1.56 watts per mean spherical cp.....	.303	.375	.546	.780	1.00	1.19	1.47	1.70
6	Acetylene.....	.386	.455	.622	.825	1.00	1.16	1.38	1.56
7	Welsbach mantle, $\frac{1}{2}$ per cent cerium.....	.495	.640	.884	1.00	1.00	1.00	.98	.98
		.4358 μ		.5460 μ		.5780 μ			
8	Mercury arc (Nichols).....	7.85		3.00		1.00			

point beyond which the average daylight curve takes more nearly the slope of the usual artificial sources. When screened from

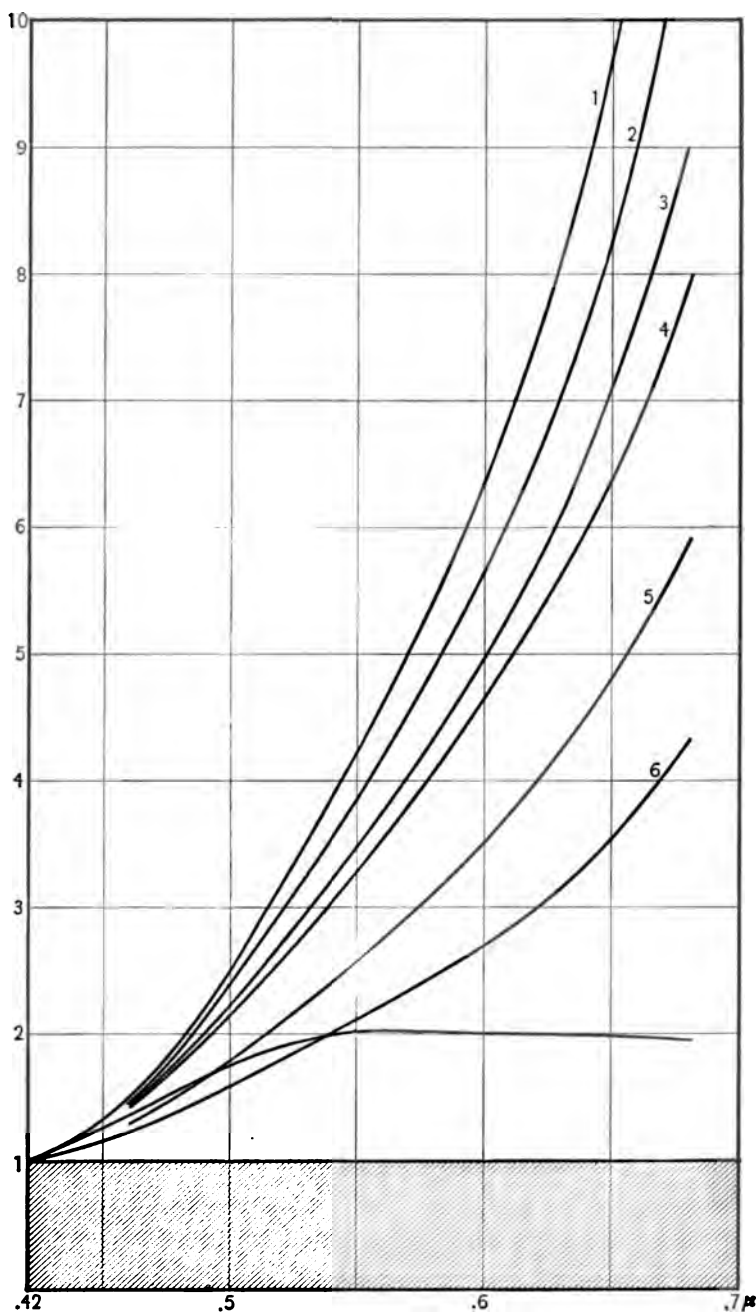


Fig.1.—Curves Showing Amount of Spectra of Various Light Sources which must be Absorbed to Leave White Light.

this point, the deficiency in blue sensation of any of the ordinary sources is less than 2 per cent, a quantity small enough to be considered negligible. A different choice of equality point will of course lead to numerically different results, a matter to be treated again in the discussion of the second method.

In Fig. 1 the data of Table II are plotted on this basis. The area of each curve above the shaded area represents the portion of the illuminant's spectrum which must be removed by absorption to reduce it to identity with the daylight spectrum. The colored glass to effect this is one whose transmission is the reciprocal of the ordinate of the curve in question at each point.

Having obtained the properly colored glass to effect the absorption, we now wish to know the intensity value of the resultant artificial daylight. This we shall compare with the intensity of the unscreened source. We do this by considering the intensity value of the different spectral colors, and the resultant loss in total intensity when the various portions of the spectrum are reduced in the proportion indicated above. The method is indicated in Fig. 2. The shaded area represents the spectrum of white light expressed in terms of luminosity; it is, in other words, the sensibility curve of the normal eye referred to white light². The areas inclosed by the outlined curves and the axis of abscissæ are the resultant of increasing the ordinates of the sensibility curve in the proportion indicated by the curves of Fig. 1. The areas of the outlined curves are therefore proportional to the intensity of the unscreened sources, the shaded area to the intensity of the white light remaining after screening. The ratio of the shaded to the unshaded area gives the value

$$\frac{\text{Intensity of White Light obtained by Screening from } .42\mu}{\text{Intensity of unscreened source.}}$$

This quantity, for which the values for several sources are given in Table III, column 6, is an indication of the loss in efficiency which may be expected upon producing artificial daylight by the use of colored absorbing screens. The figures also lend themselves to giving a value to a light source considered from the standpoint

² Abney, Phil. Trans. Roy. Soc. **193**, p. 286; 1900. This curve is for high intensities where the Purkinje effect is not noticeable.

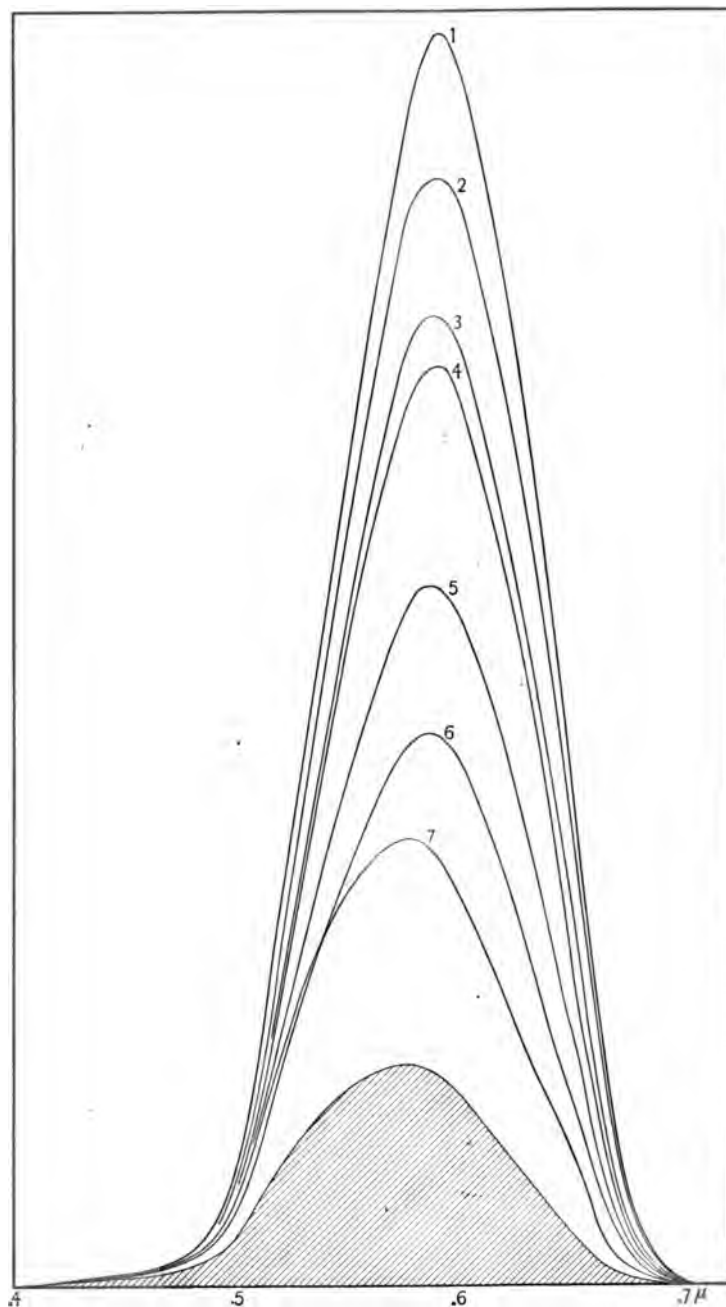


Fig. 2.—Excess of Colored Light in Various Light Sources, Represented in Terms of Luminosity.

of its approach in color to daylight. They may in fact be called the "daylight" or "white light efficiencies," as is here done.

TABLE III.

Color Sensation Values, White Sensation and White Light Efficiencies.

	Source.	1 Sensations.			2	3	4	5	6
		Red.	Green.	Blue.	Lumi- nosity.	Hue.	White sensa- tion.	White sensa- tion effici- ency.	White light effici- ency.
1	Glow lamp, carbon, 4.85 watts per mean spherical cp	46.5	39.7	13.8	44.1	.588 _μ	13.7	31.0	19.3
2	Glow lamp, carbon, 3.75 watts per mean spherical cp	45.9	39.4	14.7	43.5	.588	14.5	33.4	21.2
3	Glow lamp, metallized, 3.1 watts per mean spherical cp	44.8	39.7	15.5	42.9	.585	15.3	35.8	24.6
4	Glow lamp, tantalum, 2.6 watts per mean spherical cp	44.2	39.8	16.0	42.6	.585	15.9	37.3	26.3
5	Glow lamp, tungsten, 1.56 watts per mean spherical cp	42.8	39.1	18.1	41.5	.584	18.0	43.5	33.2
6	Acetylene	40.8	38.3	20.9	40.1	.583	20.7	51.7	42.0
7	Welsbach mantle, $\frac{1}{2}$ per cent cerium	37.4	37.8	24.8	37.4	.577	24.5	65.5	50.5
8	Mercury arc (Nichols)	22.4	23.6	54.0	23.0	.480	18.0	78.8	0
9	Average daylight	33.3	33.3	33.3	33.3		33.3	100	100

TABLE IV.

Source.	1	2	3
	Watts per mean spherical candle.	White sensation W. P. S. C.	White light W. P. S. C.
Carbon glow lamp	4.85	15.6	25.1
Carbon glow lamp	3.75	11.25	17.7
Metallized glow lamp	3.1	8.65	12.6
Tantalum	2.5	6.7	9.5
Tungsten glow lamp	1.56	3.6	4.7

In Table IV, column 3, the results are applied to the electric glow lamps in combination with their energy consumption, giving what may be called "White light watts per candle." It will be observed that the question which prompted the investigation is answered. The "White light watts per candle" of the tungsten lamp is less than the watts per candle of the "4-watt" carbon lamp, indicating that artificial daylight should be no more expensive than was electric light not long ago. This of course assumes the possibility of obtaining colored glass of exactly the absorption required.

This method of obtaining efficiency in terms of daylight, namely, by the use of colored absorbing screens, is of more practical than scientific interest. As has been pointed out, the numerical values will vary with the wave length chosen as the equality point. A method giving a relationship to daylight independent of choice of coordinates would in some ways be preferable. Moreover, the above method gives the value zero to selective sources, where by selective source is meant one of the type of the mercury arc whose spectrum consists of bright lines with no continuous background. Selective sources, however, actually excite all three primary color sensations, giving some sensation of white. The attempt to find a method applicable to such sources has led to a second method of finding color efficiency, namely, the method of color sensations, which will now be described.

SECOND METHOD.

White Light Efficiency Obtained by Means of the Primary Color Sensations.—Any color can be matched by a mixture of white light and one ray of the spectrum.³ The intensity of this white compared with the intensity of the colored source gives a white light efficiency which is obtainable with all types of selective and non-selective sources. We may imagine this accomplished experimentally in the following manner: Given a standard white light, let the intensity of the source in question be measured against the standard, say with a flicker photometer or by some other method independent of color difference. By means of a spectroscopic device, superpose on the white light colored light from a narrow region of the spectrum. Alter the intensity of the white and the wave-length and intensity of the spectral light until a visual match is obtained for both color and intensity. The ratio of the intensity of the white used in this match to the intensity of the source is a white light efficiency independent both of the character of the source and of any arbitrarily chosen "equality point."

³ Abney, "Color Measurement and Mixture," Chapter XIII. The only exceptions are furnished by colors of purple hue, in which the green sensation is smaller than either the red or the blue. These must be measured in terms of their complementaries. Purple light sources are not met with in ordinary experience, and need not be considered.

The experimental difficulties in the way of carrying out the procedure as outlined are great, chiefly because of the difficulty of producing a white light standard. The whole process may, however, be carried out indirectly by means of the color sensation curves of the normal eye, as determined by König. These curves, as modified by Exner, are reproduced in Fig. 3.⁴ They show the

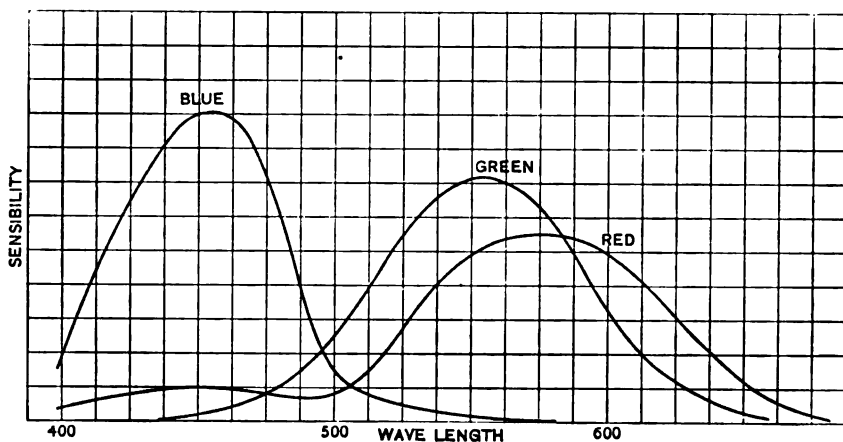


Fig. 3.

amounts of each of the primary sensations—red, green, and blue—which are excited by the various portions of the spectrum. They are drawn, according to the usual convention, with the three areas equal for white light. White then being equal excitation of the three sensations, all other colors can be expressed in terms of the relative proportions of red, green, and blue sensation. Knowing the sensation values of a color, it is then possible to find from the curves a wave-length which, with white added, has the same proportionate parts of red, green, and blue as the color. In this manner the dominant hue of the color is determined, and the amount of white light to be added to the dominant hue. It is then necessary to have the intensity values of the three sensations. Applying these to the color, and then to the white added

⁴F. Exner, Sitzungsberichte der Akademie der Wissenschaften, Wien, 8, p. 857; 1902.

to the dominant hue, the desired ratio of intensities may be obtained.

In order to apply the method it is necessary to know the sensation values of a color and the luminosity values of the three sensations. To obtain the sensation values of a color two procedures are open. If spectrophotometric measurements are available, the sensation values may be obtained by multiplying the sensation curve ordinates at each wave-length by the spectral intensity of the color as compared with the standard. A second method is to measure the color by means of a color-mixing instrument, the sensation value of whose colors are known. In the latter method we again meet the difficulty that no standard white light is available. In the present investigation the sensation values were obtained by applying the spectrophotometric values obtained above to the sensation curves. As to the luminosity values of the sensations Abney has determined that for sunlight the values are ⁵—

Red.	Green.	Blue.	
65.73	33.83	.44	(= 100.00)

These values are used in the present paper on the assumption that they will be substantially the same for average daylight.

To illustrate the method of procedure, the work for acetylene is given in full. Applying the values of Table II to the sensation curves, the sensation values (that is, the areas of the new curves, as measured with a planimeter) found for acetylene were—

Red.	Green.	Blue.
94	88	48

Giving to each sensation its appropriate luminosity value, the luminosity of the acetylene color (in arbitrary units) was 92.1. By trial it was found that if 47.6 parts of white were subtracted from the color the remaining sensation values—i. e., 46.4, 40.4, .4—were in the proportion of the sensations at .583 μ . The luminosity or intensity ratio is therefore $\frac{47.6}{92.1} = 51.7$, and the dominant hue is .583 μ .⁶

⁵ Philosophical Transactions, loc. cit.

⁶ The process of finding the dominant hue is much simplified by the use of Maxwell's color triangle.

This white light efficiency is obtainable for any type of source—for instance, the mercury arc, the values for which, calculated from spectrophotometer figures given by Nichols,⁷ will be found in Table III along with values for the sources already evaluated according to Method I.

Comparison of the white-light efficiencies obtained in this way with those from the first method shows that in every case the second efficiency is higher than the first. It is clear why this should be so, for if we call the intensity of the white light obtained by the first method Iw_1 , we have that the total color Ic (expressing it in intensity units) is equal to Iw_1 and the absorbed wavelengths represented by the curves of Fig. 1 above the shaded area. This absorbed color is itself, not being a single spectrum ray, reducible to white light, say Iw_2 and a single wave-length. We may express this fact in the following equation:

$$\begin{aligned} Ic &= Iw_1 + Iw_2 + I_\lambda \\ &= IW + I_\lambda. \end{aligned}$$

Iw represents the total white sensation; Iw_1 represents the available nonselective white light; its value may be zero, as in the mercury arc.

The two kinds of white light efficiency or color efficiency may be conveniently designated as "White Sensation Efficiency" and "White Light Efficiency," the first being always larger than the second.

Before discussing the interpretation and utility of these quantities, it is of interest to note several points in connection with the characteristics of white light and white sensation as here distinguished. If we write the above equation $Ic - I_\lambda = IW$ we express the condition for securing the largest possible amount of white sensation, namely, by subtracting light of one wave length. The nearest approach to this practicable is to subtract a wide band in the neighborhood of λ . Inspection of the sensation curves shows that with a yellowish source, such as a glow lamp, absorption of a band near $.59\mu$ would, while reducing the red and green sensations, have slight effect on the blue, while the uniform absorption of Method I wastes considerable of that already deficient

⁷ Loc. cit.

sensation. It would therefore be possible, by selective screening, to secure nearly the whole of the white sensation value existing in the source. Since the white sensation value is the only white light value possessed by selective sources, and since the white sensation value may be nearly obtained by selective screening of continuous spectrum light sources, we might use the term "selective white" to designate it, although "white sensation" is preferred by the writer. It is of interest to note that with the sources considered—acetylene, etc.—if the equality point for screening is taken as about $.455\mu$ the resultant (slightly yellowish) white is of very nearly the white sensation value. For some practical purposes, therefore, the white sensation values might be assumed to be the available white, in the case of continuous sources.

It is possible to apply the first method to the color sensation curves, for the ordinates of the curves giving the sensation value for a source may be reduced from the same equality point in the proportions demanded by the curves of Fig. 1, their areas measured before and after reduction, and the luminosity values of the sensations applied as above. This was done as a check on the sensibility curve values, and the results agreed within the errors of measurement.

In Table III are given the sensation values of the various sources, their dominant hues, white sensation and white light efficiencies. The sensation values are so reduced that their sum equals 100 in each case. Column 4 (percentage white sensation) then expresses a purely hue or color efficiency, namely, $\frac{\text{white sensation}}{\text{total sensation}}$, where the sensations are assumed of equal value in white light. This ratio, not having a close connection with the problem of screening or with the possible experimental procedure outlined under Method II, has not been considered at length.

In Table IV are given the values of "white light" and "white sensation" watts per candle for the electric glow lamps.

DISCUSSION OF RESULTS.

Two methods have been developed for obtaining a measure of the approach of a light in color to daylight. Two values have

been obtained for the quantity expressing the color efficiency. Their characteristics and interpretation are as follows:

1st. **The White Light Efficiency.**—Obtained by consideration of the white light available by the use of absorbing screens. The white light obtained is nonselective white light, spectrally identical with Nichols's "average daylight." The term "White Light Efficiency" is applied to the ratio

$$\frac{\text{Intensity of Nonselective White Light Available,}}{\text{Intensity of Source.}}$$

Its numerical value is dependent on the point of the spectrum at which absorption starts, and is zero for a bright line source. It is of practical importance as an indication of the loss in efficiency to be expected by producing "daylight" by the use of colored globes on artificial sources. As the white light obtained is identical with daylight, the "White Light Efficiency" is a good indication of a source's excellence for use where color discrimination is necessary. The larger the proportion of available white light the nearer will colors appear in their daylight values.

2d. **The White Sensation Efficiency.**—Obtained by consideration of the amount of white which with a spectrum color would match the light. The term is applied to the ratio

$$\frac{\text{Intensity of White Sensation.}}{\text{Intensity of Source.}}$$

The method is independent of any arbitrary screening point, and is applicable to all types of sources. The white sensation obtained indicates the color of the light or of a white surface illuminated by it. It gives no indication of the suitability of a source for color discrimination. The white sensation is larger than the white light given by Method I, and is as a rule only partly or not at all available as nonselective white light.

The question arises: Which of these efficiencies is it preferable to use? From a purely scientific standpoint the white sensation efficiency is preferable, because of its universal application and independence of arbitrary screening point. The white light efficiency, however, furnishes information of importance in the consideration of artificial illuminants, namely, on the merits of a

source for the purpose of illuminating colored objects. A method to include both values would be superior to either alone, and an attempt to do this graphically is given in Fig. 4, in which some of the results of Table III are embodied. The rectangles indicate the white sensation efficiencies of the several sources, the shaded portions of the rectangles the white light efficiencies. The vertical lines and scales give the dominant hues. Inspection of the figures will show that they demonstrate in a rather striking way the relative qualities of the sources from the standpoint of their color. A light of the type of II, for instance, is one of bluish hue, of about 80 per cent white sensation, but a source not suited for color discrimination; a light of the type of III is one of yellowish

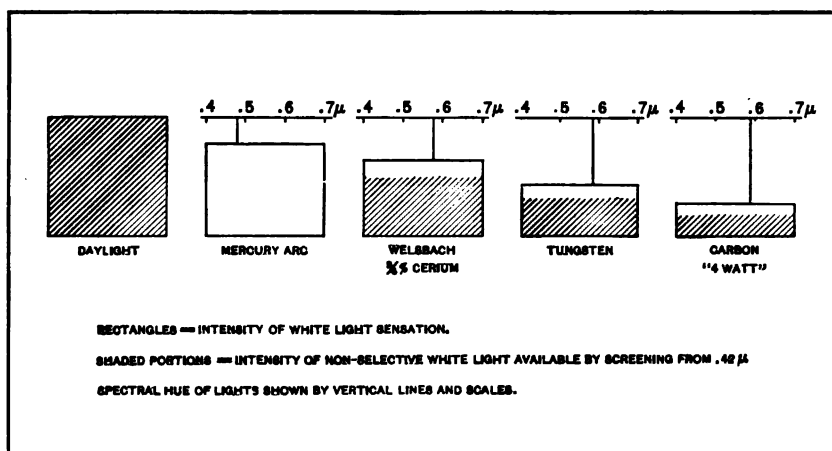


Fig. 4.

hue, with a large amount of available white, and therefore suited to color discrimination. In daylight the white sensation and white light areas are identical.

It is not claimed that this use of the white light efficiency in combination with the white sensation efficiency will lead to rigidly correct or strictly comparable results in all cases conceivable. For instance, a source consisting of a large number of bright lines evenly distributed through the spectrum would have zero white light efficiency, although it would be a pretty good light for color discrimination. This difficulty could, of course, be avoided by agreement on the number and closeness of lines which may be

considered equivalent to a continuous spectrum for purposes of illumination. On the other hand, a source containing a large amount of available white may have such a spectral distribution as to be of less pleasant effect to the eye than one of smaller white light efficiency. This is, of course, rather a question of æsthetics or physiology than of physics. If we confine ourselves to the illuminants at present in use, this combination of the two efficiencies presents the color characteristics of interest to the illuminating engineer more accurately than any classification known to the writer, and is, therefore, thought worth presenting.

The values obtained in this paper are dependent on the correctness of several sets of data: On the measurements of daylight by E. L. Nichols; on the measurements of artificial sources by the writer; on the sensibility curve of the eye as given by Abney; on the color sensation curves of König and Exner; on the luminosity values of the sensations as given by Abney. Future measurements or redeterminations may change the numerical values. The principles involved, however, remain the same, and the methods developed give, it is believed, a new and useful basis for the color comparison of artificial illuminants.

WASHINGTON, May, 1909.

COUPLED CIRCUITS IN WHICH THE SECONDARY HAS DISTRIBUTED INDUCTANCE AND CAPACITY

By Louis Cohen.

In a previous communication ¹ I have discussed the problem of coupled circuits on the assumption that the inductance and capacity of the two circuits are localized; this is equivalent to a system of two degrees of freedom, and such a system oscillates with two distinct frequencies, which were completely determined. It was also shown that such a system has two damping factors. The assumption, however, that the inductance and capacity of the secondary as well as the primary are localized does not give the conditions which correspond to those of wireless telegraphy. In a wireless telegraph system it is the antenna, which has a distributed inductance and capacity, which is the secondary, and this will represent a system of an infinite number of degrees of freedom, and consequently the system will oscillate with an infinite number of frequencies. In this communication it is proposed to investigate such a system, in order to see whether the two fundamental frequencies, as obtained in the previous paper, will be in any way modified by the assumption that the secondary has distributed inductance and capacity, and also whether the other frequencies besides the two fundamental ones will in any way influence the results as previously obtained.

It was shown in my previous paper that the resistance does not influence materially the frequency constants, and since in this problem we are concerned with frequency constants only we will, in order to simplify the mathematical treatment of the problem, neglect the resistance entirely.

¹ This Bulletin, 5, p. 511; 1909.

Let us denote by L_1 , C_1 the inductance and capacity of the primary circuit, L and C the inductance and capacity per unit length of the antenna, and L_2 and C_2 the total inductance and total capacity of the antenna. If l is the length of the antenna, then $Ll = L_2$ and $Cl = C_2$. We shall also designate by L_0 the lumped inductance at the end of the antenna, which is inductively connected with the primary, and by M the mutual inductance.

If, now, V_1 , I_1 and V_2 , I_2 are the potentials and currents of the primary and secondary, respectively, at any instant of time, then we shall have for the primary circuit the following equations:

$$\left. \begin{aligned} L_1 \frac{dI_1}{dt} + V_1 + M \frac{dI_2}{dt} &= 0 \\ C_1 \frac{dV_1}{dt} &= I_1 \end{aligned} \right\} \quad (1)$$

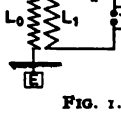


FIG. 1.

From which we obtain,

$$L_1 C_1 \frac{d^2 I_1}{dt^2} + I_1 + M C_1 \frac{d^2 I_2}{dt^2} = 0 \quad (2)$$

For the antenna we have the following relations:

$$\left. \begin{aligned} L \frac{dI_2}{dt} &= -\frac{dV_2}{ds} \\ C \frac{dV_2}{dt} &= -\frac{dI_2}{ds} \end{aligned} \right\} \quad (3)$$

From these two equations we derive the equation of propagation, which is as follows:

$$LC \frac{d^2 I_2}{dt^2} = \frac{d^2 I_2}{ds^2} \quad (4)$$

The solution of equation (4) is

$$I_2 = \{A \cos \mu s + B \sin \mu s\} e^{\lambda t} \quad (5)$$

where

$$\lambda^2 = -\frac{\mu^2}{LC} \quad (6)$$

-1
 -0
 -0
 -0
 -1
 -1
 -1
 -1
 -1
 -2
 -2
 -2
 -2
 -2
 -3
 -3
 -3
 -4
 -4
 -4
 -4
 -4

Eliminating $\frac{B}{A}$ from (11) and (12) we obtain the following:

$$-\frac{\mu}{\lambda^2 C} \cot \mu l = \frac{L_0 L_1 C_1 \lambda^2 + L_0 - M^2 C_1 \lambda^2}{1 + L_1 C_1 \lambda^2}$$

Replacing λ^2 by its value as given by equation (6) we get:

$$\begin{aligned} \cot \mu l &= \frac{\mu}{L} \frac{L_0 - L_0 \frac{L_1 C_1}{L C} \mu^2 + M^2 \frac{C_1}{L C} \mu^2}{1 - \frac{L_1 C_1}{L C} \mu^2} \\ &= \mu \frac{L_0}{L_1} \frac{1 - \frac{L_1 C_1}{L_2 C_2} \mu^2 l^2 + \frac{M^2 C_1}{L_0 L_2 C_2} \mu^2 l^2}{1 - \frac{L_1 C_1}{L_2 C_2} \mu^2 l^2} \end{aligned} \quad (13)$$

Let us assume that $L_1 C_1 = L_2 C_2$. This will be close to the condition of resonance if L_0 is small, and let us also put for brevity $\mu l = x$, then we get:

$$\cot x = \frac{L_0}{L_1} x \frac{1 - x^2 + \frac{M^2}{L_0 L_1} x^2}{1 - x^2} \quad (14)$$

If we designate $\frac{L_0}{L_1}$ by k_1 and $\frac{M^2}{L_1 L_2}$ by k_2 , equation (14) can be written in the following form:

$$\cot x = k_1 x \left\{ \frac{1 - x^2 + \frac{k_2}{k_1} x^2}{1 - x^2} \right\} \quad (15)$$

This being a transcendental equation it has an infinite number of roots. If we know the numerical values of k_1 and k_2 , we can determine the roots graphically by plotting the two curves

$$y_1 = \cot x \text{ and } y_2 = k_1 x \left\{ \frac{1 - x^2 + \frac{k_2}{k_1} x^2}{1 - x^2} \right\}$$

The points of intersection of these curves will give the roots of the equation. In plotting these curves the coefficient of coupling is assumed to be one-tenth, and therefore k_1 , which is the square of the coefficient of coupling, is 0.01. For k_1 two different values, 0.25 and 0.1, have been taken. The full-line curve in the graph corresponds to the value $k_1 = 0.25$ and the dotted curve corresponds to the value $k_1 = 0.1$.

In examining the graphs we see that for $k_1 = 0.25$ the roots have the values,

$$58^\circ, 68^\circ, 216^\circ, 380^\circ, 556^\circ$$

For $k_1 = 0.1$ the roots have the values,

$$58^\circ, 82^\circ, 290^\circ, 420^\circ, 590^\circ$$

From equation (6) we see that the frequencies are proportional to μ that is to the numbers given above. The first two roots represent the two fundamental frequencies, and the other roots represent the harmonics. From these values we see that in one case the frequency of the first harmonic is about three and a half times the fundamental, and in the other case the frequency of the first harmonic is about four times the frequency of the fundamental. In either case, however, if the system is arranged to be in resonance with the fundamental frequency, the upper harmonics will not have any influence in the working of the system.

It remains yet to see whether the ratio of the two fundamental frequencies obtained here are in agreement with the results that are obtained on the assumption that the inductance and capacity are localized in both circuits. It can be easily shown that on the assumption that the inductance and capacity are localized in both circuits, the frequency constants are given by the following expression.²

$$\lambda^2 = \frac{L_1 C_1 + L_2 C_2 \pm \sqrt{4M^2 C_1 C_2 + (L_1 C_1 - L_2 C_2)^2}}{2(L_1 C_1 L_2 C_2 - M^2 C_1 C_2)} \quad (16)$$

and

$$\frac{\lambda_1^2}{\lambda_2^2} = \frac{L_1 C_1 + L_2 C_2 - \sqrt{4M^2 C_1 C_2 + (L_1 C_1 - L_2 C_2)^2}}{L_1 C_1 + L_2 C_2 + \sqrt{4M^2 C_1 C_2 + (L_1 C_1 - L_2 C_2)^2}} \quad (17)$$

² See Fleming Principles of Wave Telegraphy, p. 210.

where L_1 and C_1 are the total inductance and capacity of the secondary. In the notation of this paper we have to replace L_1 by $L_1 + L_0$ and then make $L_1 C_1 = L_1 C_1$. We then get:

$$\frac{\lambda_1^2}{\lambda_2^2} = \frac{L_1 C_1 + L_2 C_2 + L_0 C_2 - \sqrt{4M^2 C_1 C_2 + L_0^2 C_2^2}}{L_1 C_1 + L_2 C_2 + L_0 C_2 + \sqrt{4M^2 C_1 C_2 + L_0^2 C_2^2}} \quad (18)$$

$$\frac{\lambda_1^2}{\lambda_2^2} = \frac{L_1 C_1 \left\{ 2 + \frac{L_0}{L_2} - \sqrt{4 \frac{M^2}{L_1 L_2} + \frac{L_0^2}{L_2^2}} \right\}}{L_1 C_1 \left\{ 2 + \frac{L_0}{L_2} + \sqrt{4 \frac{M^2}{L_1 L_2} + \frac{L_0^2}{L_2^2}} \right\}} = \frac{2 + k_1 - \sqrt{4k_2 + k_1^2}}{2 + k_1 + \sqrt{4k_2 + k_1^2}} \quad (19)$$

For the case of $k_1 = 0.25$ and using for k_2 the above value 0.01, we get:

$$\frac{\lambda_1^2}{\lambda_2^2} = \frac{2.25 - \sqrt{.04 + 0.0625}}{2.25 + \sqrt{.04 + 0.0625}} = 0.751$$

and

$$\frac{\lambda_1}{\lambda_2} = 0.86$$

The ratio of these two values as obtained from the curves are

$$\frac{\lambda_1}{\lambda_2} = \frac{58}{68} = 0.85$$

For the case $k_1 = 0.1$ we get:

$$\frac{\lambda_1}{\lambda_2} = 0.9$$

and the ratio of these two values as taken from the curves are

$$\frac{\lambda_1}{\lambda_2} = \frac{58}{80} = 0.72$$

From these results it would seem that the greater the ratio of the localized inductance, which is in series with the antenna, is to the inductance of the antenna, the more closely will the ratios of the two frequencies as given by (17) approach the actual conditions which take place in a wireless station.

It is self evident, of course, that by substituting the values of the first two roots, as determined graphically, in equation (6), the values of the frequency constants thus obtained will be approximately the same as those calculated by equation (16).

Now from equation (6) we have,

$$i\lambda = -\frac{\mu l}{\sqrt{L_2 C_2}}$$

The first two roots which give the values of μl are 58° and 68° or 1.012 and 1.186 radians.

Therefore, we get

$$\lambda_1 = \frac{1.012}{\sqrt{L_2 C_2}} \text{ and } \lambda_2 = \frac{1.186}{\sqrt{L_2 C_2}}$$

From equation (16) we get:

$$\begin{aligned} \lambda^2 &= \frac{2L_2 C_2 + L_0 C_2 \pm \sqrt{4M^2 C_1 C_2 + L_0^2 C_2^2}}{2\{L_2^2 C_1^2 + L_0 C_1 L_2 C_2 - M^2 C_1 C_2\}} \\ &= \frac{2 + \frac{L_0}{L_2} \pm \sqrt{4\frac{M^2}{L_1 L_2} + \frac{L_0^2}{L_2^2}}}{2L_2 C_2 \left\{1 + \frac{L_0}{L_2} - \frac{M^2}{L_1 L_2}\right\}} = \frac{2 + k_1 \pm \sqrt{4k_2 + k_1^2}}{2L_2 C_2 \{1 + k_1 - k_2\}} \end{aligned}$$

Using the corresponding values of $k_1 = 0.25$ and $k_2 = 0.01$ we get:

$$\lambda_1 = \frac{0.88}{\sqrt{L_2 C_2}} \text{ and } \lambda_2 = \frac{1.02}{\sqrt{L_2 C_2}} \quad (20)$$

It is seen from (19) and (20) that the values of the frequencies as computed by the two different methods differ by thirteen and eighteen per cent respectively. Increasing the ratio of $\frac{L_0}{L_2}$ will bring the values of the frequencies as computed by the two methods more closely together.

WASHINGTON, July 29, 1909.

EFFECT OF PHASE OF HARMONICS UPON ACOUSTIC QUALITY.

By M. G. Lloyd and P. G. Agnew.

INTRODUCTION.

Ever since the celebrated controversy between Helmholtz and König concerning the influence of the phase of harmonics, this question has remained a point of dispute between the supporters of these two great experimenters. Each of them had conducted experiments which apparently supported his case, König finding that the quality was affected by phase displacement of a harmonic, while Helmholtz found no effect.

The method used by Helmholtz¹ to produce a change in phase was elegant, but open to objection. He shaded the aperture of the resonator used to reenforce the sound. This changes the amplitude as well as the phase of the vibration, and necessitated a compensating correction by altering the distance between fork and resonator. It is thus evident that the change from one condition to another could not be made with much rapidity. König's² experiments were made with the wave siren, which has since been shown to give small quantities of extraneous harmonics.³ His results are therefore of doubtful significance.

The experiments to be described made use of electrical currents whose wave form could be controlled, a telephone receiver being used to transform these into sound waves.

APPARATUS.

Definite composition of the electrical wave was made possible by the use of a special set of generators at the Bureau of Standards,

¹ H. L. F. Helmholtz, *Sensations of Tone*, 3d ed., p. 124.

² R. König, *Quelques Experiences d'acoustique*, p. 222.

³ R. König, *Wied. Ann.* **57**, p. 339; 1896. C. Stumpf, *Wied. Ann.* **57**, p. 677; 1896.

known as the Harmonic Set. This set consists of two driving motors and eight three-phase generators, which at normal speed give frequencies of 60, 180, 300, 420, 540, 660, 780, and 900 cycles per second. They are assembled in two sections, the generators with the three lower frequencies being mounted on the shaft of one motor, while the others are mounted with the second motor. The two sections can be coupled rigidly together when desired. Each generator has rotating poles and stationary unslotted armature, and the position of the armature can be shifted by means of a worm gearing and handwheel. By connecting the generators in series, and properly exciting the fields and setting the armatures, any odd harmonic up to the fifteenth can be superposed upon the fundamental frequency in any amplitude and phase relation, thus permitting the synthesis of an endless variety of wave forms. The individual generators give waves which are approximately sinusoidal, although containing a few per cent of harmonics.

Another generating set which has been used consists of two two-phase generators of 60 and 180 cycles, respectively, connected to a driving motor. These machines give nearly pure sine waves. A third generator set furnishing three-phase and 60 cycles was also used.

Several telephone receivers were used, but most of the experiments were carried out with a Siemens & Halske instrument of 10.5 ohms resistance. The telephone was sometimes used with a high resistance in series; at other times it was placed in parallel with an adjustable carbon resistance of low value, and the whole put in series with an incandescent lamp. Variation of the carbon resistance altered the current through the telephone. In a few of the experiments separate telephones were connected to the generators, and the sound waves combined.

EXPERIMENTS.

At first the simple method of armature shift was used. Several combinations gave changes in the note sounded. But the method was neither convenient nor very sensitive, several observers and precautions to avoid suggestion being deemed necessary to firmly establish the change as taking place. The changes could be detected only with particular percentages of harmonic.

To shift the phase more quickly, a phase-shifting transformer was tried, but was found unsuitable, as slight changes in voltage and wave form accompanied the rotation of the secondary.

Finally, use was made of a second motor-generator set. By taking the fundamental from one and the harmonics from the other, and then running the machines just out of synchronism, we have one wave train slowly drifting relatively to the other, thus producing a slowly and periodically recurring change of phase.

The effect is much the same as in the arrangement used by Lord Kelvin,⁴ who took two tuning forks, one of which was an exact harmonic of the other, and changed its period very slightly by adding a small weight to it, thus giving a continuous cyclic change of phase.

TABLE I.

Cycles	Changes per lamp cycle
60	$\left\{ \begin{array}{l} 3 \\ 15 \end{array} \right.$
300	15
420	21
540	9
660	33
780	39
900	15

This method was found to be very much more sensitive than either of the others. In fact, with the armatures connected *single phase*, every combination of harmonics tried showed distinct cyclic changes in the note sounded. By controlling the speed of one of the generators the change in phase may be made rapid or slow, the rate being observed by means of a synchronizing lamp connected in series with the 60-cycle generator of each set. Relative intensities of the electromotive forces of the generators connected to the telephone could be adjusted by resistance in the field circuits. This method proved very convenient, the control being easy and flexible.

⁴ W. Thomson, Proc. Roy. Soc. Edinburgh, 9, p. 602; 1878.

Table I shows the results of using each frequency of the harmonic set with the 180-cycle two-phase generator. Changes in quality were evident in every case, and the number of cyclic changes occurring during a complete cycle of the synchronizing lamp were counted. This is the number of changes taking place while one 60-cycle generator gains a cycle upon the other. The significance of these numbers is seen when we consider the impurities present in the waves from the two generators. Thus, with 300 cycles, the possible overtones have frequencies of 900, 1500, 2100, etc. The frequency of 900 cycles is common to both, and with shifting phase these components will at times oppose, at other times reenforce, each other, thus changing the *amplitude* of this component of the combination. As the wave of this frequency shifts in phase by a complete cycle for one-fifteenth of a cycle of the 60-cycle generator, a shift of an entire cycle of the latter corresponds to a shift of fifteen entire cycles of the higher frequency, and consequently fifteen cycles of amplitude change. In each of the other cases it will be found similarly that the number of changes counted corresponds to some common harmonic of the two frequencies used. In one case (60 cycles) two distinct and recurring changes could be distinguished, each one depending upon a separate frequency, common to the two generators, namely, 180 and 900 cycles. No doubt other frequencies are common to the two generators in every case, but are not prominent enough to be heard.

It is evident that the changes in quality observed are simply beats due to the interference of frequencies common to the two sources. For an experiment to be decisive it is necessary that the two sources should have no harmonic common to them. Lindig,⁵ in 1902, found similarly that changes in quality always resulted when the two sources furnished the same harmonic. He used a telephone in conjunction with a telephone siren, and found no change in quality except when this condition was fulfilled.

Alternators giving rigorously sinusoidal waves are not obtainable. Yet it is well known that in three-phase generators the 3rd, 9th, 15th, 21st, etc., harmonics are eliminated automatically. It is to be noticed that with two machines so connected, if

⁵ F. Lindig, *Ann. d. Phys.*, **10**, p. 242; 1903.

only frequencies having ratios of 3 to 1 and 9 to 1 are used, then there will be no *common impurity*, even though they are not necessarily sinusoidal. Again, in most generators the even-numbered harmonics are entirely lacking, so that if generators with frequencies in the ratio of 2 to 1 can be used there will again be no common impurity. The first condition was attained by using the two sections of the harmonic set separately, or by using one of them with a second 3-phase generator. The second arrangement was carried out by running one generator at half speed, which also allowed other combinations to be used, which should theoretically give no common harmonics.

It will be seen that we were thus enabled to test under the three distinct conditions, in which the phase of an overtone was constantly changing in respect to the fundamental, the overtone and the fundamental containing no common impurity, viz, an odd harmonic, an even harmonic, and an overtone which was not harmonic—that is, an overtone having a pitch ratio of $3/2$, $4/3$, or the like.

When the harmonic set was run in separate parts, an oscillograph was used as a synchronizing apparatus.

It should be said that a direct synthesis of sound waves by this method does not seem to be possible without introducing distortion.

When a separate telephone was excited from each source and the two placed inside the body of a mandolin to act as a sounding board and combine the sounds, very distinct changes were observed, often of more than one period. They could not be made to disappear by any arrangement of intensities; nor even to materially decrease. Evidently these changes of tone were chiefly due to the harmonics introduced by the sounding board, just as we should expect.

Accordingly, in the final experiments the emf. waves were directly combined and applied to the telephone. This method assumes that the diaphragm gives out a sound wave having the same form as the electrical wave impressed upon the telephone. To test this assumption, the telephone was excited from a star-connected generator, and the sound analyzed by a set of Helmholtz resonators,⁶ in order to determine whether the sound wave

⁶These were very kindly loaned to us from the University of Pennsylvania by Prof. A. W. Goodspeed, to whom we are greatly indebted.

given out by the telephone contained harmonics not present in the electrical wave.

The generator was run at such a speed that one of the resonators reenforced the fundamental, and the sound was then examined by means of resonators whose natural frequencies were two, three, four, and five times as great. Four different telephones were thus examined, the results being shown in Table II. The fifth harmonic was no doubt present in the electrical wave, whereas the second, third, and fourth may reasonably be assumed to be absent. It is to be noticed that the third was always present in the sound wave, whereas the second and fourth were sometimes absent or too weak to be heard. They are less prominent in the shunt than in the series connection, and are never strong in the Siemens & Halske and Bell receivers.

TABLE II.

Experiments with Resonators.

Harmonic set, star-connected, and run slow, so that the 300-cycle generator gave about 256 cycles. With small current in the telephone, such as to require pressing against ear, no harmonics discernible.

Phone connection	Receiver	Harmonics present
Shunt.....	S. & H.....	{ 3, 4, 5 with heavy current. 3, 5 with weaker current.
Shunt.....	Elephant.....	2, 3, 4 (very faint), 5.
Shunt.....	Bell.....	2 (very faint), 3, 4, 5.
Shunt.....	Lambert Schmidt.....	3, 4, 5 or 2, 3, 5, according to current.
Series....	S. & H.....	2 (very weak), 3, 4, 5.
Series....	Elephant.....	2, 3, 4, 5.
Series....	Bell.....	2 (very weak), 3, 4, 5 (weak).
Series....	Lambert Schmidt.....	2, 3 (strong), 4, 5.

SUMMARY: 3d and 5th always noticeable, and 3d as strong as 5th; 2d and 4th sometimes absent, or too weak to be heard, and less prominent in shunt than in series connection. The 2d is never strong in the Siemens & Halske nor in the Bell.

It might seem that the distortion found by the resonators was introduced, not by the diaphragm, but by the armature reaction in passing a current from one machine through the armature of a second; but only a very small current was used, and the harmonic set is rated at 34.5 amperes. Moreover, if it were due to armature reaction, the distortion should be greater for the shunt connection, where a few tenths of an ampere were drawn, than in the series connection, where only micro-amperes were drawn; but the reverse was found.

It is thus evident that the receiver may introduce harmonics not present in the electrical wave, and thus lead to false inferences in regard to the question being studied. The best conditions are evidently obtained by using a sensitive receiver in shunt connection, excited as slightly as possible, and star-connected generators whose ratio of frequencies is an even integer. The frequencies should, moreover, be as high as possible, since the ear is more sensitive at higher frequencies and less current is necessary.

TABLE III.

Pitches of components	Intensity of sound possible without beats
60-180	Less than normal.
60-540	Less than normal.
60-900	Less than normal.
180-270	Normal.
180-390	Normal.
180-450	Normal.
180-540	Normal.
300-450	Normal.
300-900	Above normal.

In the final experiments at least one of the machines used was star connected, three phase. Even under the most favorable conditions of connection every combination tried could, by using sufficient current in the telephone, be made to show a periodic change of quality which gave the same sensation to the ear as occurred in the case of the single-phase connection described above. This could be done with the telephone either in series or in shunt, and with five different types of telephone. Lindig does not mention

this effect with strong tones. Perhaps the greater sensitiveness of the continuous slip-phase method accounts for it being noticed in our experiments and not in his.

Yet in nine different cases it was found that with less current in the telephone no change could be detected by either observer. These are shown in Table III.

By "normal" is here meant an intensity that would allow spoken words to be easily and distinctly understood, but not great enough to make the sound unpleasant—that is, about the intensity of ordinary conversation over the telephone.

In these cases it is probable that a change in quality as great as would be caused by the interference of harmonics common to the two sources, when present to the amount of even less than 1 per cent, could have been easily detected.

But in several cases beats were obtained which could not be made to disappear by decreasing the intensity of the sound, and this when, theoretically, there should have been no common harmonics present, and as far as could be told the machines were running smoothly. For example, using the two-phase machine with the harmonic set, beats were obtained with the following combinations:

90-180

180-270

180-450

Yet it will be noticed that, as indicated in the previous table, the last two combinations gave no beats when the two parts of the harmonic set were used as source. A little light may be thrown upon the discrepancy by a consideration of the second case. The 180-cycle emf. from the two-phase machine is known to contain 4 per cent of its third harmonic, corresponding to a pitch of 540. Now, if the telephone diaphragm introduced this tone independently its amplitude would be likely to be altered by the addition of this harmonic to the electrical wave or by a change in its phase relation to the fundamental. In other words, the distortion of the wave during its conversion from the electrical to the mechanical form would naturally vary with the form of the electrical wave, and particularly with any alteration in those components which are independently introduced during the conversion. Evidently these results should be discarded in favor of those obtained

under the more favorable conditions of taking both the components from star-connected machines. When so connected, and when *both machines were running smoothly under the best conditions*, no case of a change in quality was found that could not be traced to extraneous causes.

Very great care had to be observed to exclude such extraneous interference, and in many cases it was found to be impossible to do so. Several times during the course of the work results were obtained that seemed to indicate a real dependence of quality upon phase, but in each case investigation showed that the results were vitiated by bad conditions. For example, the throbs of the engine supplying power to the dynamo, which in turn supplied the direct current to the driving motor, could occasionally be heard in the telephone, and for this reason nearly all the observations were made while the energy was being taken from a storage battery. Noises in the dynamo room were sometimes transmitted through the telephone, and slip rings and commutators had to be in perfect condition or the sound became unsteady. Mechanical beats were sometimes transmitted to the telephone, as, for example, when it was attempted to obtain the combination 300-600 by running the halves of the harmonic set at slightly different speeds, the note given out by the telephone when connected to the 300-cycle machine alone contained beats which ceased when the machine of higher frequency was stopped. Unfortunately this condition prevented us from getting results with nearly all of the combinations involving even harmonics.

Thus, while the method is very sensitive, positive results should be accepted only after care has been taken to insure that no disturbing factors are present. The ideal condition for its use would be to have the machines in separate buildings and the observer in a third.

SUMMARY.

The experiments clearly support Helmholtz's view that quality is independent of phase relations, as was found by Lord Kelvin with his analogous mechanical drifting phase method, and also show that the telephone introduces both odd and even harmonics which are not present either in the emf. wave at its terminals, or in the current wave through it.

WASHINGTON, June 30, 1909.

WHITE LIGHT FROM THE MERCURY ARC AND ITS COMPLEMENTARY.

By Herbert E. Ives.

Attempts have been made to modify the color and unpleasant illuminating characteristics of the mercury vacuum arc by the addition of yellower light sources. The first light used for this purpose was the carbon glow lamp. More recently, in an article which appeared since the present investigation was started, the use of a tungsten lamp is described.¹

In the present paper is recorded an experimental investigation of the proper light source to be combined with the mercury arc in order to imitate average daylight, following which is a determination of the relative intensities of mercury arc light and the added light to secure the best effect.

In a recently published article,² the writer has considered the question of producing white light by screening some of the more common illuminants. The problem of combining lights to produce white can be treated by use of the same data and in a similar manner. As, however, the only case at present occurring of lights suitable for this purpose is that of the mercury arc and its complementary, this was reserved for separate detailed discussion and experimental test.

The complementary of the mercury arc may be determined by direct application of color theory to experimentally obtained color values, as, for instance, spectrophotometer data. First, find the spectrum hue of the mercury arc (as is done in the paper above referred to); secondly, find the complementary hue, or hue to be added to produce white; then select a light having that dominant

¹ A. J. Marshall. Transactions of the Illuminating Engineering Society, April, 1909.

² Illuminating Engineer, October, 1909.

hue. From the color-sensation values of the two sources, and the intensity values of the sensations, the candlepower relations of the two to give white are calculable. On carrying through this work, the rather unexpected result was obtained that the best combination was of the mercury arc with a Welsbach; unexpected, because the usual criticism of both illuminants is on the ground of their green color, or illuminating effect.

In the present investigation, a method similar in principle to the above-outlined theoretical discussion was used, differing from it, however, in that it affords an opportunity to make an exact experimental comparison of the combination decided upon with the standard white it is desired to imitate. The Ives colorimeter was employed, together with Maxwell's color triangle.

The Ives colorimeter, designed by Frederic E. Ives, is essentially a color-mixing machine, mixing red, green, and blue light to produce a visual match with the color measured. White being taken as equal parts red, green, and blue, other colors are numerically described by the proportionate parts of the three primaries which mix to match them. Various light sources have been measured by the writer, among them daylight of various qualities, and the artificial illuminants compared with the average of the daylight measurements as a standard.³ By means of several of the illuminants which were measured in that investigation it is possible, at any time, to determine the setting of the instrument to give "average daylight," and then compare any artificial white light with it. This possibility made available a verification of the correctness of the work here outlined; had there been no more scientific method of attaining the end, a purely experimental mixing of various sources could have been tried and the best mixture selected. As stated above, use was made of Maxwell's color triangle to guide to the proper selection of complementary to the mercury arc. As the color triangle and its properties are not widely known, a paragraph may be devoted to it.

If we represent the red, green, and blue, which mixed make white, by the vertices of an equilateral triangle, then all points within the triangle can represent mixtures of red, green, and blue in various proportions. We avail ourselves of the property of an

³ Transactions of the Illuminating Engineering Society, November, 1908.

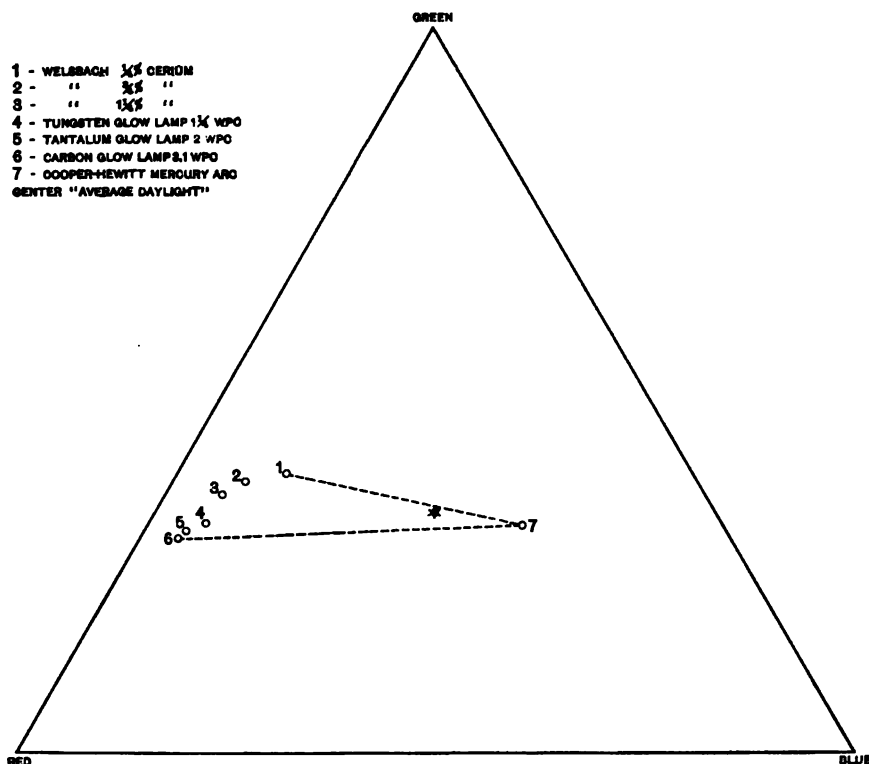


Fig. 1.

TABLE I.

Colorimeter Measurements of Illuminants.

		Red	Green	Blue	Cp to 1 cp of Hg light	Efficiency of combination
1	Welsbach $\frac{1}{4}$ per cent cerium.....	48.7	38.3	13.0		
2	Welsbach $\frac{3}{4}$ per cent cerium.....	54.0	37.4	8.6	.57	
3	Welsbach $1\frac{1}{4}$ per cent cerium.....	57.5	35.5	7.0		
4	Tungsten glow lamp $1\frac{1}{4}$ wpc.....	61.4	31.7	6.9	.54	0.8 wpc
5	Tantalum glow lamp 2.00 wpc (similar in color to Gem 2.5 wpc).....	64.6	30.4	5.0		
6	Carbon glow lamp 3.1 wpc.....	65.7	29.8	4.5	.50	1.4 wpc
7	Cooper-Hewitt mercury arc.....	24.1	31.4	44.5		

equilateral triangle, that the sum of the perpendicular distances of any point from the three sides is constant, and is equal to the altitude. We first express the red, green, and blue representing a color, so that their sum equals the altitude of the triangle, in convenient units. If, then, the vertical distance of a point from the side opposite, say, the red vertex, represents the amount of red, and similarly for green and blue, every color is represented by a point in the triangle. Colors matched by mixing two primaries lie on the sides; white (equal parts of the three primaries) lies at the center. The most important property for our present investigation is this: Mixtures of two colors are represented by points on the line joining the two; therefore two colors which lie on opposite ends of a line through the center are complementary, i. e., will mix to produce white.

In Fig. 1 is given a color triangle in which the red, green, and blue are the primaries of the Ives colorimeter when used as described in the paper referred to. The various light sources measured are plotted from the figures in Table 1, where the values are expressed so that their sum = 100 = altitude of triangle. These figures are from new measurements made for this paper.

An interesting fact brought out by this diagram is that all of the ordinary light sources, from the 3.1 watt carbon lamp to the "greenest" Welsbach ($\frac{1}{4}$ per cent cerium), lie so nearly opposite the mercury arc that a respectable white may be produced by mixing any one of them with it in the proper proportion. The light lying most nearly opposite the mercury arc is the Welsbach mantle, as supplied for residential lighting, viz, $\frac{3}{4}$ per cent cerium; of the glow lamps the tungsten is nearest the ideal complementary.

The results obtained from the diagram were checked experimentally for the three cases considered of most interest—the Welsbach, the tungsten, and the 3.1-watt carbon lamp. The first two not only come the nearest to the proper hue, but because of their efficiency and cheapness are distinctly practical possibilities. A Cooper-Hewitt lamp was provided with an adjustable cardboard shutter so that more or less mercury light could illuminate the plane of measurement. The Welsbach and the glow lamps were placed, in succession, directly before the tube and immediately below the opening in the shutter. In this way

it was possible to vary the mixing proportions at will. The colorimeter was adjusted by measurements on glow lamps of known color values, so that one-half its field showed "average daylight," then the shutter over the mercury tube was raised or lowered (intensity match being preserved by opening or closing the comparison slit of the colorimeter) until the nearest match was found. With the Welsbach and tungsten it was found possible to obtain an extremely close match, the Welsbach match tending slightly toward green and the tungsten toward pink, but by such small amounts that the slight differences of adjustment in several such experiments made at one time one and at another time the other mixture appear a perfect match with the standard. The carbon glow lamp formed, with the mercury arc light, a decidedly pinkish mixture. The indications of the color triangle were therefore entirely borne out.

For practical purposes, the important information to be derived from such an experiment is the candlepower relations between the two lights to be mixed. To obtain these it was necessary to measure the relative intensity of each component illumination as actually obtained on the surface observed during the measurement. The most convenient method was to extinguish each of the two lights of a mixture in turn and measure the illumination from the remaining one. As unit of intensity, the green light of the colorimeter was taken, that is, the red and blue slits were closed, the green slit opened until an intensity match was found, and the opening of the slit read. This, of course, involved comparison of two widely differently colored fields. By taking numerous readings and checking the results by the fact that the sum of the two intensities agreed closely with the intensity of the mixture measured in the same way, sufficient accuracy was obtained, it is believed, for the present purpose.

The values found were as follows: To one candlepower of mercury arc light should be added, to make white, .57 cp of Welsbach light; .54 cp of tungsten; .50 cp of 3.1 carbon light. Practically, therefore, we may say that we must add to the mercury arc a little over half its candlepower of Welsbach, tungsten, or carbon glow lamp light, to secure white.

It is interesting to compare these figures with those given by Mr. A. J. Marshall in the installation above referred to. There the proportions were: Cooper-Hewitt, 200 cp; tungsten, 80 cp; or a smaller proportion of tungsten than here found best.

Several points in connection with the character of this composite white light should be emphasized. It must not be forgotten that daylight is extremely variable in color, and that the combination of lights here investigated can only claim to be a more definite guide than has hitherto been given to the proportions for a match of the integral color to an average daylight. Further, it is, of course, only the integral color that appears white to the eye; analysis reveals the mercury line spectrum superposed on a continuous background strongest in the red. Because of the partly noncontinuous character of the spectrum of such a compound light, we must expect some deficiencies when it is used as an illuminant of colored objects. The chief defect of the mercury arc alone—its entire deficiency in red—should, however, be largely overcome. Because of the peculiarity of the eye that it quickly adapts its scale of color values to the color of the illuminant, it is much more important, in the writer's opinion, that the illuminant should preserve certain color values to which the eye is particularly sensitive than that its integral color should be a perfect subjective white. In other words, the eye quickly adjusts itself to the belief that a light is "white" even if the light is measurably yellower or pinker than a standard such as average daylight or sunlight, but rebels at a distortion of its new scale of color values. Therefore our preference should be given to that illuminant of this character which favors preservation of color values to which the eye is particularly sensitive, such as the color of flesh, lips, and other commonly observed objects.

The combinations under discussion were investigated for their color rendering values by observing their effect on various colored silks, plaids, colored prints, flesh, etc. The color values were, on the whole, found to be very well preserved; immeasurably more so than with the mercury arc alone, and much better for blues than with the yellower sources alone. A slight accentuation of purples and a graying of very deep reds was observable with both

combinations (Welsbach and tungsten), but ordinary reds, browns, yellows, greens, and blues appeared normal. The tungsten lamp, with its larger proportion of deep red, is appreciably better in this combination than is the Welsbach, judged by this criterion. For the best results in color rendering, apart from the integral color of the mixture, a larger proportion of the tungsten light rather than a smaller might be recommended, because, as remarked above, the resultant pinkish character of the white would be less noticed by the eye than the disturbance of the scale of color values which occurs with a deficiency of continuous spectrum background. The behavior of the carbon lamp is similar to the tungsten, although the integral color is too pink to be called a good white.

The results of the above investigation are as follows: The Welsbach and the tungsten glow lamp are shown to be most nearly complementary in color, of the ordinary illuminants, to the mercury arc; the proportions in which their light should be added to the light of the mercury arc to produce white are determined; study of the color rendering values of the resultant mixtures indicates that the tungsten combination is the more desirable, and that the proportions given may be considered a lower limit to the amount of tungsten light to be added for satisfactory illumination of colored objects. The carbon glow lamp is also investigated and compared with the others in this combination.

An approximate idea of the efficiency of the combination of mercury arc and glow lamps is easily obtained. For the tungsten lamp combination we find, assuming 1.25 wpc for tungsten and .55 wpc for the Cooper-Hewitt, that 1.54 cp is obtained at an expenditure of 1.22 watts, or an efficiency of .80 wpc. For the 3.1 carbon lamp we obtain, similarly, 1.4 wpc. Therefore the tungsten-mercury combination is not only nearer white, but much more efficient than the carbon lamp combination used at first. The Welsbach-mercury combination can not be expressed in this way, but consideration of expense makes this combination compare closely with the mercury-tungsten one.

WASHINGTON, August 1, 1909.

THE REGULATION OF POTENTIAL TRANSFORMERS AND THE MAGNETIZING CURRENT.

By M. G. Lloyd and P. G. Agnew.

The formulas now in general use for calculating the regulation of potential transformers involve the magnetizing current. Since the magnetizing current is flowing both at no load and at full load, and since regulation depends only on the difference in ratio between no load and full load, we should expect, a priori, the regulation to be independent of the magnetizing current. That it has no appreciable effect will be shown by developing the formula for regulation from the consideration of the vector diagram, from a treatment of the problem by the symbolic method, and by experimental results.

The regulation is defined as the change in secondary voltage between no load and full load (with constant primary voltage) expressed as a percentage of the secondary voltage on full load. In practice it is sometimes wrongly expressed in per cent of the no-load value, but the error in so doing will seldom amount to more than one-tenth per cent.

The regulation may also be expressed as the change in ratio of terminal voltages $\left(\frac{\text{primary}}{\text{secondary}}\right)$ between no load and full load divided by the ratio at no load.

The difference between the induced and the terminal voltage of a transformer winding is due to its resistance and its leakage reactance. If these be known for both windings, the voltage drop is thereby determined, and the regulation may be calculated. Direct measurements of resistance and of impedance drop of voltage are easily made by well-known methods, and furnish the required data.

The relations involved may easily be seen by reference to figure 1, which is a vector diagram of the quantities concerned. Let Φ represent the magnetic flux, E' , the voltage induced in the secondary winding, and I_s the secondary current. The total current-

Now, for convenience, let us keep in mind that the primary current I_1 can be regarded as made up of two parts, I in reversed phase with the secondary current and I_0 , the exciting current. The primary resistance drop can also be divided into two parts, $I r_1$ and $I_0 r_1$, of which the first is in reversed phase with the secondary resistance drop and may be lumped with it as IR . The difference between the terminal voltages is thus made up of $I_0 r_1$, IR , IX , and the impedance drop of the exciting current, $I_0 x_1$. Resolve each of these into two components, parallel and perpendicular to E'_1 , for ease in combining. The parallel components are

$$I_0 r_1 \sin \gamma + IR \cos \theta + IX \sin \theta + I_0 x_1 \cos \gamma$$

The perpendicular components are

$$I_0 r_1 \cos \gamma + IR \sin \theta - IX \cos \theta - I_0 x_1 \sin \gamma$$

If we designate the two components of I_0 as M and F , and the ratio of turns by n , we have

$$E_1^2 = (nE_2 + Fr_1 + IR \cos \theta + IX \sin \theta + Mx_1)^2 \\ + (Mr_1 + IR \sin \theta - IX \cos \theta - Fx_1)^2$$

Dividing by E_1^2 and extracting the square root, we get

$$1 = \frac{nE_2}{E_1} + \frac{Fr_1 + IR \cos \theta + IX \sin \theta + Mx_1}{E_1} \\ + \frac{1}{2E_1^2} (Mr_1 + IR \sin \theta - IX \cos \theta - Fx_1)^2$$

with close approximation.

For no load, we have

$$1 = \frac{nE_{2,0}}{E_1} + \frac{Fr_1 + Mx_1}{E_1} + \frac{1}{2E_1^2} (Mr_1 - Fx_1)^2$$

Since the term in quadrature is always very small in comparison to the total voltage, we get by subtraction of this equation from the previous

$$\frac{nE_{2,0} - nE_2}{E_1} = \frac{IR \cos \theta + IX \sin \theta}{E_1} + \frac{1}{2E_1^2} (IR \sin \theta - IX \cos \theta)^2$$

Multiplying by $100 \frac{E_1}{nE_2}$ we have the regulation expressed in per cent.

$$100 \frac{E_{2,0} - E_2}{E_2} = \frac{100}{nE_2} \left[IR \cos \theta + IX \sin \theta + \frac{1}{2E_1} (IR \sin \theta - IX \cos \theta)^2 \right]$$

where nE_2 may be substituted for E_1 without appreciable error. For non-inductive load this becomes¹

$$100 \left[\frac{IR}{nE_2} + \frac{1}{2} \left(\frac{IX}{nE_2} \right)^2 \right]$$

For approximate work E_1 may be used in place of nE_2 , and the reactance term neglected for noninductive load. If the reactance drop is not over 3 per cent, this term will affect the result by less than 0.1 per cent. If it is desired to get the result correct to 0.01, it is necessary to use nE_2 , which requires that the ratio of turns, n , be known, or that the ratio of voltages be measured and the approximate value of the above expression be used to determine the value of nE_2 to be used in a more exact computation. In some transformers, n is the same as the ratio of voltages given on the name plate, but in others, such as instrument transformers (where accurate values are most desired) the turns are slightly altered to give the nominal ratio at full load, or at half load.

For most practical purposes it is entirely sufficient to know the regulation to 0.1 per cent. It is customary in some quarters to compute the values of regulation to 0.01 per cent, whereas the conditions of use are not specified sufficiently closely to warrant it. Thus the cold resistance may be used in the computation, whereas the regulation after being loaded for some time might be the quantity desired. So that unless the temperature is specified, it is meaningless to consider regulation to 0.01 per cent.

It is evident from the above that although the exciting current affects the ratio slightly, it has an entirely inappreciable effect upon the regulation, or change of secondary voltage with load. The formulas published by some of the manufacturing companies are in error in this respect, since they contain the magnetizing current as one of the quantities affecting the regulation. These formulas are generally complicated to an unnecessary extent, also, by retaining the radical sign, instead of simplifying the expression as above. In the *General Electric Review* for December, 1908, one writer computes tables showing the alleged effect upon regula-

¹ This formula is correctly given by LaCour in Arnold's "Wechselstromtechnik," Vol. 1.

tion of different values of magnetizing current, while as a matter of fact the magnetizing current has no such effect. Whatever objections there may be to the large magnetizing currents sometimes observed with cores of silicon steel, it does not consist in disordered regulation.

An example which recently came to our notice was in the case of an instrument transformer rated at 200 watts, 8660/100 volts, 50 cycles, which required an exciting current of 0.737 ampere on the low-voltage side. The core loss was 35.8 watts; the impedance drop (primary side), 84.8 volts; primary resistance, 1600 ohms; secondary resistance, 0.175 ohm. From these observed values the following are computed:

$$R = 2910$$

$$IR = 67.2$$

$$M = 0.00744$$

$$F = 0.00413$$

$$IX = 51.7$$

Substituting these values in the above equation for noninductive load, we get for the regulation, 0.78 per cent. Making the same calculation by the formula² used in the tables mentioned above, we get for the regulation 1.10 per cent. The difference is 0.32, which represents the error introduced by using the wrong formula.

The regulation was experimentally determined by measuring the ratio at no load and at full load. A differential null method³ was used giving an accuracy of about one part in five thousand. The values of ratio found were 86.52 at no load and 87.20 at full load, giving a value of regulation of $100 \frac{.68}{86.52}$ or 0.79 per cent.

In another transformer of the same type and capacity, but higher primary voltage, the exciting current measured on the secondary was 3.92 amperes, and it consequently presents an extreme case. The regulation as determined experimentally on non-inductive load was 0.29 per cent. The value according to our formula was 0.28 and by the other formula 2.56, or nine times as large as the correct value.

²Substituting our notation, this formula is

$$100 \sqrt{\left(1 + \frac{IR}{E_1} + \frac{MX}{E_1}\right)^2 + \left(\frac{IX}{E_1}\right)^2} - 100$$

³For a description of this method, see Agnew and Fitch, this Bulletin, p. 281.

In power transformers the exciting current is relatively smaller and the resistance drop relatively larger, and the errors are consequently much less in magnitude. It is probably due to this fact that incorrect formulas have been used for so long without exciting suspicion.

In order to test the effect of varying magnetizing current upon the same transformer, some measurements were made at different voltages upon a 600-watt, 60-cycle, 120/120-volt transformer. The results are given in Table I. At 80 volts the exciting current was 0.27 and the ratios at no load and full load were 1.0007 and 1.0348. At 150 volts the exciting current was 1.33 and the ratios 1.0018 and 1.0199. Full load in each case consisted of lamps

TABLE I.

Primary volts	Secondary amperes	Exciting amperes	Ratio	Regulation drop
130	5.0	0.752	1.0220	2.70
130	0.0		1.0012	
140	5.0		1.0207	2.70
140	0.0	0.973	1.0014	
120	5.0		1.0236	2.71
120	0.0		1.0010	
100	5.0	0.580	1.0281	2.73
100	0.0		1.0008	
80	5.0		1.0348	2.73
80	0.0	0.27	1.0007	
150	5.0		1.0199	2.72
150	0.0		1.328	1.0018
			Mean.....	2.715

carrying five amperes. It is at once evident that exciting current affects ratio, since the latter is not constant for no load. To determine whether the exciting current affects the regulation, we must determine whether the difference in actual drop varies in the two cases. The difference in ratio between no load and full load gives the drop in terms of the voltage; multiplying this by the voltage gives the actual drop. At 80 volts this amounts to $.0341 \times 80 = 2.728$. At 150 volts it is $.0181 \times 150 = 2.715$, an

agreement within one-half of one per cent of the value of the drop, or .01 per cent of the total voltage. This makes it evident that if the exciting current could be changed without altering the voltage (say by changing the core) the ratio of terminal voltages would be changed, but the regulation would not be changed, since the changes in ratio at no load and full load would be of equal magnitude.

The above difference of .01 per cent may well be attributed to slight changes in resistance from heating during the experiment, although the changes were made as rapidly as possible from one condition to the other. Readings were also taken at other intermediate voltages, in irregular order, and all gave the same result within .01 per cent of the mean.

The derivation of the same formula for regulation by the use of complex quantities, in place of the geometrical or vector method, follows:

r_1 = primary resistance; r_2 = secondary resistance. $R = r_1 + n^2 r_2$.

x_1 = primary reactance; x_2 = secondary reactance. X = total reactance referred to primary.

$F + jM$ = exciting current.

$P + jW$ = load current (on primary side).

n = ratio of turns.

E_1 = terminal primary voltage.

$E + jQ = E_2$ = terminal secondary voltage.

$E_1 - (r_1 + jx_1)(F + jM + P + jW)$

$= n(E + jQ) + n(r_2 + jx_2)(nP + njW)$

$E_1 - r_1(F + P) + x_1(M + W) = nE + n^2(r_2P - x_2W)$

or

$$\begin{aligned} E_1 - r_1F - RP + x_1M + XW &= nE \dots\dots\dots (1) \\ -r_1(M + W) - x_1(F + P) &= nQ + n^2r_2W + n^2x_2P \end{aligned}$$

or

$$-r_1M - RW - x_1F - XP = nQ \dots\dots\dots (2)$$

From (1) we have for the drop in voltage in phase with E_1

$$E_1 - nE = r_1F + RP - x_1M - XW$$

For no load

$$E_1 - nE_0 = r_1 F - x_1 M$$

From (2) we have for the drop in voltage in quadrature

$$-nQ = r_1 M + RW + x_1 F + XP$$

For no load

$$-nQ_0 = r_1 M + x_1 F$$

For regulation it is sufficient practically to take differences of real and imaginary parts before combining. Then for the regulation drop in phase and in quadrature, we have

$$n(E_0 - E) = RP - XW$$

$$n(Q_0 - Q) = RW + XP$$

and the regulation in per cent is

$$\begin{aligned} & \frac{100\sqrt{(nE_2 + RP - XW)^2 + (RW + XP)^2}}{nE_2} - 100 \\ &= 100\sqrt{\left(1 + \frac{RP}{nE_2} - \frac{XW}{nE_2}\right)^2 + \left(\frac{RW + XP}{nE_2}\right)^2} - 100 \\ &= 100\left[\frac{RP - XW}{nE_2} + \frac{1}{2}\left(\frac{RW + XP}{nE_2}\right)^2\right] \end{aligned}$$

For non-inductive load, $W = 0$, and this reduces to

$$100\left[\frac{RP}{nE_2} + \frac{1}{2}\left(\frac{XP}{nE_2}\right)^2\right]$$

It should be borne in mind that the proper algebraic sign must accompany numerical values of M and W when these are introduced into the formula. These will be positive for leading and negative for lagging values. Consequently M is always negative and W usually so.

WASHINGTON, June 21, 1909.

THE DETERMINATION OF THE CONSTANTS OF INSTRUMENT TRANSFORMERS.¹

By P. G. Agnew and T. T. Fitch.

The determination of the ratio and phase angle of instrument transformers is an important problem, and one in which considerable precision is required for technical work. The principle of the method here described is the same for both current and potential transformers, being an application of the potentiometer method.

POTENTIAL TRANSFORMERS.

If a high noninductive resistance be connected in parallel with the primary, the secondary voltage may then be applied to it potentiometer fashion, as shown in Fig. 1, by connecting one ter-

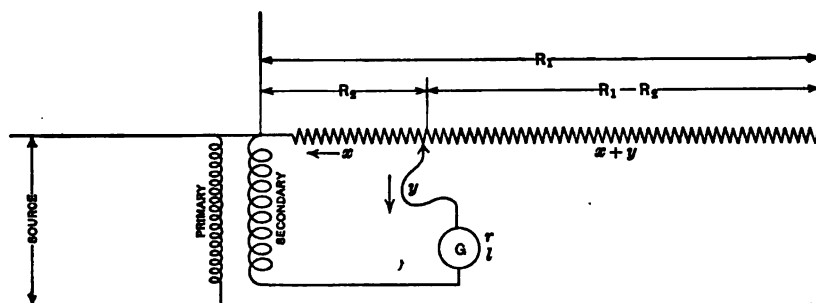


Fig. 1.

minal in a reversed sense to one end of the resistance and applying the other terminal, through a detector G, to the resistance. If there were no phase difference between the primary and secondary voltages, a point could be found at which no current would flow

¹ Paper read before the Washington meeting of the American Physical Society, April 27, 1909. A preliminary account of the work appeared in the *Electrical World*.

through the detector. But, in general, there is a phase difference, so that the emf. at the terminals of G can only be reduced to a minimum value Q , Fig. 2, which is in quadrature with the primary

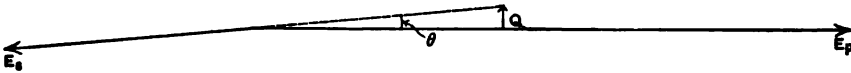


Fig. 2.

emf. If, then, we had an idiostatic voltmeter that would read to 0.001 volt with an accuracy of 1 or 2 per cent, we should have to adjust for a minimum only and then read the value of Q , the ratio of the total resistance to the part tapped off giving the voltage ratio of the transformer and the ratio of Q to E_p giving $\tan \theta$. But as such an instrument does not seem to be available, some indirect method has to be used. The method first tried was to balance the small quadrature component Q , using as a source of emf. a second phase from a two-phase machine, and making use of a telephone as detector. The capacity effects of the various parts of the circuit reduced the accuracy at low frequencies on account of the greater sensitiveness of the telephone to the harmonics present. Either a thermo-galvanometer or a vibration galvanometer gave greater accuracy, but neither was satisfactory as regards ease of manipulation. A method considerably more accurate is to use a vibration galvanometer and only a single source of emf., placing variable inductances in the two parts of the high resistance circuit and alternately adjusting the resistances and inductances for zero deflection, and then calculating the difference in the phase angles by which the emf. leads the current in the two parts of the high resistance.

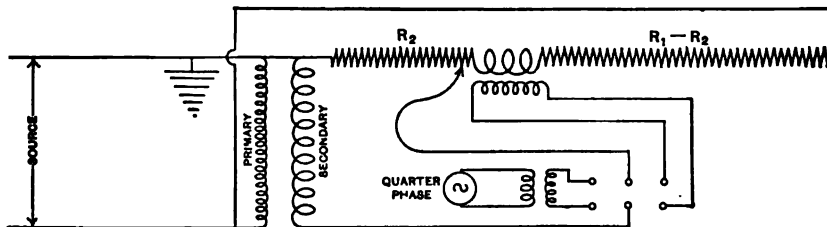


Fig. 3.—Two-dynamometer method for test of potential transformers.

The method finally adopted as most satisfactory was to use as detectors two electrodynamicometers (Fig. 3). The field coil of one

was placed in series with the high resistance, while the field of the second was energized by a current in quadrature from a two-phase machine. The operation is to throw the switch to the right and adjust R_2 for no deflection. The current in the moving coil is then not necessarily zero, but is in quadrature with that in R_1 . Then throw the switch left and read the deflection which measures the quadrature component Q . The first setting gives the ratio

$$\frac{E_1}{E_2} = \frac{R_1}{R_2} \quad (1)$$

while the deflection of the second dynamometer gives the phase angle²

$$\tan \theta = \frac{Q}{E_2}$$

The latter instrument draws a very small current from the network of conductors, and hence for the phase angle measurement has to be calibrated as a voltmeter under the conditions of use. At first sight it would seem that its multiplier should be R_2 only (Fig. 3). But it will be shown by analysis, as seems evident from a close study of the figure, that $R_1 - R_2$ is in parallel with R_2 , so that for calibration they are so connected.

Also if the time constant of R_2 differs appreciably from that of the total R_1 , then the emf. at its terminals will not be in phase with E_1 , and a correction equal to the phase difference will have to be applied to the value of the phase angle obtained by deflection.

In order to establish these results analytically, and at the same time to investigate the magnitude of the possible corrections due to residual inductances, let us assume the phase angle of the reversed secondary voltage is positive if it leads the primary. This accords with the customary geometrical convention in dealing with complex quantities. Then in Fig. 1, let

R_1 = total resistance in the primary circuit.

R_2 = resistance included in the secondary.

r = resistance of the detecting dynamometer.

L_1, L_2, l = corresponding inductances.

$E_2 + iQ$ = secondary emf., E_2 being the part in phase with E_1 , Q the part in quadrature.

² Mr. L. T. Robinson, in the Proc. A. I. E. E., July, 1909, p. 981, describes a method involving the same potentiometer principle, but the component in quadrature is rectified by a rotating commutator and measured with a direct current instrument.

θ = phase angle.

x, y = vector currents in the directions shown.

$p = 2\pi$ times the frequency.

$i = \sqrt{-1}$.

Then applying Kirchhoff's law to the two circuits

$$\begin{aligned} E_1 &= x(R_1 + ipL_1) + (x + y)[R_1 - R_2 + ip(L_1 - L_2)] \\ &= x(R_1 + ipL_1) + y[R_1 - R_2 + ip(L_1 - L_2)] \end{aligned}$$

and

$$E_2 + iQ = x(R_2 + ipL_2) - y(r + ipl).$$

Solving the last two equations for y ,

$$y = \frac{\begin{vmatrix} R_1 + ipL_1 & E_1 \\ R_2 + ipL_2 & E_2 + iQ \end{vmatrix}}{\begin{vmatrix} R_1 + ipL_1 & R_1 - R_2 + ipL_1 - ipL_2 \\ R_2 + ipL_2 & -r - ipl \end{vmatrix}}.$$

In expanding the determinant, since L_1 , L_2 , and l are small quantities, we may neglect terms containing their squares or products, whence

$$y = \frac{E_2 R_1 - E_1 R_2 - QpL_1 + i(QR_1 + E_2 pL_1 - E_1 pL_2)}{-R_1 r - R_1 R_2 + R_2^2 + i(-R_1 pl - r pL_1 - R_2 pL_1 + 2R_2 pL_2 - R_1 pL_2)}.$$

Rationalizing the denominator, and again neglecting squares and products of small quantities,

$$\begin{aligned} y &= \frac{(E_2 R_1 - E_1 R_2 - QpL_1)}{-R_1 r - R_1 R_2 + R_2^2} \\ &\quad - i \frac{(E_2 R_1 - E_1 R_2)(-R_1 pl - r pL_1 - R_2 pL_1 + 2R_2 pL_2 - R_1 pL_2)}{(-R_1 r - R_1 R_2 + R_2^2)^2} \\ &\quad + i \frac{(-R_1 r - R_1 R_2 + R_2^2)(QR_1 + E_2 pL_1 - E_1 pL_2)}{(-R_1 r - R_1 R_2 + R_2^2)^2} \end{aligned} \quad (2)$$

Since the current through the dynamometer (see Fig. 3) is only very slightly out of phase with E_1 , we may, for the condition of zero deflection in the series dynamometer, set the real part of y equal to zero.

$$E_2 R_1 - E_1 R_2 - QpL_1 = 0$$

Dividing by $E_2 R_2$

$$\frac{E_1}{E_2} = \frac{R_1}{R_2} - \frac{QpL_1}{E_2 R_2}$$

The last term is an extremely small correction to the ratio of the resistances, and is entirely inappreciable, amounting to less than one part in 50,000 in the most unfavorable case examined.* Strictly, the ratio is

$$\frac{E_1}{\sqrt{E_2^2 + Q^2}} = \frac{R_1}{R_2} \cos \theta$$

but as the phase angle of a good potential transformer designed for use with instruments will always be less than 30', and as the cosine of 30' differs from unity by only one part in 25,000, we may take as the ratio

$$\frac{E_1}{E_2} = \frac{R_1}{R_2} \quad (1) \text{ bis}$$

Q , which gives a measure of the phase angle, may be determined from the quadrature component of y in equation (2). It is to be noticed that the numerator of the second term of (2) reduces to zero, since for the ratio balance

$$E_2 R_1 - E_1 R_2 = 0$$

Hence

$$\begin{aligned} y_{(4)} &= \frac{QR_1 + E_2 pL_1 - E_1 pL_2}{-R_1 r - R_1 R_2 + R_2^2} \\ \therefore Q &= -y_{(4)} \left(r + R_2 - \frac{R_2^2}{R_1} \right) - \frac{E_2 pL_1 - E_1 pL_2}{R_1} \\ \therefore \theta = \tan \theta = \frac{Q}{E_2} &= -\frac{y_{(4)}}{E_2} \left(r + R_2 - \frac{R_2^2}{R_1} \right) + \left(\frac{pL_2}{R_2} - \frac{pL_1}{R_1} \right) \quad (4) \end{aligned}$$

The expression $\left(\frac{r + R_2 - R_2^2}{R_1} \right)$ is the effective resistance of the moving coil circuit, the two parts of the high resistance R_2 and $R_1 - R_2$ being in parallel, but the whole in series with r .

Hence the numerator of the first term of (4) is the emf. measured by the quarter phase dynamometer. The last term is a constant correction depending on the difference in the time constants of the two parts of the high resistance R_1 . With high grade resistances this correction may easily be made less than one minute at 60 cycles.

* If R^1 , R^2 , and r have the same time constants, there are no corrections to either ratio or phase angle.

The calibration is easily accomplished without the use of direct current by keeping a known resistance in series with the field of the quarter phase dynamometer, with a switch so that the emf. at its terminals may be thrown into the potential circuit, while the transformer is disconnected from the source. This introduces a known emf. in the moving coil circuit which gives the calibration as voltmeter. The two parts of the high resistance are thrown in parallel by short-circuiting both windings. In fact, since the resistance of the windings is nearly always negligible in comparison to the other parts of the circuit, and as the calibrating current divides between the two paths in such a manner as to neutralize their magnetizing effects upon the core, the deflection of the dynamometer is the same whether the windings are short-circuited or not.

This two-dynamometer method has the advantages of giving a uniform scale, practically independent adjustments for the two settings, a high sensibility, and the ratio can be determined at zero load on the secondary. The correction factor for any slight departure from exact quadrature of the field in the second dynamometer is inappreciable except for the most precise measurements, in which case the adjustment may either be made or easily corrected for.

With the instruments used in obtaining the ratio and phase angle curves shown below, about 0.05 ampere was drawn through R_1 , and a constant current of 0.5 ampere maintained in the field of the quarter phase dynamometer. For the safety of the operator the point of connection of the two transformer windings is connected to earth, and the portion $R_1 - R_2$ put out of reach. (Fig 4.) This particular arrangement gave a maximum sensibility of one part in 20,000 in ratio, and about 0.1 minute in phase angle, but the absolute accuracy is of course less, being limited by the accuracy of the high resistances in the ratio determination and by the residual reactances in the multipliers in the determination of phase angles.

The method has been compared with the differential voltmeter method described by Rosa and Lloyd ⁴ for the ratio determina-

⁴ This bulletin, 6, p. 1, 1909.

tion, and with a modification of their method for the phase angle measurement, in which a quarter phase current from a machine was substituted for the condenser current used by them. The results agreed within the accuracy of these methods, as did also a second method of measuring phase angles, described by the same authors, in which the phase angle was read off directly as the angle by which a stator armature of a generator was shifted to give null settings on a dynamometer, whose field was excited by this generator and whose moving system was energized first by the primary, and then by the secondary of the transformer under test, the transformer being fed by a second generator rigid on the same shaft.

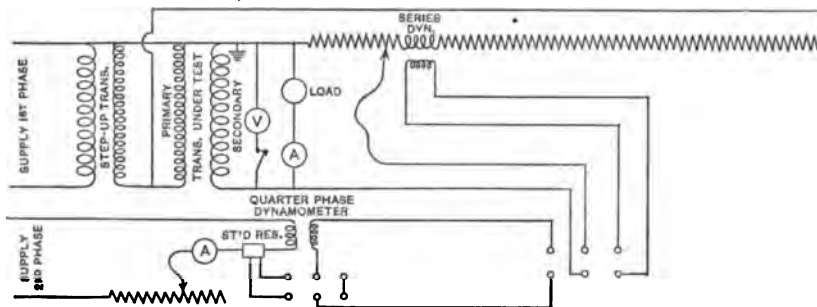


Fig. 4.—Complete diagram of connections for test of potential transformer.

A check not requiring the use of a dynamometer was furnished by the method mentioned above, of using a vibration galvanometer as detector, and placing variable inductances in the two parts of high resistance. This requires a simultaneous balance of resistance and inductance. It is fully as sensitive as the dynamometer method. The values of ratio given by the two methods agreed precisely, but the phase angles showed a discrepancy of about $0.4'$. While this difference is too small to have any practical significance, it appeared as a constant difference, and was thought to be due to a charging current into the galvanometer.

In Figs. 5 to 9 are shown load curves of typical potential transformers, designed especially for use with measuring instruments, from different makers. The full-line curves are for noninductive load, while the isolated points shown in some instances are for the highly inductive load of the potential coil of a watthour meter,

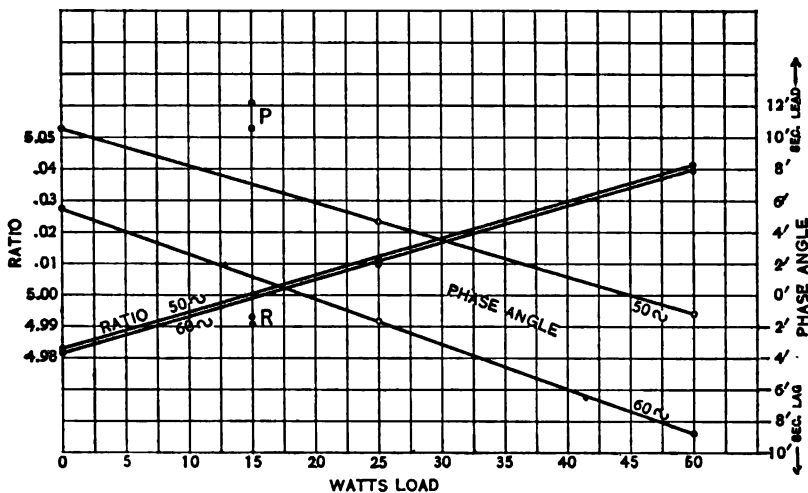


Fig. 5.—Potential transformer M_1 . Isolated points P and R are phase angle and ratio plotted in volt-amperes for inductive load, 20 per cent power factor.

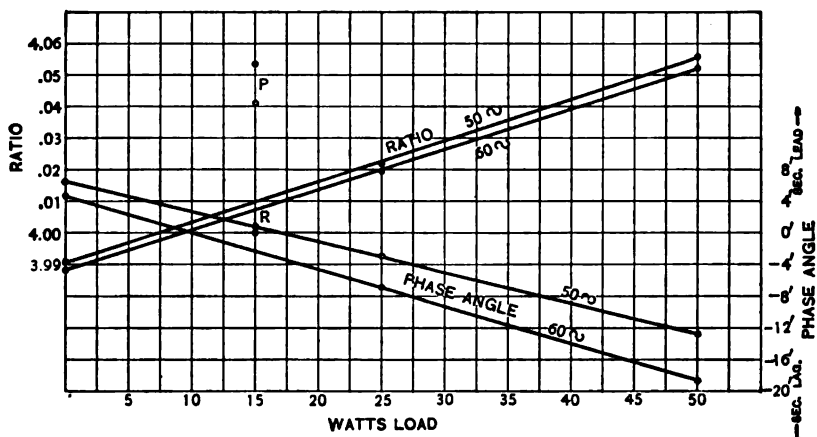


Fig. 6.—Potential transformer M_2 . Isolated points P and R are phase angle and ratio plotted in volt-amperes for inductive load, 20 per cent power factor.

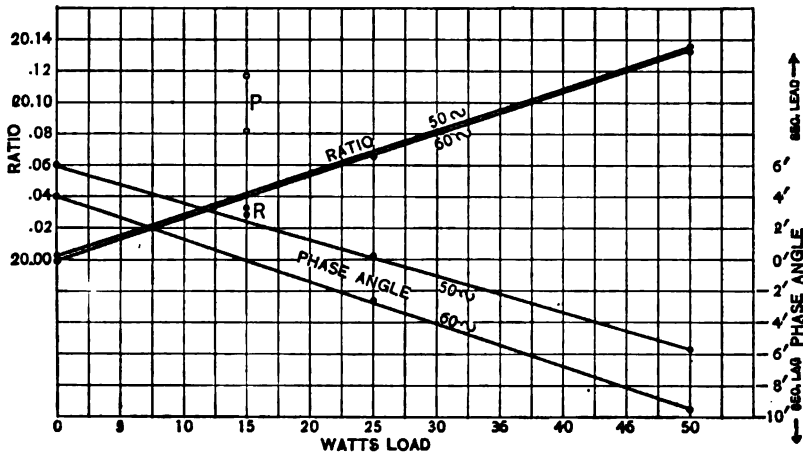


Fig. 7.—Potential transformer N. Isolated points R and P are ratio and phase angle plotted in volt-amperes for 20 per cent power factor.

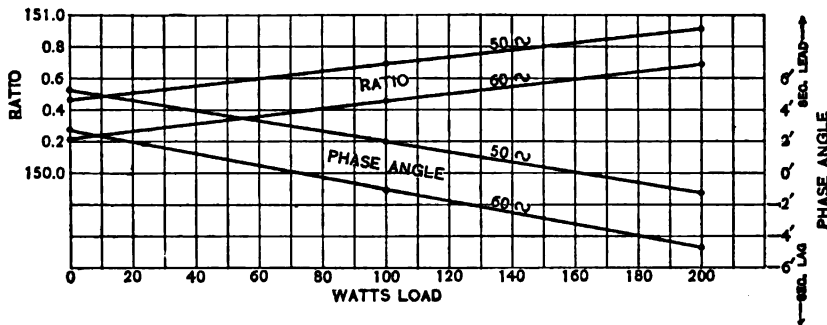


Fig. 8.—Potential transformer O.

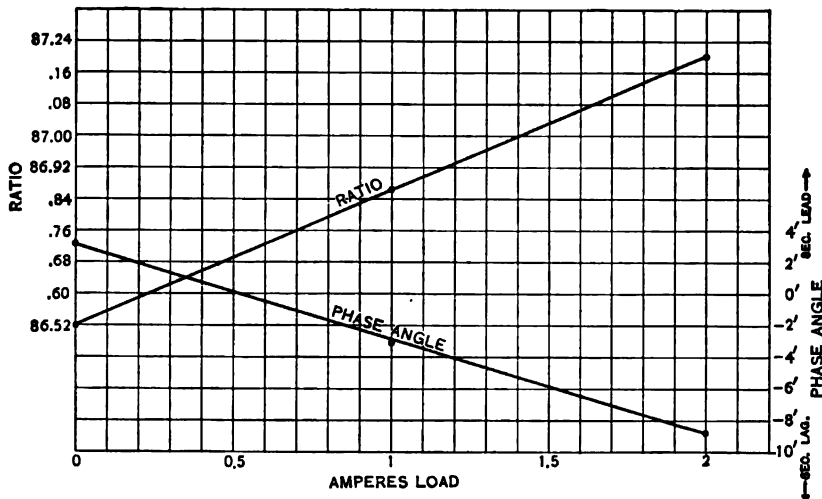


Fig. 9.—Potential transformer P. Frequency 50 cycles.

being plotted with voltamperes as abscissas for comparison. Descriptive data for these transformers is collected in Table I.

TABLE I.
Data for Potential Transformers.

Transformer	Voltage	Nominal ratio	Rated frequency	Watt rating	Resistance	
					Primary	Secondary
M₁	550	5/1	50	50	30.5	1.54
M₂	440	4/1	50	50	30.5	1.88
N	2,200	20/1	50	50	460	0.393
O	16,500	150/1	50	200	1,255	0.112
P	8,660	86.6/1	50	200	1,600	0.175

That a modern, well-designed potential transformer is an instrument of high precision may be seen from a study of these curves. The regulation of the better transformers is less than 1 per cent, and in some cases is but 0.3 per cent.

This refers to the nominal load ratings given by the maker, which are 50 watts for the smaller and 200 watts for the larger sizes. But if the transformer be used only in connection with a limited number of instruments the load will be but a few watts, and consequently the regulation for actual load is very much smaller. For example, the ratio of a small transformer will be changed less than 0.2 per cent by connecting a 150-volt voltmeter to the secondary, and the ratio of a large transformer would be changed only 0.02 per cent. For special purposes makers supply transformers rated as low as 10 watts, in which case the changes would be larger.

Both the ratio and the phase angle curves for constant voltage are strictly linear. The phase angle is so small that for all ordinary technical uses it may be neglected, as it brings in an appreciable error only at extremely low-power factors, and even in this case the other instruments used in a power measurement will usually limit the accuracy rather than the potential transformer.

Fig. 10 shows the effect of variation of voltage upon ratio and phase angle. From this and from the other curves it is seen that the ratio is affected only to a very slight degree by considerable variations in voltage and frequency. In case the core is worked at too great a flux density, so that the magnetizing current becomes excessive, the effect of change of voltage or of frequency may become appreciable. This was the case with transformer O, which took a magnetizing current more than three times as great as is customary for transformers of like range.

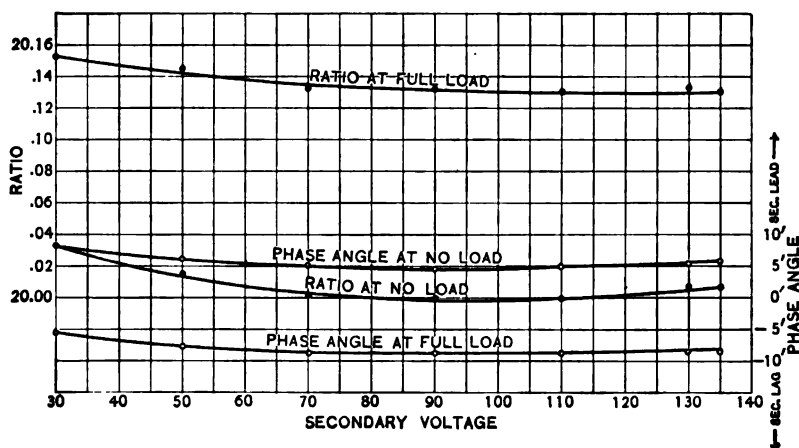


Fig. 10.—Showing effect of change of voltage upon ratio and phase angle. Potential transformer N.

DIFFERENTIAL TEST.

The same method may be used to determine differences in ratio and in phase angle between the transformer under test and a second one used as a standard by putting their primaries in parallel on the same source and measuring the ratio of their secondary voltages and the phase angle between them. This avoids the necessity of working on the high-voltage side. The sensibility goes up enormously, as the resistance $R_1 - R_2$ is here reduced to a few ohms, thus lowering the resistance in series with the moving coil of the dynamometer. With the same instruments the sensibility, when used differentially, was a part in a million in ratio, and 0.05" in phase angle, though, of course, the *absolute* accuracy is not at all increased. In fact, by using the pivoted movements of portable dynamometer voltmeters, sensibility greater than is

needed in any technical work may be obtained. Hence the use of standard transformers whose ratio and phase angle are to be determined once for all is proposed.

This method could easily be made to read differences in ratio directly in hundredths of a per cent and differences of phase angle directly in minutes. By bringing out several secondary taps a single transformer could be made to cover a considerable range. For example, if a 60,000-volt transformer had its primary divided in four sections, it could be connected for 60,000, 30,000, or 15,000 volts, and by determining its voltage-ratio curve for different frequencies, it could be used as a standard for testing transformers down to three or four thousand volts. Its ratio and phase angle would have to be determined for but one resistance load on its secondary.

Experience has shown that carefully designed transformers having series-parallel connections have the same phase angle, no matter what the grouping, and that factors connecting the different ratios are very accurately 2 and 4. Hence such a transformer could be tested to 15,000 volts and the ratio and phase angle assumed for the higher voltage connections.

CURRENT TRANSFORMERS.

The problem of the current transformer has been a more difficult one, for the resistance and inductance of a sensitive instrument placed in the secondary circuit will change both ratio and phase

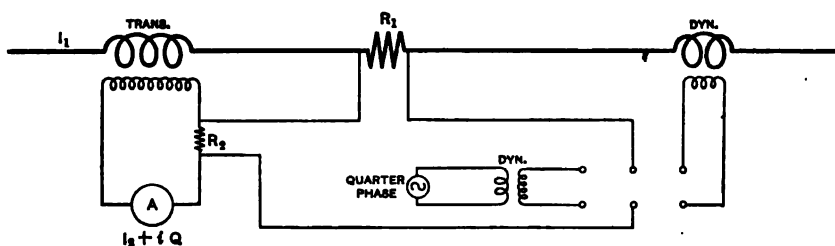


Fig. 11.

angle very appreciably, so that results thus obtained do not correspond to the conditions of use. But the same potentiometer principle used in the case of the potential transformer may be applied. An electro-dynamometer and a low noninductive resistance R_1 (Fig. 11) are placed in series with the primary of the

transformer, and a second noninductive resistance, R_2 , of such a value that R_2/R_1 is approximately the ratio of transformation, is placed in the secondary circuit. The potential terminals of these resistances are joined so as to oppose the emfs. impressed upon them, the leads being brought to a double throw switch so that the resultant emf. may be applied to the moving coil of either the series or of the quarter phase instrument. The switch is thrown to the right and R_2 adjusted (by shunting) to give no deflection. This gives the ratio of the currents. The switch is then thrown to the left and the component of emf. in quadrature, Q , measured by deflection. This gives the phase angle.

In developing the relations we shall find that a correction must be applied to the phase angle measurement due to the difference in the phase angles of the shunts. As in the case of the potential transformer, let

R_1, R_2, r = the resistances of the shunts in the primary and secondary, and in the moving coil circuit, respectively.

L_1, L_2, l = the corresponding inductances.

I_1, I_2, x = the corresponding currents.

θ = the phase angle. Then

$$I_1(R_1 + ipL_1) - (I_2 + iQ)(R_2 + ipL_2) = x(r + ipl)$$

$$x = \frac{I_1R_1 + iI_1pL_1 - I_2R_2 - iI_2pL_2 - iQR_2 + QpL_2}{r + ipl}$$

or, neglecting terms containing products or squares of L_1, L_2 , and l ,

$$x = \frac{I_1R_1r - I_2R_2r + QrpL_2 - QR_2pl}{r^2 + p^2l^2} + \frac{i(I_1rpL_1 - I_2rpL_2 - QR_2r - I_1R_1pl + I_2R_2pl)}{r^2 + p^2l^2} \quad (5)$$

For the balance for the determination of ratio, the real part of (5) becomes zero.

$$I_1R_1r - I_2R_2r + QrpL_2 - QR_2pl = 0$$

$$\frac{I_1}{I_2} = \frac{R_2}{R_1} + \frac{Qp}{I_2R_1r}(R_2l - L_2r) = \frac{R_2}{R_1} \left[1 + \frac{Q}{I_2} \left(\frac{pl}{r} - \frac{pL_2}{R_2} \right) \right]$$

The last term, like the corresponding one in (3) for potential transformers, is negligible, being less than one part in 5,000 at full load in the most unfavorable case examined.⁵ Hence we may take as the ratio of the currents in phase

⁵ If R_1, R_2 , and r have the same time constants, there are no corrections to either ratio or phase angle.

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}$$

and for the actual ratio, taking account of phase angle

$$\frac{I_1}{\sqrt{I_1^2 + Q^2}} = \frac{R_2}{R_1} \cos \theta \quad (6)$$

For most work the correction due to the cosine factor may be neglected, as it amounts to but a part in 6,000 for a phase angle of 1° , and is but a part in 1,000 at 2.5° , and the phase is large only at light loads where the accuracy needed is not as great.

In the measurement of Q , the deflection is due entirely to the quadrature component of x .

From (5)

$$x_{(i)} = \frac{(I_1 r p L_1 - I_2 r p L_2) - Q R_2 r - (I_1 R_1 p l - I_2 R_2 p l)}{r^2 + p^2 l^2}$$

Since for the ratio setting $(I_1 R_1 - I_2 R_2)$ is made zero, we have

$$x_{(i)} = \frac{p r (I_1 L_1 - I_2 L_2) - Q R_2 r}{r^2 + p^2 l^2}$$

From which, if we neglect $p^2 l^2$ in comparison with r^2 ,

$$Q = \frac{1}{R_2} (-x_{(i)} r) + \frac{p(I_1 L_1 - I_2 L_2)}{R_2}$$

$$\therefore \theta = \tan \theta = \frac{Q}{I_2} = \frac{-x_{(i)} r + p(I_1 L_1 - I_2 L_2)}{I_2 R_2}$$

Now, since $I_2 R_2 = I_1 R_1$, we have, finally,

$$\theta = \frac{-x_{(i)} r}{I_2 R_2} + \left(\frac{p L_1}{R_1} - \frac{p L_2}{R_2} \right) \quad (7)$$

The numerator of the first term represents the emf. measured by the quarter phase dynamometer, while the expression in the parenthesis represents a constant correction due to difference in the time constants of the two shunts.

Thus for the most accurate work, constant corrections have to be applied to the measured phase angle both in the case of the potential transformer (equation 4) and in the case of the current transformer (equation 7). With the assumptions made above in regard to signs; these corrections, when positive, increase the angle of secondary load over the value that would be calculated by the deflection alone.

It should be noted that the negative sign before the first term of the right members of equations 4 and 7 is without significance so far as giving a sign to an experimental value is concerned, since it would require the interpretation of the direction of a dynamometer deflection. In order to determine whether a given deflection means a lagging or a leading secondary voltage in the case of a potential transformer, we have only to note whether a small noninductive load added to the secondary increases or decreases the deflection, since a noninductive load always tends to rotate the vector representing the secondary emf. backward. Normally, θ starts at no load with a small positive value (secondary leading) and changes to a negative value for full-rated non-inductive load. An inductive load makes the secondary lead still more.

In the case of the current transformer the secondary current usually leads, θ becoming negative only for very inductive loads. The angle of lead is always greater for a small current than for full-load current.

For the most accurate work the moving coil of the series dynamometer should be set at the point of zero mutual inductance, and that of the quarter phase dynamometer worked as near the point of zero mutual inductance as practicable. Also, considerable care must be taken to keep the leads twisted together to avoid induction from stray fields.

With the dynamometers used, a maximum sensibility, in the case of good transformers at full-load current, of about one part in 20,000 in ratio and 0.1' in phase angle was obtained by using from 0.1 to 0.4 volt drop in the shunts, but of course the final accuracy is less. This requires but a few hundredths of an ohm to be placed in the 5-ampere secondary of the transformer, which is comparable to the resistance of connecting leads as ordinarily used. The energy necessary to operate the instruments is taken chiefly from the primary instead of from the secondary.

As the sensibility in making the ratio settings varies inversely as the square of the current, dynamometers having two current ranges were ordinarily used in order to keep up the sensibility at low loads. The sensibility of the measurement of phase angle varies inversely with the load.

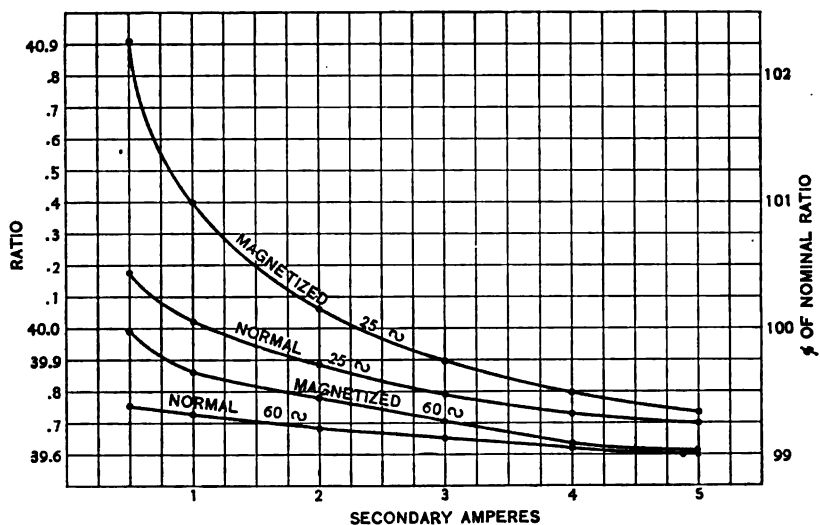


Fig. 26.—Showing effect of magnetic history on the ratio of current transformer A.

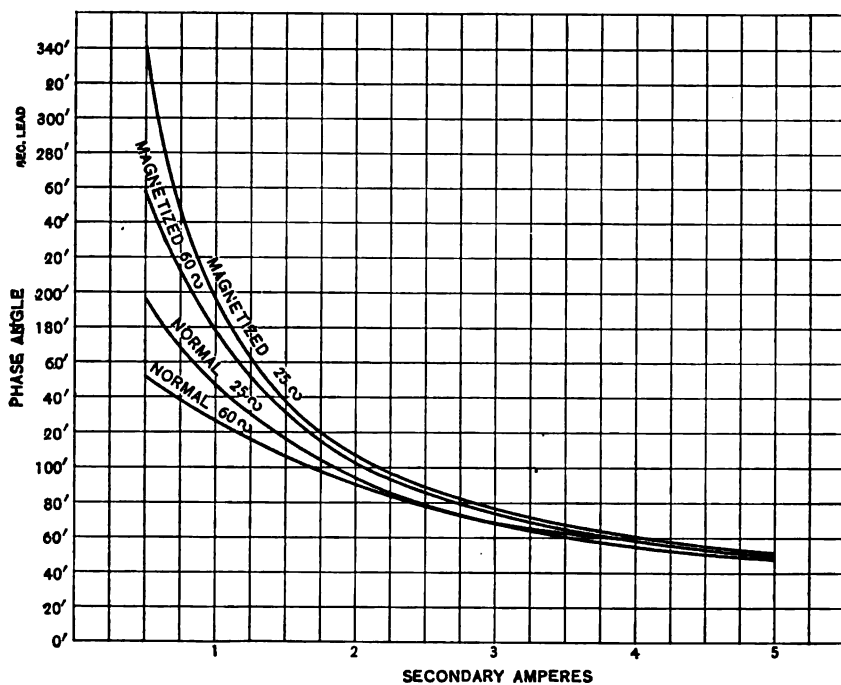


Fig. 27.—Showing effect of magnetic history on phase angle, current transformer.

The multiplier of the quarter phase instrument was usually adjusted to give a deflection of 1 centimeter per millivolt, and the calibration was made by introducing the emf. at the terminals of a resistance of 0.01 ohm placed in series with the field coil, as explained above.

In Figs. 12 to 25 are shown ratio and phase angle curves for different types of secondary loading. For convenience of reference both the actual ratio and the ratio expressed in percentage of nominal ratio are given on the vertical scale.

TABLE II.
Data for Current Transformers.

Transformer	Primary current	Nominal ratio	Rated frequency	Watt rating	Working voltage	Secondary resistance
A	200	40/1	25-125	40	15,000	0.463
B	200	40/1	25		12,000	0.353
C	10	1/1	25		220	0.050
D	250	50/1	25-100		12,000	0.204
E	200	40/1	25-125	40	15,000	0.70

EFFECT OF MAGNETIC HISTORY.

It does not seem to have been before noted that both the ratio and the phase angle of a current transformer may be changed by the past magnetic treatment it has received.

If a direct current of full-load value be passed through either winding, the flux rises to an abnormally high value, and when the circuit is broken the iron does not return to a neutral state, but is left in a state in which its differential permeability is less. Hence, when it is again used to measure an alternating current, it draws a higher magnetizing current than before, and both its ratio (primary to secondary current) and its phase angle are increased. The effect is precisely the same if the secondary be accidentally open-circuited while alternating current is passing through the primary, although the amount by which the calibration curves will be shifted depends upon the point in the cycle at which the circuit happens to be closed. Figs. 26 and 27 show the effect of such treatment in raising the entire curves, both of ratio and of phase angle, in typical cases.

This effect has been found to be very appreciable in every transformer investigated, and is also present with every condition of secondary load. As indicated by theory, it is greater the greater the impedance in the secondary circuit, and decreases as the load increases. The change in ratio at full load is nearly always about 0.1 per cent, but at one-tenth load it may be anywhere from one-half to five or even ten per cent, depending upon the quality of the transformer core and the amount of resistance and inductance in the secondary.

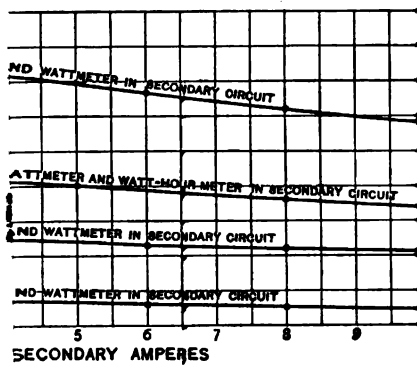
This is no doubt a frequent source of serious error in measurements requiring the use of current transformers. It is clear that the secondary of a current transformer should never be open-circuited while current is passing through the primary, and that the direct current used in calibrating standards, sometimes used with them, must not be allowed to pass through the transformer.

Fortunately, the effect may be entirely removed, in case the secondary has been accidentally open-circuited, by demagnetizing the core. This may be conveniently done by opening the secondary, passing full load, or an overload alternating current through the primary, and continuously reducing it to zero.

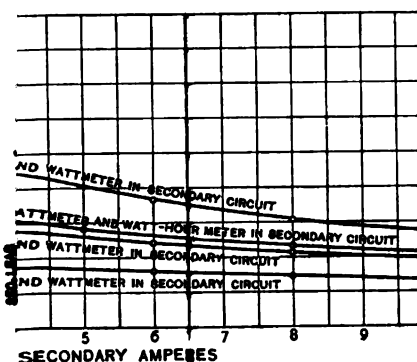
DETERMINATION OF THE INDUCTANCE OF SHUNTS.

The determination of the residual inductances of low resistance shunts presents great difficulties if attempted by ordinary bridge methods, as the inductance of the connecting leads is many times the inductance to be measured. For example, the inductance of a millimeter wire 1 centimeter long is 4.5×10^{-9} henry. A 0.001 ohm shunt having the same inductance will show a phase angle of 6' at 60 cycles.

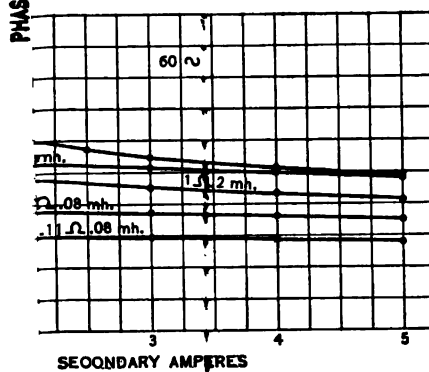
The quarter-phase dynamometer can be used to compare the inductances of two shunts, if a known current be passed through them in series (Fig. 28). If the inductance of one of them is known, and if a phase-shifting device is at hand, the comparison may be conveniently made by calculating the reactance emf. of the standard shunt at the given current and frequency, and then shifting the phase of the field of the dynamometer to give the calculated deflection when the shunt terminals are connected to the moving coil. This brings the two currents into exact quad-



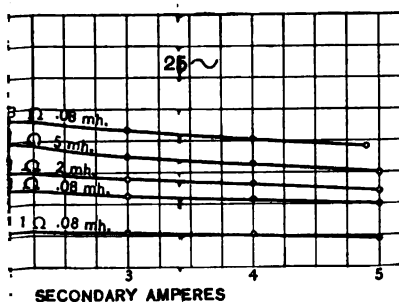
curves, current transformer C.



angle curves, current transformer C.



es, current transformer D. 60 cycles.



es, current transformer D. 25 cycles.

nature. (Care must be taken that the deflection is in the proper direction, or the emf. may contain a resistance drop twice as great as the reactance drop, as well as the reactance drop. The two will be opposed so that the deflection may have the proper magnitude, but be in the wrong direction. One may make certain of having the correct connection by lagging the field current, when the deflection should decrease.) Now, if the moving coil be connected to the unknown shunt, the deflection measures the reactance emf. directly, or the deflections are proportional to the inductances and are independent of the resistances.

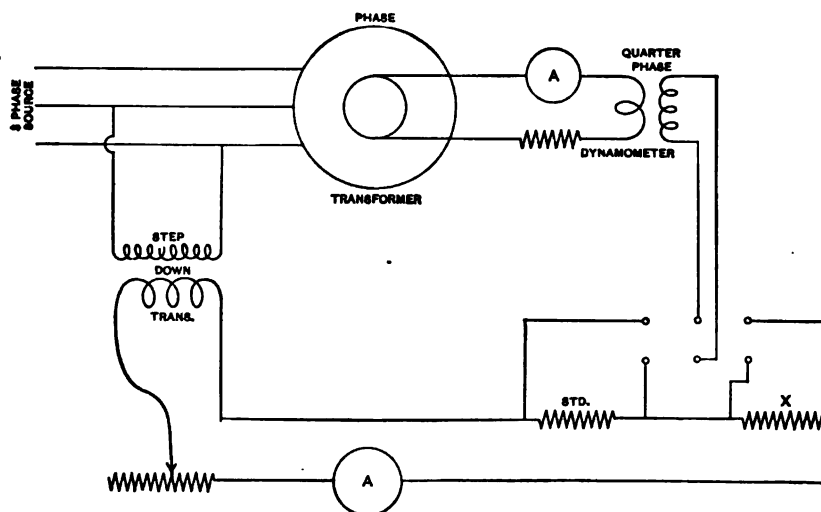


Fig. 28.—Connections.

As a check on the method, two shunts were constructed of forms that could be calculated, one concentric and one a flat shunt, doubled back upon itself. The comparison agreed to within 1 per cent of the calculated values. The values of the inductances compared were 3×10^{-8} and 126.8×10^{-9} henry.

Values of the phase angle of heavy low resistance "noninductive" shunts found by this method varied from 3 to 13 minutes at 60 cycles.⁶

WASHINGTON, July 5, 1909.

⁶ We are indebted to Mr. H. B. Brooks for valuable suggestions during the course of the work, and to Mr. W. H. Stannard for assistance in taking readings and in the preparation of the data.

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CONTENTS

	Page
Selective Radiation from Various Solids, II W. W. Coblentz	301
Luminous Efficiency of the Firefly Herbert E. Ives and W. W. Coblentz	321
Luminosity and Temperature P. G. Nutting	337
A Theoretical and Experimental Study of the Vibration Galvanometer	
. Frank Wenner	347
Specific Heat of Some Calcium Chloride Solutions Between -35° C and $+20^{\circ}$ C	
. H. C. Dickinson, E. F. Mueller, and E. B. George	379
On the Definition of the Ideal Gas Edgar Buckingham	409



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1910

SELECTIVE RADIATION FROM VARIOUS SOLIDS II.

By W. W. Coblenz.

While it is evidently important to improve upon the accuracy of the results already at hand by reinvestigating the known emissive properties of matter, it is equally important to investigate unknown substances. Believing that only through an immense amount of routine work will any marked advance be made in our knowledge of the laws governing the spectral partition of energy at various temperatures, it is intended from time to time to examine and place on record the behavior of representative examples of various groups of chemical compounds. Many of these substances will probably need no further investigation, but there are already a considerable number known, having unusual emission spectra, which will deserve a detailed investigation at some future time. The immediate application (particularly the commercial application) may not always be apparent, just as was the investigation of the reflecting power of metals made some six years ago, which until recently has had but little application.

In previous papers¹ attention was called to the fact that the intensities of discrete spectral lines increases in proportion to the energy input, i. e., with the temperature, while in the case of the solids investigated, which have banded emission spectra, the distribution of intensities among the emission bands varies with rise in temperature. The distribution of energy seems to follow a law similar to that of a complete radiator in which the wave length of maximum intensity decreases with rise in temperature. In other words, if we draw the envelope through the points of maximum intensity of the emission bands, then the point of maximum intensity of the envelope shifts toward the

¹ Coblenz: this Bulletin, 4, p. 533; 5, p. 159.

shorter wave lengths with rise in temperature, just as is true of substances, e. g., metals, emitting a perfectly smooth continuous spectrum. This was well demonstrated in the previous investigation of the isochromatics of the Nernst glower and certain other substances. The isochromatics were shown to have a double curvature, at the temperature at which the wave length of the maximum of the envelope coincided with the "isochromatic" wave length. This is in marked contrast with the growth of intensity of emission lines in gases, in vacuum tubes, in which the effect of (vapor) density is negligible. In the latter all the isochromatics² have the same trend (i. e., none change in their curvature), showing that, for the range investigated, there is no shift in the wave length of maximum emission but simply a gradual increase in intensity of all the lines with rise in temperature.³ This is corroborated by Nutting and Tugman⁴ by photometric comparisons of helium lines. On the other hand, the high-pressure mercury lamps⁵ show a variation in the distribution of intensities among the emission lines, with rise in temperature, and hence in the density of the mercury vapor.

The following emissivity curves of talc illustrate to an unusual degree the gradual shift of the maximum intensity of emission bands toward the shorter wave lengths without the merging into a continuous spectrum which was observed in the Nernst glower.

RADIATION FROM MINERALS.

TALC ($\text{H}_2\text{Mg}_3\text{Si}_4\text{O}_{12}$).

In Fig. 1 are given a series of spectral-energy curves of a layer of powdered talc rendered incandescent on a heater tube as in previous experiments. Curves *a* and *b* belong to a different series from the rest. The temperatures (roughly estimated) varied from 500° C., curve *a*, to about 900° in curve *e*. In Fig. 2, these same curves are shown on a larger scale. There are three sharp emission bands, at 2.42 μ , 2.75 μ , and 3.35 μ , respectively, which deserve especial notice. In curve *a* the band at 3.35 μ is as intense as the

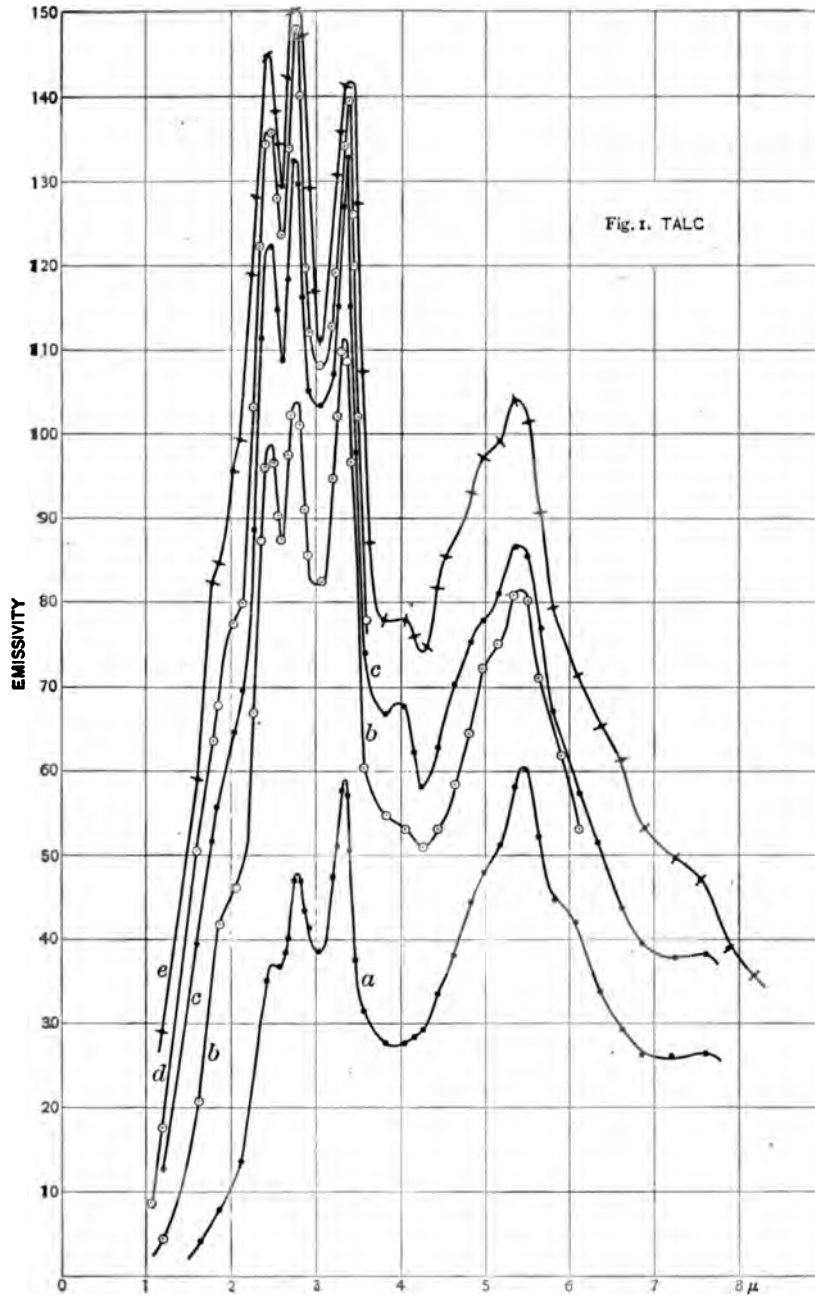
² Coblenz: Carnegie Publication No. 35, p. 318.

³ See also this Bulletin, 5, pp. 181 et seq.

⁴ Nutting and Tugman, *Nature*, 81, p. 189; 1909.

⁵ Pflüger, *Ann. der Phys.* (4), 26, p. 789; 1908.

Küch and Retschinsky, *Ann. der Phys.*, 20, p. 563, 1906; 22, pp. 595 and 852; 1907.

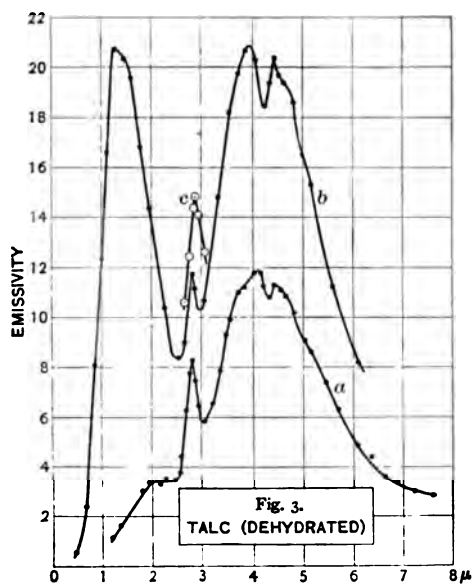
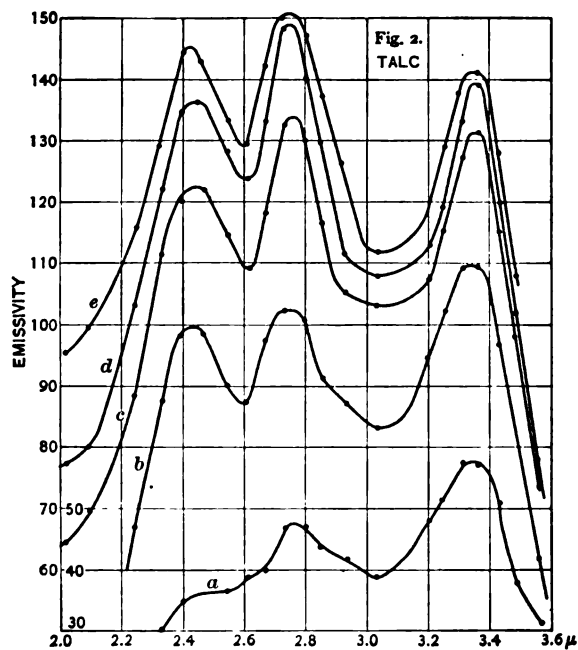


one at 5.4μ , while in all of the remaining curves the spectral region at 2μ to 3μ is the more intense. In this region the band at 3.35μ , curve *a*, is far more intense than those of shorter wave length, but with rise in temperature the bands at 2.42μ and 2.75μ become the most intense. In no case does the position of the maximum of the well-defined emission bands shift appreciably with rise in temperature. In Fig. 3 are given two energy curves of a rod of dehydrated talc, heated electrically as in previous experiments. The rod was made in an oxyhydrogen flame and was about 4 mm thick. Curve *b* gives the spectral energy distribution of the hottest side of the rod including the bright electrically conducting streak. Curve *a* is for the coolest side. As a result of dehydration, the maximum at 2.75μ in talc is shifted to 2.79μ , the products after dehydration being the oxides of magnesium and silicon. In Fig. 3, curve *c* is for oligoclase, previously described, with a sharp maximum at 2.87μ . In Fig. 4, curves *c* and *d* give the spectral emission of a similar rod of dehydrated talc heated to a low red. In curve *d* the temperature is somewhat higher, and the intersection of the two curves is due to the difference in the scale of ordinates.

MAGNESIUM SILICATE (MgSiO_3).

This is a sample of the pure material obtained from Dr. Allen, of the Geophysical Laboratory. The rod was electrically heated as in the case of the dehydrated talc. The spectra of the two substances (Fig. 4) are quite similar to magnesia;⁶ but in the minor details there is a marked difference, the 2.8μ silica band being weaker and the bands at 3.7 , 4.1 , and 4.5μ being more prominent in the pure magnesium silicate than in the dehydrated talc. The weakness of the silica band at 2.8μ is no doubt due to the fact that, in the pure silicate, the ratio of MgO to SiO_2 is 1 : 1, while in the dehydrated talc the proportion of silica is greater. Although these spectra seem to be the composite of the emission bands of the individual oxides, it is not generally admitted that the substances are solid solutions, but true chemical compounds. This will be noticed again on a subsequent page.

⁶ See this Bulletin, 5, p. 169.



DIASPORE ($\text{AlO}(\text{OH})$).**HYDROTALCITE** ($\text{Mg}_3\text{Al}(\text{OH})_6 + 3\text{H}_2\text{O}$).

In Fig. 5 are given the emission spectra of diaspore, curves *a* and *b*, temperatures 500° to 600° and of hydrotalcite, curve *c*, and temperature 400° . The spectra have no marked bands, except diaspore at 2.78μ .

CYANITE (Al_2SiO_5).

In Fig. 6, are given the spectral emissivity curves of cyanite (powder, on heater) the estimated temperatures varying from 500° , curve *a*, to 900° , curve *c* (not drawn to same scale). In this substance, the spectrum of alumina seems to predominate, in marked contrast with the preceding.

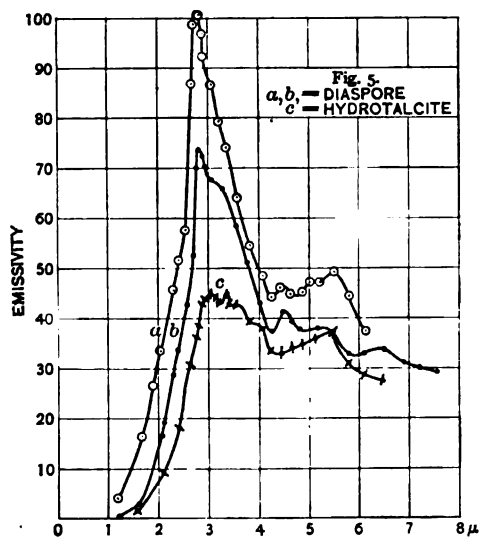
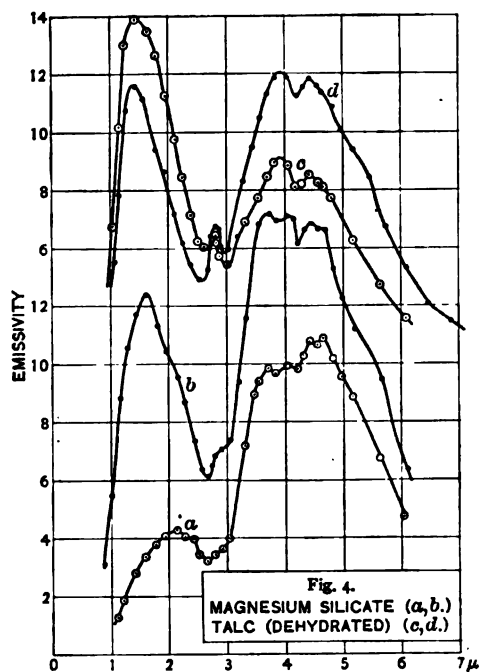
ZIRCON (ZrSiO_4).

Two spectral energy curves of powdered (brown color) zircon are given in Fig. 7, the temperatures being perhaps 600° and 700° , respectively. The scale of ordinates of curve *a* are twice those of curve *b*. Numerous small emission bands are noticeable, which are somewhat shifted from the absorption bands of a transparent plate, having maxima at 2.1μ , 3.1μ , and 3.6μ , respectively.

TOPAZ [$(\text{AlF})_2\text{SiO}_4$].

In Fig. 8 are given the energy curves of powdered topaz on a heater tube, the temperatures ranging from perhaps 500° to 900° , curves *a* to *c*. In this illustration the scale of curve *a* = 3.6 *b* = 6.7 *c*. The examination was made to compare with previous curves⁷ of a rod of topaz made in an oxyhydrogen flame, hence probably decomposed. Although the temperature of the powder was from 500° to 600° lower than that of the rod previously examined, the spectral region with an intense sharp maximum at 2.8μ is higher than the region at 4.5μ , which is just the opposite of the radiation curves of the rod. This is strong evidence that in the rod the mineral was decomposed into the constituent oxides previously examined. In Fig. 9 are given the spectral radiation curves of an electrically heated rod made of alumina and about 1 per cent feldspar, curve *a*, and a rod of alumina and 1 per cent

⁷ This Bulletin, 5, p. 172.



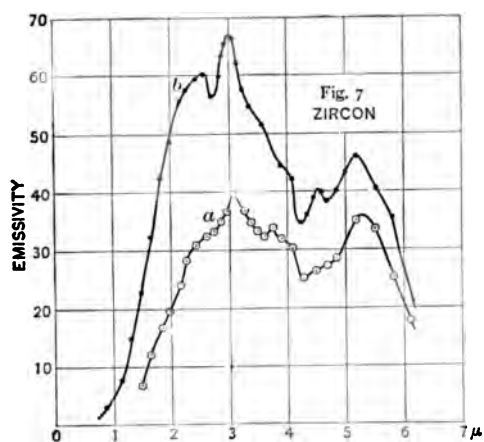
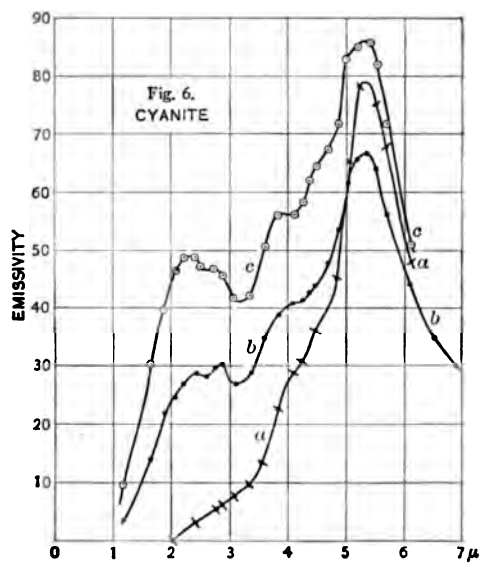
silica, curve *b*. The radiation curves are very similar to those obtained on the decomposed topaz, and seem to indicate a solid solution instead of a true chemical compound; for the spectra appear to be the composite of those of the individual oxides, while in a true compound the individuality of the spectra of the constituents is destroyed and an entirely new spectrum is formed. An excellent example of the latter is zircon, Fig. 7, which has an entirely different spectrum from the pure oxides. Our knowledge of the constitution of mixtures of oxides having high melting points is extremely limited. Whether mixtures of these oxides form a definite compound or simply a solid solution is a mooted question. To melt them in a crucible and keep them pure is a difficult undertaking. The methods of analysis here employed may therefore be of use in furnishing qualitative information as to the physical condition of mixtures of the highly infusible oxides.

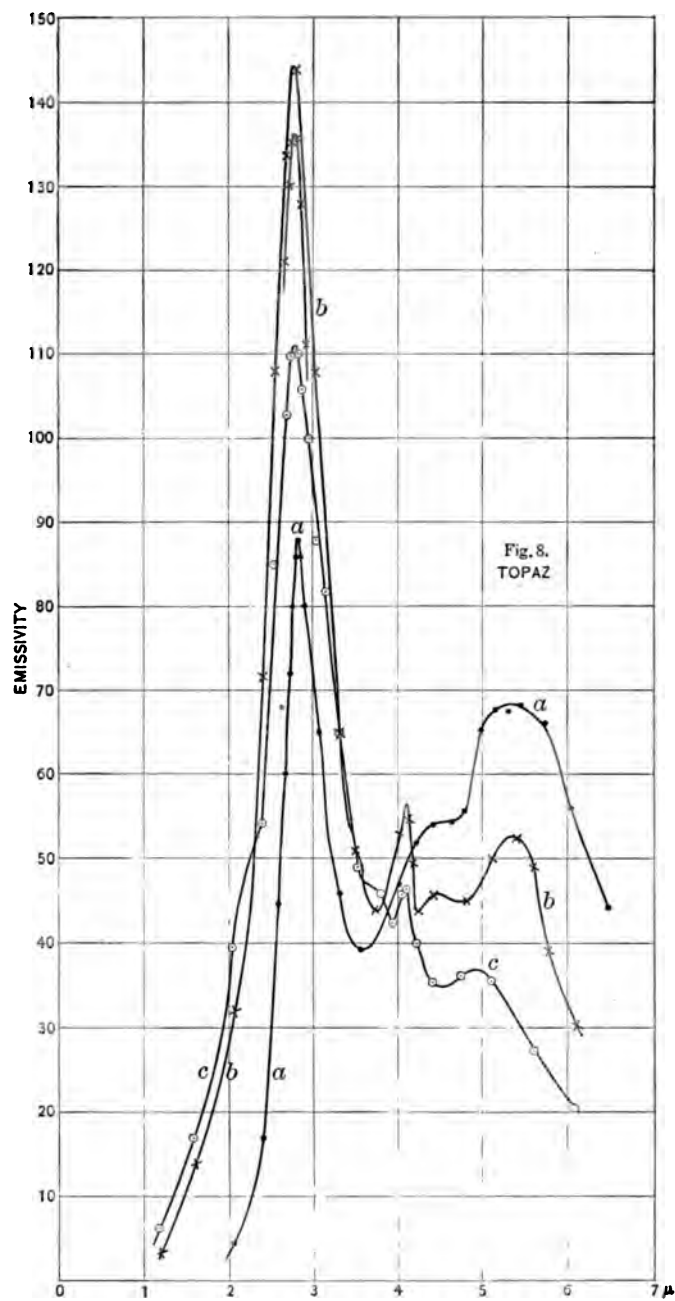
RADIATION FROM METALS.

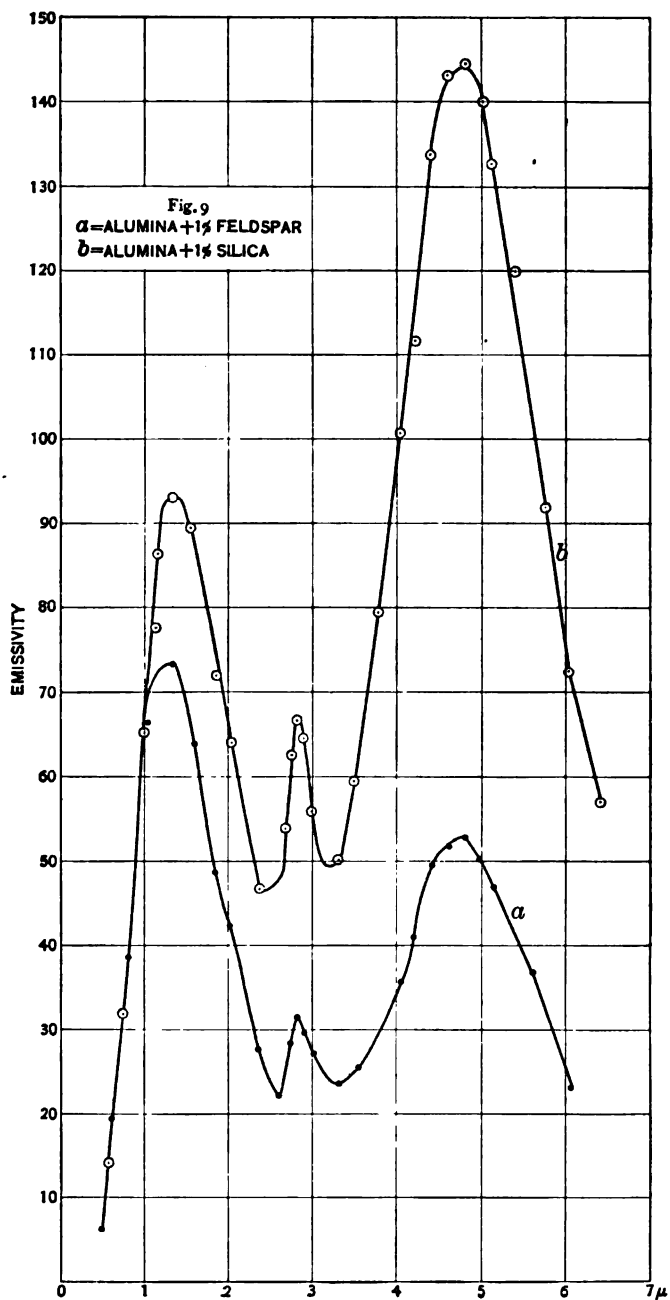
The question of the luminous efficiency of incandescent filaments has occupied the attention of experimenters from the earliest work on the so-called glow lamp. Evans⁸ described comparisons of the radiation from bright and from black incandescent lamp filaments of carbon, in which he demonstrated very clearly the superiority of the former as a source of light. He selected for this purpose two filaments of similar size and structure, upon one of which he deposited a silver-gray coating of carbon and upon the other a hard coating of carbon of the color of lamp black. He showed that for the same energy input the bright filament emitted the more light. The lamp bulbs containing the black filaments were found much hotter than those containing the silver-gray filaments. He remarks: "I have little doubt that the loss of efficiency when black was due to the energy supplied being radiated in large quantities as heat waves from the blackened surfaces, which these surfaces when bright would not emit." Weber⁹ showed that the total radiation of the untreated carbon was very much greater than the flashed when operated

⁸ Evans, *Proc. Roy. Soc.*; Feb. 18, 1886.

⁹ Weber, *Phys. Rev.*, **2**, pp. 112 and 197; 1894.







upon the difference in their reflectivity in the visible spectrum. It might, therefore, have been better to set the two filaments to the same emissivity bolometrically, in the infra-red, and observe their difference in emissivity in the visible spectrum, instead of the reverse process herewith presented. The filaments (if metals) would then be operating at more nearly the same temperature than on a color match. The luminous efficiency would then be obtained at more nearly the same temperature, which is an important desideratum. Instead of a spectrobolometer for making this intensity match in the infra-red it may be possible to use a screen which has a narrow transmission or reflection band, in the infra-red, but opaque to the visible spectrum.

In view of the great importance attributed to the so-called selective emission of the metal filament lamps in the infra-red it is worth while to discuss some recent data on this subject before describing the present results. Nyswander¹³ has published a series of spectral energy curves of an incandescent tungsten filament which apparently indicates bands of selective emission in the infra-red, but which in reality are due to atmospheric absorption bands, and also to probable inaccuracies in the so-called slit width correction which is difficult to determine at 1.7μ , in fluorite prisms. In his earlier work the writer experienced the same difficulties, but after eliminating them, the spectral energy curves of a uniformly heated cavity, as well as those of metal filaments, were always found to be smooth and continuous except at 3.1μ , which band seems to be due to atmospheric oxygen. From this and from our knowledge of the reflectivity of metals it seems quite certain that in pure metals the spectral radiation curves do not show indentations due to bands of selective emission. Instead of indentations, the emissivity of the metal filaments is greatly and uniformly suppressed in the infra-red, as compared with a complete radiator. (See fig. 10—incorrectly drawn at 0.7μ .)

In the present investigation the carbon and tungsten lamps were matched in color by several observers in the photometry section of this Bureau, and hence considerable accuracy can be claimed for it. The treated carbon filament was of the single loop type while the tungsten filament was a single loop "series lamp." To

¹³ Nyswander, *Phys. Rev.*, **28**, p. 438; 1909.

attain greater accuracy straight filaments should be used. The adjustments and procedure in the experiments was the same as in the previous paper. The only changes introduced was a more sensitive bolometer; and the prism was covered with a diaphragm

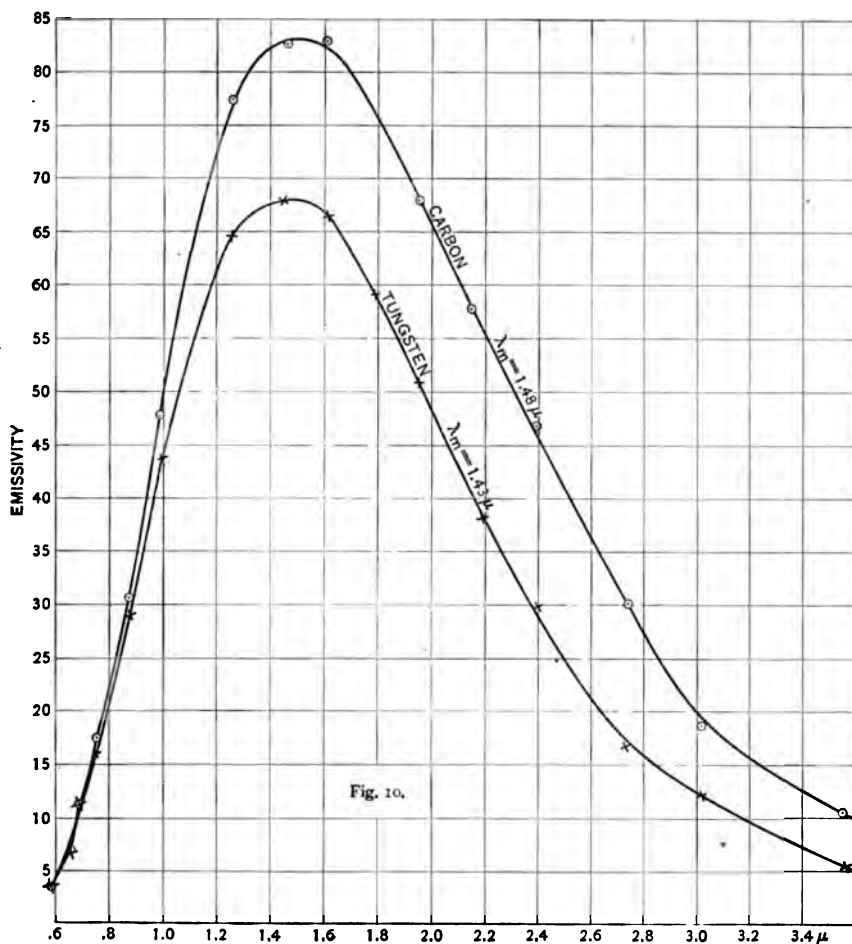
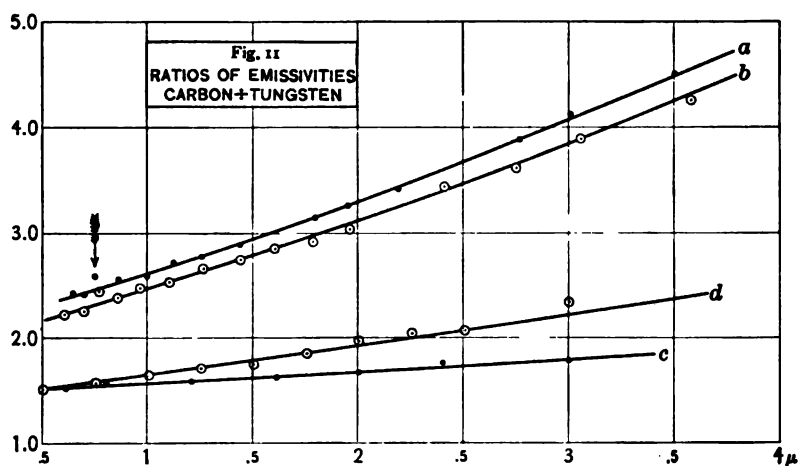


Fig. 10.

having an opening 8 mm wide to insure that, whatever the adjustments in the lamps, the same area of prism face was covered by the two filaments. In this manner it was hoped that in varying the adjustments, the same ratios of intensities would be maintained after correcting for variation in sensibility of the bolometer. But

the curvature of the carbon filament did not admit uniform illumination of the prism face, so that while the ratios for various adjustments gave parallel curves (Fig. 11, *a* and *b*) an additional factor would have to be introduced in order to superpose the curves. This, however, has nothing to do with the present question of the rapid increase in the ratios as we leave the visible spectrum.

In Fig. 10 are given the spectral distribution of energy of a treated carbon and a "series" tungsten lamp having filaments about the same diameter. The wave-length of maximum emission of the carbon lamp was $\lambda_{max} = 1.48\mu$ and of the tungsten lamp it was $\lambda_{max} = 1.43\mu$. Beyond 2.5μ the curves are not com-



parable on account of the absorption of the glass bulb. The observations on the tungsten lamp were multiplied by a common factor which reduced the intensities to equality at 0.58μ . Here the deflections are small, and while the errors of observations are large, the effect is insignificant in the energy curve. At 0.75μ the energy curves already depart from equality by about 5 per cent. Here the errors of observation are small because of the large deflections which are at least 5 times those at 0.58μ . In three series of measurements made on different carbon lamps, the energy curves showed an abnormal rise in value at 0.75μ . This rise in emissivity is similar to the spectrophotometric observations of Nichols,¹⁴ but

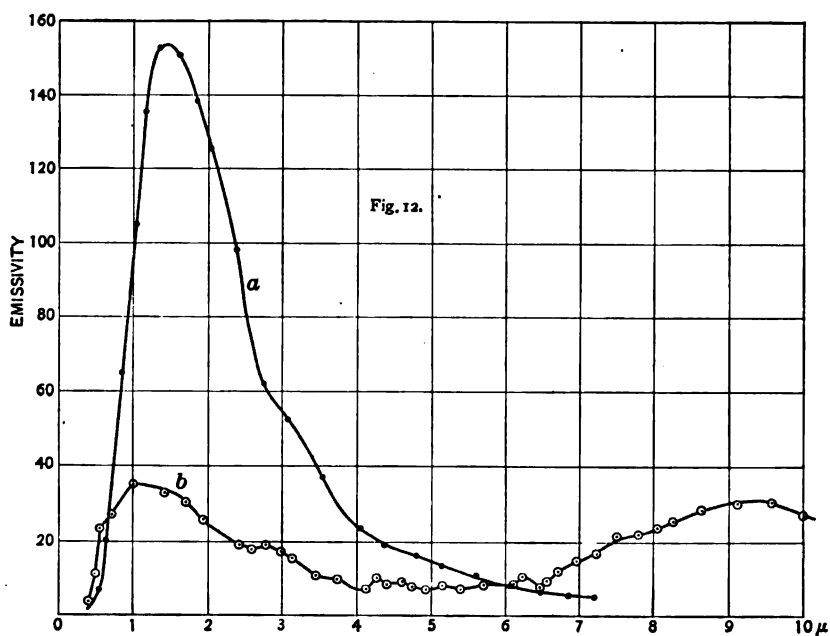
¹⁴Nichols, E. L., Phys. Rev., **18**, p. 65; 1901.

whether it is a reality would require further investigation. This should be done, using a larger dispersion, since the radiometric observations can be made with considerable accuracy in this region, where the spectrophotometer becomes untrustworthy.

In Fig. 11 are given the observed ratios of the emissivities of these two lamps. It will be noticed that at 0.75μ the ratios deviate from the general series by a greater amount than can be attributed to experimental error. Curves *a* and *b*, Fig. 11, give the ratios of the emissivities (carbon $\div$ tungsten) for a color match when the carbon lamp was operated on about 4.6 w. p. c. (lamp stationary; plane of the filament perpendicular to photometer); curves *c* and *d* are similar observations made at an earlier date, when the color match was less accurately determined and the energy input was different. Curves *a* and *b* are parallel, as they should be, since all the conditions remained the same except that the lamps were readjusted before the slit in the two series. That these two curves do not coincide is due to lack of uniform illumination of the prism in the two series, as explained on a previous page. If the color match gives us an accurate knowledge of existing conditions, then the bolometrically observed radiation curves of the visible spectrum should be parallel, but not necessarily of equal intensity. The ratio of intensities should therefore be the same throughout the visible spectrum, i. e., the curves should not slope. The curves herewith presented demonstrate conclusively that for a color match (and intensity match at 0.52μ) the bolometer shows a difference amounting to 3 to 5 per cent at 0.70 to 0.75μ . The farther the observations extend into the infra-red the greater the ratio of emissivities (carbon $\div$ tungsten), and whatever doubts we have regarding the accuracy of the observations in the visible spectrum, from the observations beyond 0.8μ there can be no doubt that the eye is incapable of distinguishing variations in emissivity in the deep red, when two lamps are apparently matched in color.

As a finale to this discussion of the inapplicability of the color match (or for that matter any other device than a spectroradiometer) for predicting what happens in one part of the spectrum from observed conditions in another region, let us select the Welsbach mantle and also the same material formed into an electrically heated rod. Let us suppose that it is difficult to estimate

the efficiency of the mantle; we therefore select the electrically heated rod and determine the watts per candle. Assuming that the emissivity of the oxides is dominated by surface conditions, as in metals (in which the absorption coefficient is extremely large as compared with oxides), we decide from our observations on the electrically heated rod that the same relation of watts per candle must hold in the mantle. We might even conclude that, on account of the cellular structure of the mantle, it is "blacker" than the smooth rod, and hence is less efficient as a source of light.



To what absurdities such assumptions can lead will be noticed in Fig. 12, which shows the distribution of energy in the spectrum of the Welsbach mantle, curve *b* (data from Rubens), and of the same material (99 p. c. thoria 1 p. c. ceria) in the form of a rod heated electrically. In the latter, curve *a*, Fig. 12, the vertical scale is one-third that of the mantle. It will be observed that the mantle is almost lacking in infra-red radiation. The reason for this is not difficult to find. The oxides are "transparent media," and hence the emissivity is a function of the thickness rather than of the reflecting power (high absorption coefficient), as in metals. The

small amount of cerium-oxide, which has a high absorption for all the visible spectrum except the yellow and red (combined with a high temperature), is sufficient to produce a saturated radiation in the region of short wave-lengths, but in the infra-red the emissivity is low, due to the lack of thickness of the mantle. On the other hand, the (electrically) heated rod is so thick (1.2 mm in this experiment) that, although the absorption coefficient is low beyond 1μ , the emissivity is high, due to the thickness of the radiating layer.

WASHINGTON, August 14, 1909.

LUMINOUS EFFICIENCY OF THE FIREFLY.

By H. E. Ives and W. W. Coblentz.

In connection with an investigation of various sources of illumination, it was deemed of scientific interest to examine the light emitted by the common firefly. It has been an open question whether the narrowness of the spectrum of the firefly is due to its low intensity (i. e., whether the spectral energy extends into the infra-red) or whether the light emitted consists of only a narrow yellowish green band which terminates abruptly in the red and in the blue.

About twenty years ago, Langley and Very¹ made a photometric and bolometric study of one species of firefly, "*Pyrophorus Noctilucus*," found in Cuba. They found that the radiation of this insect consisted of a band in the yellow-green of the spectrum, which decreased rapidly in intensity toward the red and the blue. The spectrophotometric work was confessedly unsatisfactory, being done visually and hence hindered by the unsteady character of the light, as well as by its low intensity. The authors satisfied themselves that the spectrum, as compared with sunlight, was shorter, not extending far into the blue, or beyond the red, and that its maximum lay at 0.57μ . The accuracy of the visual work was not sufficient, because of the difficulties encountered, to draw an accurate energy curve, or to detect minute spectral structure if it existed. Because of the small amount of energy and of its unsteady character, direct determination of the energy in the visible spectrum by bolometric means was hopeless.

¹ Langley and Very: "The Cheapest Form of Light." Amer. Jour. Sci., 3d ser., vol. 11; 1890. Reprinted in Smithsonian Misc. Coll. No. 1258; 1901.

In 1902 (see *Annals Astrophys. Obs.*, Vol. 2, p. 5) further experiments were made on the relative amount of heat and light in the radiation of the Cuban firefly. No heating whatever could be detected by the bolometer. "A portion of the flame of a standard sperm candle, equal in area to the bright part of the insect, gave, under the same circumstances, a bolometric effect of such magnitude that had the heat of the insect been $\frac{1}{8000}$ as great as this from the candle, it would certainly have been recognized." The candlepower of the insect was $\frac{1}{1000}$ candle.

I. THE QUALITY OF THE LIGHT EMITTED BY THE FIREFLY.

In the present investigation an attempt was made to secure a more exact idea of the energy distribution in the firefly light in the visible spectrum by means not available to the previous investigators. In place of visual observations use was made of photographic plates sensitive to the whole visible spectrum. The following considerations lead to the use of photography. With a radiation meter it is possible to measure radiant energy in the visible spectrum directly, provided the source measured is steady and has sufficient energy to affect the instrument. The energy must be considerable, since instruments of this kind are many thousand times less sensitive than the eye or photographic plate for radiation less than 0.65μ in wave length. With the spectrophotometer it is possible to measure the distribution of visible radiant energy, even if of low intensity, indirectly, by comparing the source with one of known energy distribution, provided again the source measured is steady. If the source measured is unsteady as well as weak neither method of measurement is applicable. For cases of this kind the photographic plate as a record of incident energy is the only resort. Its great advantages are, first, that all portions of the spectrum are recorded simultaneously, so that unsteadiness of total light is no disadvantage; second, the action of the plate is integrative, and therefore the obstacles imposed by low intensity may be overcome by long exposure. A further advantage for the present purpose is that the plates used, Wratten and Wainwright "Panchromatic," are sensitive to all colors of the spectrum and therefore record the whole spectral extent of the firefly light (if limited as indicated by Langley's work).

The apparatus consisted of a large prism spectrograph having a dispersion in the yellow sufficient to separate the yellow mercury lines by about half a millimeter, and giving a spectrum of about seven centimeters length in the visible. A helium tube and a carbon glowlamp, which was carefully calibrated for watts per candle, served for comparison sources, the one for a wavelength scale, the other as a standard of energy distribution.

The fireflies were of the species common in Washington early in July, "*Photinus Pyralis*." These are much smaller than the West Indian genus, averaging about 12 to 15 mm in length, of which length only about one-half is luminous. The light ordinarily consists of a faint glow, which at periods of a few seconds flashes out with greatly increased intensity. Under the microscope the light-giving portion is seen, when not giving out the bright flash, to be made up of numerous small irregularly scintillating points, reminding one of the spinthariscopes. It was at first attempted to secure sufficient light by inclosing a number of the insects in a small cage with a white wall from which the light would be reflected into the slit. It was found, however, that the fireflies in captivity quickly lost their desire to flash, and this scheme was abandoned. The only satisfactory method proved to be to hold the insects in the fingers, one or two at a time over the spectroscopy slit. The best specimens would flash as frequently as every three seconds until tired, when others would be substituted for them. Others, after a period of flashing, would yield a steady glow of considerable intensity. With a fairly wide slit, and taking the fireflies as they came, from two to six hours were required to obtain a satisfactory photograph. The work was done evenings and extended over about two weeks before three satisfactory negatives were obtained, one of long exposure, one of short, and one (short exposure) with a very narrow slit. The latter was taken with a view of detecting any irregular structure.

In Fig. 1 is reproduced the long exposure, together with the spectrum of the carbon glowlamp and of the helium tube. The firefly light is confined to a band in the yellow green, ending in the blue green on one side, in the red on the other. The limit on the red side is so far from the limit of sensitiveness of the plates as to leave no doubt that the firefly's spectrum does end within the

limits of the visible spectrum. Langley concluded from his bolometric work that there is no emission of energy in the infra-red. If this is true the complete energy curve may be obtained from the energy emitted in the visible spectrum. From the exposure made with a very narrow slit there is no evidence of fluted or other irregular structure.

The distribution of energy was obtained by measurements of the density of the photographic plates. As is well known, the relation between exposure and density is not the same for all exposures. For a limited range density is proportional to exposure, beyond this range saturation sets in, still farther away reversal. It is therefore necessary to know accurately the exposure equivalent of each density on the plate measured; in other words, to know the "characteristic curve." With this end in view a series of exposures of known relationship were made on the carbon glow lamp, in which the densities covered practically the whole range of those in the best two firefly negatives. All the negatives, it should be noted, were developed under identical conditions, by time, in complete darkness, and should therefore be as nearly comparable as possible without all the exposures in question being made on one plate. The densities of all negatives were measured by means of a Martens polarization photometer, mounted on a dividing engine, in such manner that strips 1 mm wide and 1 mm apart were compared for blackness against the clear unexposed plate. One of the helium lines served as reference mark, and the points of measurement were the same on all plates.

From these observations density curves were plotted as shown in Fig. 2, where the glow lamp curves correspond to exposures of 2, 4, 6, 8, 12, 20, 30, 60, 120, and 240 units; the two firefly curves represent a very long exposure, giving the limits of the spectrum, and a short exposure giving the maximum of the insect's light. Where the firefly exposure curves cross the glowlamp exposure curves the corresponding exposure units may be read off directly. For other points the characteristic curve must be drawn from the glowlamp points, and the firefly value determined by interpolation on the curve. By this procedure the unequal sensitiveness of the plates for different wave lengths is taken care of, and also any change in the density-exposure relationship for different colors.



Fig. 1.

- A. Spectrum of "4 watt" carbon glow lamp.
- B. Spectrum of fire-fly, "*Photinus pyralis*."
- C. Helium vacuum tube spectrum.



Fig. 6.

From the longest firefly exposure a set of points were so determined for the fainter portions of the energy curve; the stronger portions lay beyond any of the comparison exposure curves, in the

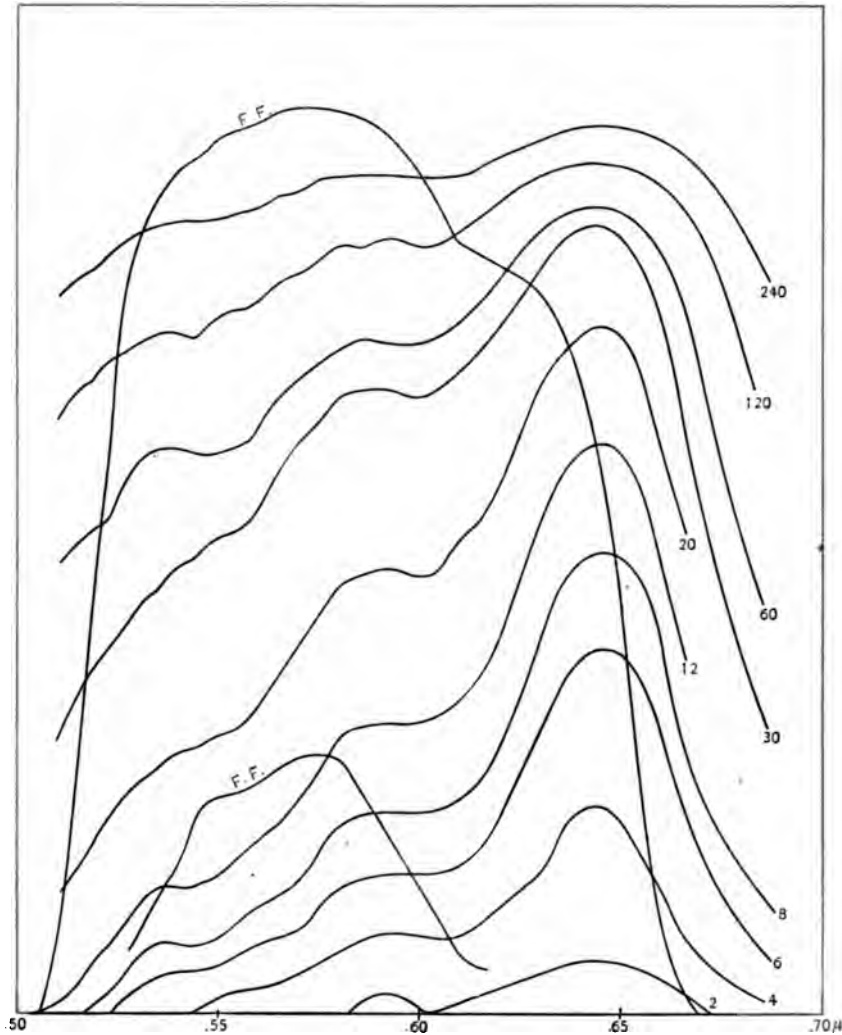


Fig. 2.—Density curves of photographic plates of firefly (F. F.) and glow lamp.

region of saturation, and were therefore unsuitable for measurement. From the short exposure, in which the densities were all below the region of saturation, the intermediate points of the

curve were determined. The two curves overlapped for a short range, from which the multiplying factor to bring them to a common unit was determined. From the values so obtained the ratio $\frac{\text{firefly light}}{\text{glowlamp light}}$ was derived for each point of density measurement. That is, there was obtained the spectrophotometric relationship between the two. The numerical values are given in Table I, column 5.

At this stage it becomes necessary to know the energy distribution of the carbon glowlamp. This can be obtained directly by radiometric methods, the source having the qualities both of steadiness and comparatively large amount of energy.

The spectrophotometric measurements of the complete energy distribution of the carbon glowlamp employed were made by two methods (the one suited best for the longer wave lengths, the other for the shorter), which gave very concordant results over their common range. In the first method an improved and highly sensitive bolometer, a fluorite prism, and mirror spectrometer² were employed to obtain the spectral distribution of energy from 0.55μ to 4μ in the infra red, where the glass walls of the lamp become opaque. The numerical values of the spectral intensities as obtained by this method are given in Table I, column 3. The lamp (a 110-volt, 16-candlepower, 4 watts per mean horizontal candle, treated carbon filament) was operated at a fixed voltage corresponding, as determined by careful calibration, to 3.99 watts per mean horizontal candle. For the type of filament (oval anchored) this means 4.83 watts per mean spherical candle, or 2.60 lumens per watt. The unit of candlepower is the new International Candle, 1.6 per cent lower than the unit as hitherto maintained at the Bureau of Standards. The black body temperature for red light ($\lambda = 0.66\mu$) as observed with an optical pyrometer was $1790^{\circ} \pm 20^{\circ}$ C. The wave length of maximum energy is $\lambda_{\max} = 1.41\mu$; the computed radiation constant is $\alpha = 5.7$, which means that the emissivity of this treated filament is of the order of $(\alpha - 1) = 4.7$ th power of the absolute temperature. The observations were corrected for the variation in the reflecting

² For adjustment of apparatus, lamp, computations of radiation constants, etc., see papers by Coblentz, this Bulletin, 4, p. 391; 5, p. 339.

power of the silvered mirrors of the spectrometer (data from Hagen and Rubens) on the assumption that the mirrors were perfectly new, which was not the case. Hence the observations are to some extent undercorrected toward the violet.

TABLE I.

Emissivity of Carbon Glowlamp			λ	Firefly Glowlamp	Firefly energy dis- tribution
λ	Glass Prism	Fluorite Prism			
0.50 μ	3.9	. . .	0.500 μ	0	0
.52	5.5	2.6	.512	.65	.03
.54	7.5	. . .	.517	1.0	.05
.56	10.0	5.1	.521	2.0	.10
.58	12.9	. . .	.526	4.0	.22
.60	16.1	8.2	.531	8.9	.53
.62	20.1	10.1	.537	12.0	.79
.64	24.8	. . .	.545	16.0	1.15
.65	27.3	13.6	.550	16.3	1.26
.66	29.7	14.9	.555	16.0	1.31
.68	35.3	. . .	.560	15.5	1.365
.70	41.5	20.8	.565	15.1	1.40
.8		39.1	.575	13.0	1.40
.9		57.9	.578	12.0	1.35
1.0		73.4	.580	10.7	1.26
1.1		88.8	.586	9.0	1.095
1.2		100.5	.589	8.0	1.02
1.3		106.2	.598	6.0	.86
1.4		107.6	.608	4.0	.62
1.5		107.0	.621	2.0	.355
1.6		104.5	.627	1.0	.19
1.8		95.1	.637	.65	.135
2.0		82.8	.643	.4	.09
2.2		69.4	.647	.26	.06
2.4		56.0	.652	.20	.05
2.6		42.5	.659	.13	.035
2.8		30.6	.665	.06	.015
3.0		23.5	.670	0	0
3.5		12.5			
4.0		5.4			
4.5		.12			

The second method of measuring the distribution of energy (in the visible spectrum) was by means of a Rubens thermopile and a Fuess two-prism monochromatic illuminator, in which the lenses had a focal length of only about 10 cm. This gave a more intense spectrum and at the same time a larger dispersion than obtained in the preceding apparatus. This combination also eliminated the question of the reflecting power of the mirrors, which is difficult to determine in the visible spectrum. The numerical values are given in the second column of Table I. The complete energy curve, Fig. 3, was made by combining the two sets of observations

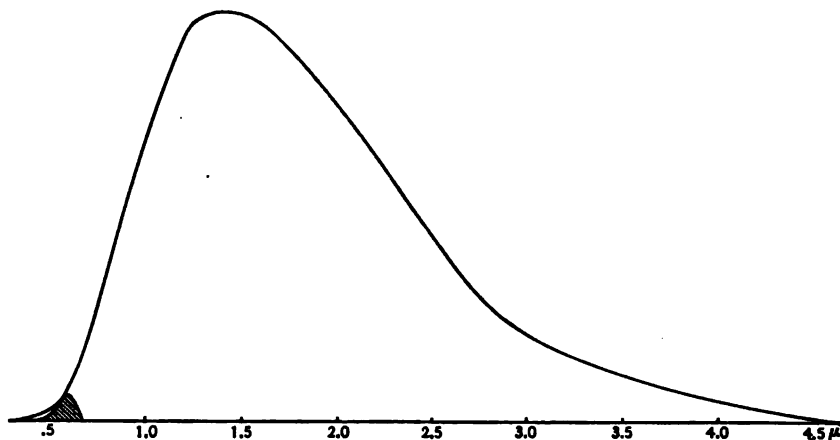


Fig. 3.—Luminous efficiency of the 4-watt carbon glow lamp. (Shaded area $\rightarrow$ total area.)

reduced to equality at 0.6μ . The two curves so obtained agreed over their common range as closely as one would expect, considering the difficulties in obtaining such measurements.

Having already obtained the spectrophotometric relationship between the firefly and the glowlamp (by photographic means) and now knowing the distribution of energy of the glowlamp, we can at once obtain the distribution of energy in the spectrum of the firefly by multiplying the energy values of the glowlamp by the ratio $\frac{\text{firefly light}}{\text{glowlamp light}}$ at each wave length. In Fig. 4 are given the two curves—the energy of the glowlamp in the visible spectrum, and the energy of the firefly. The points on the latter are

obtained by comparison with the smoothed curve of the glow-lamp; equality is assumed at 0.59μ . In Table I, column 6, are given the energy values so derived at wave lengths corresponding

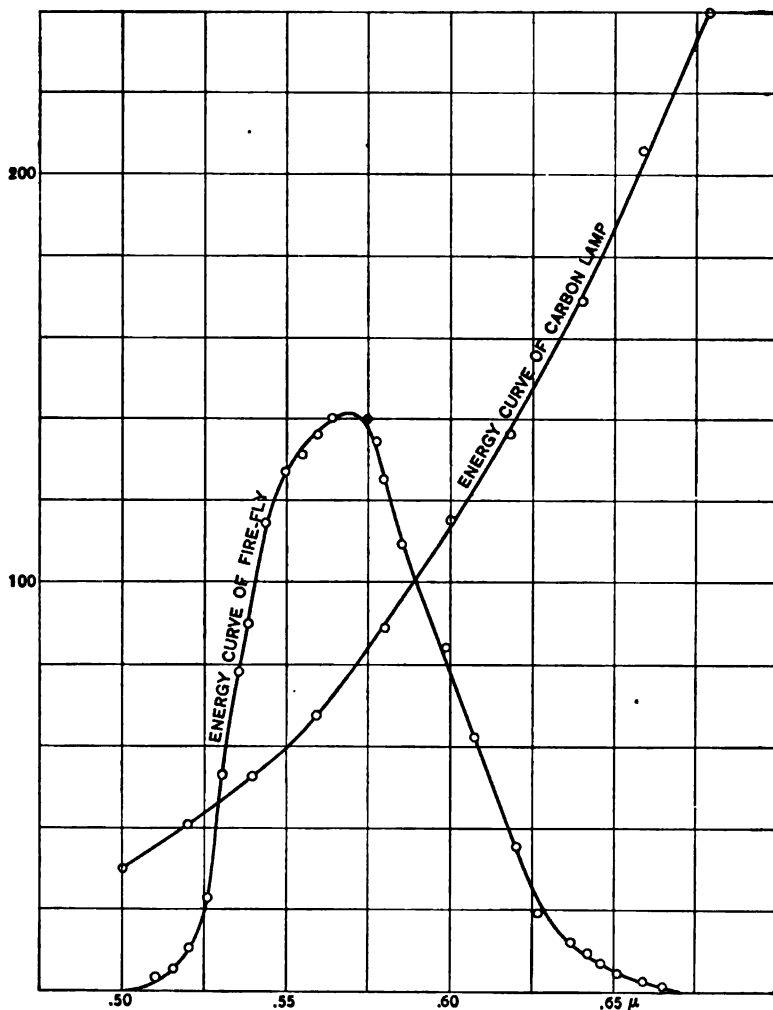


Fig. 4.

either to points of density measurement or to points of intersection of density curves, as explained above.

It will be seen that the maximum of light emission is in the part of the spectrum to which the eye is most sensitive, the yellow

green, at 0.57μ , and the extent of the spectrum is from 0.50μ to 0.67μ . Very much longer exposure might have shown greater length. It must be borne in mind that the measurements were made on extremely small areas, subject to errors from such local unevennesses in the plates as might occur³ and therefore it is believed no particular significance is to be attached to the apparent slight concavity on the red side of the maximum or the twisted appearance of the maximum itself. The greater extension on the red side makes it similar in appearance to the complete radiator. Since the distribution of energy in the spectrum of the sun is approximately uniform through this region, the spectrum of the firefly as compared with sunlight has very closely the same character as the energy curve shown.

It is of interest to compare the luminous efficiency of the firefly with that of the carbon glow lamp, and indirectly, with that of other luminants. If we know the relative visual intensity values of the different colors of the spectrum of a source of known energy distribution (the sensibility curve of the eye if the source gives a normal or equal energy spectrum) we can calculate the intensity value of any given energy distribution, and from it obtain the ratio
$$\frac{\text{light (radiated energy} \times \text{visual sensibility)}}{\text{radiated energy}}$$
 in common units. Koenig has determined the intensity distribution of a gas flame, and from the energy distribution of the flame as determined by Langley, Koenig's observations are expressible in terms of a spectrum of uniform energy distribution. These are given by Nutting⁴ and the values for intensities above the Purkinje effect are here used.⁵ The procedure is to multiply the energy at each point by the fraction representing its visual intensity value. In Fig. 3

³ A defect in the plate made it necessary to omit a density measurement at 0.57μ .

⁴ This Bulletin, 5, p. 261. The value unity is given to the maximum of intensity ($.565\mu$).

⁵ These intensity values are of course dependent on the accuracy of the gas flame energy values. These are much lower in the blue than for the glow lamp here used, which would suggest, since a gas flame is rather bluer than the 4-watt lamp, that a redetermination of the quantities in question by more sensitive means would change the intensity values used. The effect on the final values for luminous efficiency in this paper would not be altered appreciably by changes of the extent indicated by this apparent discrepancy.

this is done for the glowlamp and in Fig. 5 for the firefly. The shaded area in each case represents the energy available as light expressed photometrically or in terms of intensity. The ratio of

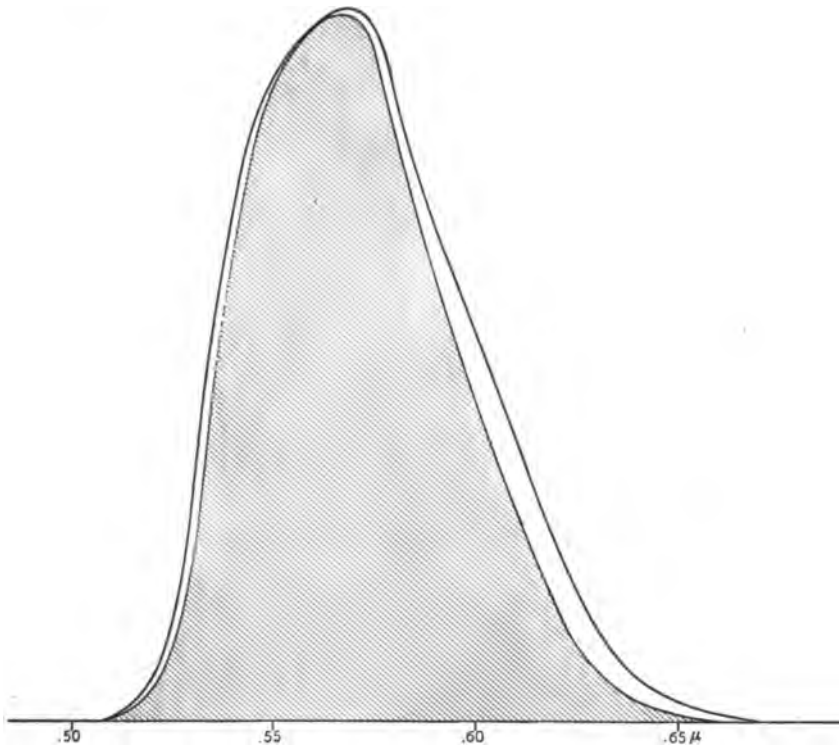


Fig. 5.—Luminous efficiency of the firefly. (Shaded area to total area.)

the shaded to the whole area gives the luminous efficiency, that is

$$\frac{\text{light (radiated energy} \times \text{visual sensibility)}^6}{\text{total radiated energy.}}$$

For the glowlamp it is 0.43 per cent; for the firefly 96.5 per cent, these numbers representing the relative amounts of light (measured on a photometer) for equal amounts of radiated energy—a

⁶ This method of calculating luminous efficiency was first suggested, it is believed, by C. E. Guillaume (Soc. Int. Elect. Bull., 5, pp. 396-400; May, 1905). The numerical values are not to be compared with those obtained from the older method of calculating this quantity, in which the ratio of radiated energy between 0.38μ and 0.76μ to the total radiation is sought, a quantity having no direct relationship with efficiency as ordinarily understood in connection with artificial illuminants.

striking illustration of the wastefulness of artificial methods of light production. From the specific consumption of the tungsten lamp (1.6 w. p. s. c.) and the mercury arc (0.55 w. p. s. c.) we obtain by comparison with the carbon filament that their luminous efficiencies are 1.3 per cent and 3.8 per cent. The most efficient artificial illuminant therefore has about 4 per cent of the luminous efficiency of the firefly.

We may express these comparisons in a different way if we make one assumption. Let us assume all the applied energy in the glow-lamp to be used in producing radiation, none being lost by convection or conduction in the evacuated bulb, base connections, and glass walls. The glowlamp has an efficiency of 4.83 watts per mean spherical candle, so we may say the firefly has an efficiency of .02 watts per candle, a quantity, because of our assumption, representing the upper limit of the specific consumption. This is to be compared with the tungsten lamp at 1.6 watts per candle and the mercury arc at 0.55 watts per candle.

These figures are, however, to a certain extent misleading, for they give no weight to the question of color. The light of the firefly would not be acceptable as an illuminant for general purposes because of its green hue and small spectral extent. The color values of objects illuminated by it would be distorted to a greater extent than by the mercury arc. All hues would be submerged in a nearly uniform green. Considered from the point of view of the amount of white sensation as compared with the total luminous sensation⁷ the firefly is very inefficient. This is of course a necessary consequence of the relationship between color and visual intensity, the most efficient light—disregarding the requirements of color—would be a single spectral line in the green. Its luminous efficiency, on the basis used above, would be 100 per cent, but it would be quite unsuitable for ordinary illumination. We may say, therefore, that the firefly has carried the striving for efficiency too far to be acceptable to human use; it has produced the most efficient light known, as far as amount of light for expenditure of energy is concerned, but has produced it at the (inevitable) expense of range of color. The most efficient

⁷ H. E. Ives, *Illuminating Engineer*, September, 1909.

light for human use, taking into account both color and energy-light relationship, would be a light similar to the firefly light containing no radiation beyond the visible spectrum, but differing from it by being white.

II. A FLUORESCENT SUBSTANCE FROM THE FIREFLY.

It was observed by one of us (Coblentz) that the milky fluid which exudes from the firefly (*Photinus pyralis*) on the slightest touch contains a substance which emits a deep blue fluorescence when the solution is concentrated and free from albuminous matter. In the impure dilute solution the fluorescence is not unlike ordinary kerosene, only very much fainter. This fluorescent substance exists in all parts of the body, the least being present in the head and prothorax, the abdomen (light emitting segments) containing a greater amount, while the thorax and wing covers (which are rather soft) contain the most. The wing covers are bordered with a transparent yellowish-brown margin in which the white fluid, containing the fluorescent substance, is found in the greatest abundance. The fluid exudes in a large drop from the margin of the wing on slight pressure, which fact was taken advantage of in collecting a sample of the pure material. Exposed to the air, the white fluid thickens into a sticky mass, insoluble in ether and alcohol, but soluble in water when fresh. The fluorescent substance is soluble in these liquids, and, in the later experiments was extracted from the insect by means of ether. To this extract water was added, the albuminous matter was precipitated with lead acetate, and the solution was concentrated by boiling. The solution was then neutralized with potassium oxilate to prevent turbidity while photographing the fluorescence spectrum, which was obtained with the same apparatus and photographic plates used in photographing the light emitted by the firefly. The cadmium spark seemed to produce a greater fluorescence than magnesium or aluminum and was used as an exciting source. The absorption appears to be very great, as indicated by the fact that the fluorescence extends but a few millimeters into the solution. The fluorescence spectrum is shown in Fig. 6, and is marked for its intensity and extent. The spectrum shows no structure, and seems to be of the same nature as are the common banded spectra of solutions and of solids.

Substances showing fluorescence are rather common (see Kayser's *Spectroscopie*, Vol. 4), and the present data are too meager for discussion. The isolation of the fluorescent constituent is a chemical problem.

The life of this insect is less than two months, and further experiments must be deferred. Other species must also be examined to learn whether this is a common constituent of fireflies.⁸ There are, perhaps, 25 species of fireflies in this country, while the number of various animals emitting light is very great. Certain light-emitting animals are said to contain a phosphorescent constituent, but no record of a fluorescent constituent has come to the writer's notice. While it possibly has no significance, it seems rather remarkable that a highly fluorescent body fluid should be accidentally present in this light-giving insect, the two spectra being complementary, and the emitted light being of longer wave length than the fluorescent light. The range of the fluorescence spectrum extends from 0.38μ to 0.51μ ($\Delta\lambda = 0.13\mu$ about) while the emitted light extends from 0.51μ to 0.67μ ($\Delta\lambda = 0.16\mu$ about). The maximum of the fluorescence spectrum occurs at about 0.41μ , the plate having a region of maximum sensitivity at 0.43μ , which shifts the maximum of the fluorescence spectrum toward the long wave lengths.

At first there seemed to be no economic reason why fireflies (at least this species) should be burdened with such a large amount of body fluid, especially in the margin of the wing covers. According to Mr. H. Barber, of the National Museum, entomologists are of the opinion that the freely exuded fluid, which has an unpleas-

⁸ NOTE.—Since the completion of this paper Coblenz has found that this fluorescent substance occurs in a nonluminous genus (*Ellychnia corrusca*, Linn.) of the family of true fireflies. Through the courtesy of the U. S. National Museum an opportunity was given him to examine several species of elaters (*Pyrophorus noctilucus*, etc.) from Cuba and Guatemala which emit a yellow-green light. In all cases the outer horny covering of the insect, the chitinous layer, was found to be perfectly transparent in the regions covering the luminous spots, while elsewhere the integument was a deep reddish brown. From this it is evident that the color of the emitted light is not due to absorption in passing through the chitinous layer, and that it is the sum total of all the light produced. That the red light, said to be emitted by some insects, is the result of absorption seems highly improbable; for it would be poor economy to produce light covering a wide range of the spectrum and then absorb all but the part emitted, whether the emitted part be red, green, or blue.

ant odor and taste, is present simply as a protection against bats and other insectivorous animals. In view of the fact that an insect, which is as conspicuous as is the firefly, would be an easy prey without some protective device, such as an ill-tasting body fluid, this is not an unreasonable explanation.

SUMMARY.

The spectrum of the firefly "*Photinus Pyralis*" has been photographed on plates sensitive to the whole visible spectrum.

The spectrophotometric curve of the firefly, as compared with a carbon glowlamp, has been obtained by comparison of the photographic densities of the negatives of the firefly and of the glowlamp.

The distribution of radiant energy in the carbon glowlamp has been determined, and by means of the spectrophotometric relationship the distribution of radiant energy of the firefly derived. The light was found to consist of an unsymmetrical structureless band in the yellow green of the spectrum, with a maximum at 0.57μ , and extending to 0.51μ and 0.67μ .

The luminous efficiency of the firefly is calculated as 96.5 per cent, as compared with 0.4 per cent for the carbon glowlamp, and about 4 per cent for the most efficient artificial illuminant. This value is obtained on the assumption that there is no infra-red radiation, other than that due to the natural body heat, as Langley had concluded from his bolometric study of the Cuban genus. There are reasons for believing that the light emitted is due to a physiological-chemical reaction, not necessarily accompanied by heat waves of low frequency such as occur in a purely thermal (low temperature) radiation. But even if heat waves of low frequency are generated in the photogenic cells, they could not pass out into space on account of the opacity of the outer integument, the chitinous layer, of the insect, which probably can transmit radiation to wave lengths as great as 1.5μ . Now, from our present knowledge of emission and absorption, and from the fact that the emission band in the visible spectrum stops abruptly at 0.7μ , it appears highly improbable that there is appreciable radiation between 0.7 and 1.5μ , hence our assumption of no infra-red radiation can not lead us far astray. Nevertheless, by the exami-

nation of this region of the spectrum, by photographing the undispersed light of the firefly, after it has passed through red glass, as indicated by one of us elsewhere, it is hoped to obtain a qualitative test of this question. It should be noted that we have found the *radiant efficiency*, which is apparently very high. Whether the processes of physiological chemistry, which are probably involved in this light production, are equally efficient is an entirely different question.

WASHINGTON, August 1, 1909.

LUMINOSITY AND TEMPERATURE.

By P. G. Nutting.

With light expressed in terms of radiation and radiation a given function of temperature and wave length, light or luminosity is expressible directly in terms of temperature. These relations of light to radiation and of radiation to temperature have recently received considerable attention. Assuming them to be known, let us see what may be done toward establishing the direct relation of luminosity to temperature. The numerical values of some of the constants involved are not known with sufficient precision to permit of very precise numerical deductions, but the general relations developed show the interrelations of the various quantities involved. The relation of greatest practical interest is that of luminous efficiency to temperature.

Previous work has been limited to portions of the general problem. There are some good experimental determinations of the amount of light (monochromatic and total) emitted by bodies at known temperatures, but no one appears to have attacked the general problem directly.

Paschen and Wanner¹ in 1899 used a photometric method for determining the second constant of the Wien-Paschen radiation function

$$J = c_1 \lambda^{-5} e^{-c_2/\lambda T}$$

c_1 , c_2 , and α being constants, λ wave length, and T absolute temperature. Working at constant wave length, they wrote this in the form

$$\log J = r_1 - \frac{r_2}{T}$$

¹ Paschen and Wanner, Sitzber. d. Berliner Akad., 2, pp. 5-11; 1899.

or for two different temperatures T_1 and T_2 ,

$$\log \frac{J_1}{J_2} = r_2 \left(\frac{1}{T_2} - \frac{1}{T_1} \right),$$

r_1 and r_2 being constants. Assuming light proportional to radiation, they computed r_2 for four different wave lengths from their data. The limitations of this assumption we shall discuss later on.

Lummer and Kurlbaum² studied the photometric brightness of a hole in a uniformly heated opaque envelope as a function of temperature. They compared the total light with that from a constant source. They expressed their results in the form

$$\frac{H_1}{H_2} = \left(\frac{T_1}{T_2} \right)^x$$

H being brightness, T temperature, and x a variable having the following values:

$T = 900$	1000	1100	1200	1400	1600	1900
$x = 30$	25	21	19	18	15	14

They state that x approaches the value 12, but this is not apparent from the plotted curve, and later Rasch showed (see below) that their equation was essentially defective.

Lummer and Pringsheim³ then took up the absolute measurement of the total light in Hefners emitted by a square millimeter of a perforation in such an opaque envelope.

They found

Temp. C.	Light/mm ² .
1175°	0.0042 H
1325	0.0220
1435	0.0635

and extrapolated

1500° C	0.1 H (about)
1700	0.5
1800	1.0

² Lummer and Kurlbaum, *Verh. d. Phys. Gesell.*, **2**, pp. 89-90; 1900.

³ Lummer and Pringsheim, *Phys. Zs.*, **8**, pp. 97-100; 1901.

but did not attempt any formulation of their results. This work was later greatly extended by Nernst,⁴ using an iridium furnace cross-checked by an auxiliary heated rod of oxide. He obtained for the light (Hefners) per sq. mm.

Temp. absolute.	Light H/mm^2 .	Temp. absolute.	Light H/mm^2 .
1464°	0.005	2357°	4.0
1524	0.01	2446	6.0
1685	0.05	2516	8.0
1764	0.1	2571	10.
1982	0.5	2619	12.
2092	1.0	2680	15.
2217	2.0	2763	20.

His measurements extended to about 2290° absolute. The above values were obtained from the relation

$$\log H = B - \frac{A}{T}$$

where H is Hefners per mm^2 , T is absolute temperature, $B = 5.367$ and $A = 11230$, the constants being determined from observations between 1400° and 2200°. Temperature measurements were made with a Wanner pyrometer, an instrument based on the monochromatic Paschen-Wanner relation mentioned above. His observations then indicate that the monochromatic and total light emission of a complete radiator between 1400° and 2200° differ only by a constant factor.

Rasch⁵ appears to have been the first to attempt a rational foundation for the relation of luminosity to temperature. Since retinal action is essentially a chemical process, he reasoned that the thermal equation of chemical equilibrium should apply, the luminous sensation being related to "retinal temperature" and this again to the rate of absorption of energy. In van't Hoff's reaction isochore

$$\frac{dK}{K} = - \frac{q}{R} \frac{dT}{T^2}$$

⁴ W. Nernst, *Phys. Zs.*, **7**, pp. 380-383; June 1, 1906.

⁵ E. Rasch, *Ann. d. Phys.*, **14**, pp. 193-203; 1904.

he replaces the equilibrium constant K by brightness H and the constants $-q/R$ by κ . Hence by integration

$$\log H = c - \frac{\kappa}{T}$$

$$\log \frac{H_2}{H_1} = \kappa \left(\frac{1}{T_1} - \frac{1}{T_2} \right)$$

This indicates that the quantity x in Lummer and Kurlbaum's equation (see above) is $x = \kappa/T$. Hence it is not a constant nor does it approach a constant value. It will be noticed that Rasch's equations are the same in form and but slightly different in interpretation from those of Paschen and Wanner.

Rasch computed a mean value of $\kappa = 26750$ from Lummer and Kurlbaum's data. Lucas,⁶ from the relation $\kappa = c_2/\lambda$ and c_2 (the second constant of the Wien-Paschen equation) = 14500, finds for λ the value 0.542μ , an equivalent mean wave length in dealing with total light.

Goldhammer⁷ from the data of König on visual sensibility and the work of Rasch just mentioned, derived the function

$$V = \left(\frac{\lambda_0}{\lambda} e^{1 - \frac{\lambda_0}{\lambda}} \right)^m$$

to represent the ratio of light to radiation at each wave length, λ_0 being the wave length of maximum sensibility and m a constant varying from 100 to 300 for different individuals. Both λ_0 and m he found to vary somewhat with the intensity of the radiation. Goldhammer recognized that it was necessary to introduce this auxiliary function of radiation, the *visibility* (V), to express light in terms of radiation. He multiplied the above function by the Wien-Paschen function $J = c_1 \lambda^{-5} e^{-c^2/\lambda T}$ and thus obtained for the total light from a complete radiator

$$H = a \left(\frac{T}{T + \delta} \right)^n$$

a , δ and n being constants. He did not attempt to construct a visibility function to represent the average human eye. The

⁶ R. Lucas, Phys. Zs., 6, 19-20; Jan. 1, 1906.

⁷ D. A. Goldhammer, Ann. Physik., 16, 621-652; April, 1905.

one he did construct contains the large exponent m and is always steeper on the blue side of the maximum, while the experimental curves are steeper on either side in about equal numbers. It differs only in its constants from the Wien-Paschen function for non selective radiation.

The writer⁸ has developed a visibility function of the form

$$V = V_0 e^{-\kappa(\lambda - \lambda_m)^2}$$

to represent the sensibility of the average human eye. Here V_0 is a constant depending upon the units employed, κ is 4.6 at low intensities (<0.05 meter-candle) and 2.0 at high (>10 m. c.); while λ_m , the wave length of maximum sensibility, is similarly $.511\mu$ at low intensities (rod vision) and $.565\mu$ at high (cone vision) intensities.

If the light is expressed in meter-candles and the radiation in meter-watts per unit wave length, then V_0 is the number of candles per watt at the wave length of maximum sensibility, $\lambda = \lambda_m$, for here $L = EV$ becomes $L = EV_0$, E being the radiation. To find this ratio of light to radiation it is only necessary to measure the same monochromatic radiation as light and as power. This ratio the author (l. c.) found to be 13.5 candles-watt, while Drysdale⁹ found 16.7. König's data on visual sensibility indicate that this ratio is a constant except for intensities ranging from about 0.3 m.-c. up to 30 m.-c., which is the range during which the transition from rod to cone vision occurs.

Visibility then is the constant of proportionality between radiation and light. It varies enormously with the wave length, but at constant wave length is constant at very high and very low intensities, with an uncertainty slightly larger than the possible error in its determination through a small range of moderate intensities.

Let us now apply the visibility function to known types of radiation. The best known is the non-selective radiation from the interior of a nearly closed cavity having opaque walls at a uniform temperature. This is known to be related to wave

⁸ Electrical World, June 27; 1908.

This Bulletin, 5, pp. 261-305; 1908

⁹ C. V. Drysdale, Proc. Roy. Soc. 80, pp. 19-25; 1907.

length and the temperature of the radiator by the Planck function

$$E = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1}$$

This is difficult to use, but if it is expanded in a power series the first term is the Wien-Paschen function

$$(1) \quad E = c_1 \lambda^{-n} e^{-c_2/\lambda T}$$

which is sufficiently accurate in dealing with the short wave lengths of the visible spectrum. Multiplying this by the visibility function

$$(2) \quad V = V_0 e^{-\kappa (\lambda - \lambda_m)^2}$$

the product is the desired relation between luminosity and temperature

$$(3) \quad \begin{aligned} L &= E V \\ &= c_1 V_0 \lambda^{-n} e^{-c_2/\lambda T - \kappa (\lambda - \lambda_m)^2} \end{aligned}$$

Hence the isochromatic relation (λ constant) is of the form

$$(4) \quad \log L = A - \frac{B}{T}$$

where

$$A = \log (c_1 V_0 \lambda^{-n}) - \kappa (\lambda - \lambda_m)^2$$

and

$$B = c_2/\lambda$$

The value of the constant B is seen to be independent of the form of the visibility function while the constant A is not. Further, the form of (4) does not in any way depend upon the form of the visibility function, but only upon the form of the emission function $E(\lambda, T)$.

From (4) may be deduced the relative temperatures at which the various spectral colors appear. For $L=1$ the temperature at which complete radiators will attain a specific luminosity of one light unit (the threshold value in this case) per unit wave length will be given by

$$T = \frac{B}{A}$$

from which T may be calculated for each λ .

Equation (4) assumes a constant wave length, hence it can apply to but a limited spectral region at a time. The reason that it has been found (by Nernst and others) experimentally to hold approximately for total light emission is that for a complete radiator, 90 per cent of the light is confined to a spectral region only 0.1μ broad, because of the narrow form of the visibility curve. The fact that both κ and λ_m vary considerably with intensity is of no consequence so long as the product $\kappa(\lambda - \lambda_m)^2$ remains constant.

The relation between total light emission and temperature is obtained by integrating $EVd\lambda$ from zero to infinity. In the form (3) above it is not readily integrable but using the alternative form¹⁰ of visibility function

$$(5) \quad V = V_0 \left(\frac{\lambda_m}{\lambda} \right)^\nu e^{\nu \left(1 - \frac{\lambda_m}{\lambda} \right)}$$

the product EV may be easily integrated in the form

$$(6) \quad L = \int_0^\infty EVd\lambda = c_1 V_0 \lambda_m^\nu e^\nu \Gamma(n + \nu - 1) \left(\frac{c_2}{T} + \nu \lambda_m \right)^{-(n + \nu - 1)}.$$

which is of the simple form

$$(7) \quad L = A \left(\frac{B}{T} + 1 \right)^{-p}$$

where

$$\begin{aligned} A &= c_1 V_0 \lambda_m^\nu e^\nu \Gamma(n + \nu - 1) (\nu \lambda_m)^{-\nu} \\ &= c_1 V_0 \left(\frac{B}{c_2} \right)^{n-1} \sqrt{2\pi\nu} \frac{\Gamma(n + \nu - 1)}{\Gamma(\nu + 1)} \end{aligned}$$

$$B = c_2 / \nu \lambda_m$$

$$p = n + \nu - 1$$

This expression for the light emitted applies to any body whose radiation is expressible by equation (1) viewed by any eye whose chromatic sensibility may be represented by equation (5); that is, it is of general application to nonselective radiation.

¹⁰This Bulletin, 5, p. 277; 1909.

Defining luminous efficiency F as the ratio of the total light emitted to the total energy radiated from the same area we have from (1) and (7)

$$(8) \quad F = \int_0^\infty E V d\lambda : \int_0^\infty E d\lambda \\ = A \left(\frac{B}{T} + 1 \right)^{-\nu} : \alpha T^{n-1}$$

since

$$\int_0^\infty E d\lambda = c_1 \left(\frac{T}{c_2} \right)^{n-1} \Gamma(n-1) = \alpha T^{n-1}$$

Equation (8) may be reduced to

$$F = V_0 \sqrt{2\pi\nu} B^{n-1} \frac{\Gamma(n+\nu-1)}{\Gamma(\nu+1) \Gamma(n-1)} \frac{T^\nu}{(B+T)^{n+\nu-1}}$$

a form more convenient for computation, by means of the relation

$$x! = x^x e^{-x} \sqrt{2\pi x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots \right)$$

Since ν is greater than 100 the series in parenthesis has the value unity to within .001.

The luminous efficiency has a maximum value

$$(9) \quad F_m = V_0 \left(\frac{n-1}{n+\nu-1} \right)^{\frac{1}{2}}$$

at a temperature

$$(10) \quad T_m = B \frac{\nu}{n-1} = \frac{c_2}{(n-1)\lambda_m}$$

The numerical values of all the constants in (6) and (8) have been determined, but in no case with great precision. From König's data on visual sensibility,¹¹ ν may be calculated to be 120 ± 10 , a pure number with an uncertainty of about 10 units. The wave length of maximum visual sensibility is $0.565 \pm 0.005 \mu$. General average values of ν and λ_m for a large number of individuals might lie outside even these limits of uncertainty, but it is hardly probable. The values here quoted apply to cone vision. For low intensities (rod vision) ν is about 300 while $\lambda_m = 0.511 \mu$, but the value of ν_0 has not been determined, so that we are unable

¹¹ This Bulletin, 5, p. 279; 1908-9.

at present to compute the luminosity of very faint sources and weak reflected light.

For enclosed radiators of the "black body" type $C_2 = 14500 \pm 300$, when wave lengths are in microns (μ) and temperature in centigrade degrees absolute. $n = 5$, an integer so far as we know. For other bodies, n varies from 5.5 to 7. Coblentz¹² for an untreated carbon filament found $n = 5.2$ to 6.5 (decreasing with increase in temperature) for platinum $n = 6$ to 8, "metallized" carbon $n = 5.8$ to 6.0, silicon carbide $n = 6.4$. For lamps run at normal voltage he found for

Metallized carbon	$n = 6.1$
Tantalum	$n = 6.3$
Tungsten	$n = 6.6$
Osmium	$n = 6.9$

For radiation from a cavity or "black body," then, the temperature of maximum luminous efficiency is

$$T_m = \frac{14500}{(5-1) \cdot 565} = 6420^\circ \pm 200^\circ \text{ abs}$$

the value of that efficiency being

$$F_m = 15 \left(\frac{5-1}{5+120-1} \right)^{1/2} = 2.7 \pm 0.2 \frac{\text{cand.}}{\text{watt.}}$$

The value $\nu_0 = 15.0$ is the mean of the values of Drysdale (16.5) and the author (13.5), and the uncertainty in the result is due chiefly to the uncertainty in this value.

This result indicates that such a radiator could give at best (6420°) only about 0.18 as much light as it would if all its radiation were confined to the yellow-green.

At a temperature of 2000° abs. (1727° C.) the computed luminous efficiency is

$T = 2000^\circ$	$C_2 = 14500$	$\nu = 120$	$V_0 = 15$
$n = 5.0$	6.0	7.0	
$F = .055$	.209	5.05 cand./watt.	

The first case ($n = 5$) corresponds to a perfect radiator or black body. This table is made on the assumption of C_2 constant. But

¹² This Bulletin, 5, p. 339; 1908-9.

little is known as to its variation with temperature and with the substance radiating. Lummer and Pringsheim's results indicate that for a given substance, C_2 varies in proportion to n , so that

$$\lambda_{max}T = C_2/n = \text{const.}$$

and they give for platinum $C_2 = 15600$. For platinum, then, at $2000^\circ \text{ abs.} = 1727^\circ \text{ C.}$, just below the melting point, $F = .121$ cand./watt.

The variation of luminous efficiency with temperature

$$\frac{\partial E}{\partial T} = F \left(\frac{\nu}{T} - \frac{n + \nu - 1}{B + T} \right)$$

is not very rapid at high temperatures, so that (8) could hardly be applied to advantage to the measurement of temperatures, nor could (8) be applied to the determination of the visual constant ν . The variation of F with ν is for

$T = 2000$	$n = 5$	$\nu_0 = 15$	_____
$\nu = 120$	110	100	
$F = .051$	.058	.062	cand./watt.

Its most useful field appears to be in the determination of C_2 , with which F varies rapidly.

The agreement of the above calculated efficiencies with observed values are sufficiently close to give formula (8) considerable weight. The rapid increase in computed efficiency with the constant n , shown above, is quite in accord with experience with the new glow lamps, and Coblenz's values of this constant for such lamps.

WASHINGTON, August 14, 1909.

A THEORETICAL AND EXPERIMENTAL STUDY OF THE VIBRATION GALVANOMETER.

By Frank Wenner.

CONTENTS.

	Page.
1. Introduction.....	347
2. Theory.....	349
3. Effect of Harmonics.....	356
4. Effect of Inductance.....	358
5. The use of a Transformer.....	358
6. The System of Units.....	359
7. Method of Determining Constants.....	360
8. Basis of the Equations.....	361
9. Method of Tuning.....	363
10. Galvanometers and their Constants.....	364
11. Characteristics of the Instruments.....	366
12. An Illustrative Problem.....	369
13. Design.....	375
14. Summary.....	377

1. INTRODUCTION.

The vibration galvanometer is an instrument for the detection of alternating currents or electromotive forces. It differs from other instruments for the same purpose mainly in having the moving system tuned to the frequency of the current or electromotive force to be investigated. In one of the later designs the construction differs from the ordinary D'Arsonval galvanometer only in having a very narrow coil and having bifilar suspensions of adjustable length and tension so as to give a period of from about 50 to about 750 vibrations per second. As the deflection of the instrument changes sign as often as the current changes direction, the image formed by the mirror of a line source of light appears as a band, unless the frequency is very low. The width of the band is a measure of the amplitude of the vibration.

If the period or frequency of the current is changed while the free period of the instrument is kept constant, it is found that the instrument has a high sensibility at and near the resonating frequency, while at frequencies differing by more than a few per cent from the resonating frequency the sensibility is very small. The instrument, therefore, responds only slightly to the harmonics in the current or electromotive force wave, so that usually no error is introduced when measurements and calculations are made on the assumption of a pure sine wave. This property makes the instrument especially suitable for those measurements in which frequency enters and in which null methods are used.

Max Wien¹ pointed out some of the advantages of a tuned instrument when he described his optical telephone, a vibration galvanometer in which the essential part of the moving system consists of a piece of soft iron on a very flexible diaphragm. The soft iron armature is actuated by one or two bipolar telephone magnets of the usual construction. The current to be investigated, passing through the coils, causes a relative displacement between the magnet and armature, and this is communicated to a mirror in such a way as to give an angular displacement to the latter. The displacement is read by the broadening of a line image. The mirror is on a spring, and for a maximum sensibility the period of this system, as well as the period of the current to be investigated, is adjusted to the period of the diaphragm. Wien showed experimentally that the deflection is proportional to the current.

Rubens² next brought out an instrument in which the mirror and soft iron piece are mounted together on a metal suspension instead of a diaphragm. The bipolar telephone magnets are so placed and their coils are so connected that the current causes an angular displacement of the soft iron armature and mirror. The frequency can be adjusted by either a change in length or tension of the suspensions. This instrument is more flexible than the optical telephone and a somewhat higher sensibility is claimed for it.

Max Wien³ later developed an instrument having a very small permanent magnet mounted on a brass wire suspension, the

¹ Wied. Ann., **42**, p. 593, and **44**, p. 681; 1891.

² Wied. Ann., **56**, p. 27; 1895.

³ Ann. der Physik., **309**, p. 425; 1901.

lengths and tension of which may be changed so as to give a considerable range in the number of vibrations per second. The coils are wound on a core of soft iron wire, and the pole faces of this core are very close and parallel to the small magnet. The sensibility claimed for this instrument is 16 times that of the optical telephone.

Rosa and Grover⁴ in their work on the measurement of inductance pointed out certain advantages in the use of a vibration galvanometer and give a curve showing the change in sensibility with a change in tuning for an instrument which apparently had two free periods.

Wells⁵ showed experimentally that the deflection is proportional to the electromotive force. He also plotted a curve showing the change in the observed deflection as the free length of the suspension was varied.

In 1907 Campbell⁶ described a very convenient instrument of the D'Arsonval type. This instrument is suitable for frequencies of from about 50 to about 750 vibrations per second and is easily adjustable over this range. In sensibility this instrument compares favorably with Wien's when both are adjusted to the same frequency. Concerning the theory of the instrument, Campbell points out that the sensibility is independent of the moment of inertia and inversely proportional to the frequency. He also states that the damping is "both mechanical and electrical" and should be kept small.

2. THEORY.

In any galvanometer, a current through the winding produces a force tending to cause a relative displacement between the fixed and moving system. The resulting relative displacement causes a part of the flux from the magnet to cut through the winding, and thus develops a counter electromotive force. The energy available for producing a deflection is the time integral of the product of the current and the counter electromotive force. As galvanometers are ordinarily used, this energy is small, due to the

⁴ This Bulletin, 1, p. 291; 1905.

⁵ Phys. Rev., 23, p. 504; 1906.

⁶ Phil. Mag., 14, p. 494; 1907.

Electrician, 60, p. 60; 1907.

fact that after the moving system has reached a displacement equal to the final deflection the time integral of the product of the counter electromotive force and current is either equal to or less than zero.

If the motion is only slightly damped, a reversal of the current at the end of the first deflection will cause a second deflection in the opposite direction larger than the first. This is due to the fact that the motion of the system again causes a counter electromotive force, and thus more electrical energy is converted into mechanical energy. However, not all this energy is added to the system, since some is lost by air friction, induced currents, etc. By reversing the current at the end of each deflection the successive deflections can be increased until the average rate at which energy is lost by damping becomes equal to the average rate at which electrical energy is converted.

In the work that follows we shall consider first the case where the impressed electromotive force and current waves are both of a sine form and later the effect of harmonics. With a sine wave current and electromotive force we may assume a simple harmonic vibration, and a sine wave back electromotive force.

In use the free period is adjusted to or near that of the current to be investigated. If this adjustment is good the displacement of the moving system leads the driving current by 90° , and since the back electromotive force depends upon the rate of change of this displacement it is in phase with the current. If, however, the tuning is not good the moving system leads the current by an angle greater or less than 90° , and consequently the phase angle between the current and the back electromotive force is not zero. As the power converted depends upon the cosine of this phase angle, the tuning may have a marked effect upon the deflection.

As the amplitude of the vibration increases the back electromotive force increases and unless the resistance is very high it has a marked effect on the value of the current. If the three vectors—the current, impressed electromotive force, and back electromotive force—are in the same line, the power converted and consequently the deflection is a maximum, if after a steady condition is reached the back electromotive force is equal to one-half the impressed electromotive force. Since the back electromotive force is pro-

portional to the field strength it may happen that an increase in the field strength results in a decrease in the amplitude, just as in a direct-current motor an increase of the field strength may result in a decrease in the speed. For a given set of conditions the deflection of the vibration galvanometer is a maximum for a particular field strength, just as for a given set of conditions the speed of a direct-current motor is a maximum for a particular field strength.

Since when a steady condition is reached the power lost by damping is equal to the electrical power converted, a decrease in the damping must, if the other quantities are kept fixed, result in an increase in the amplitude of the vibration. This increase may or may not be large, just as in the direct-current motor a decrease in the load may or may not result in a material increase in the speed.

The relations between the amplitude of the vibration, current, electromotive force, frequency, etc., may be expressed in terms of the working constants, the more important of which are: (a) The current sensibility, (b) the electromotive force sensibility, (c) the power sensibility, (d) the ratio of the back to the impressed electromotive force, (e) the time constant for current indication, (f) the time constant for electromotive force indication, (g) the resonance range for current indication, and (h) the resonance range for electromotive force indication. By resonance range is meant the fractional part by which the period of the current or the impressed electromotive force is changed to reduce the deflection by half.

These working constants depend upon the intrinsic constants of the instrument which are: (a) The moment of inertia, (b) the moment of damping (by which is meant the ratio of the retarding torque to the angular speed when the coil is on open circuit), (c) the moment of restoration (the ratio of the torque, tending to bring the moving system to the null position, to the angular displacement), (d) the moment of displacement (the ratio of the torque produced by the current to the current), and (e) the resistance.

The intrinsic constants as here given and defined refer only to an instrument in which the displacement is angular. The following equations, however, are not limited to this kind of a displacement but are equally applicable (except for the constant where an

arbitrary unit of displacement is chosen) to instruments of the Einthoven or string type.

The equations giving the relation between the various intrinsic constants, the deflection, time, and current or electromotive force will be stated and solved under certain limiting conditions. The symbols used are as follows:

θ = angular displacement.

t = time.

α = moment of inertia.

β = moment of damping.

γ = moment of restoration.

ψ = moment of displacement.

ρ = resistance of galvanometer.

r = resistance of complete galvanometer circuit.

i = instantaneous value of current.

I' = maximum value of current.

I = effective value of current.

w = power.

f = frequency of current or impressed electromotive force.

f_o = natural free frequency of moving system.

$p = 2\pi f$.

$p_o = 2\pi f_o$.

ϕ = amplitude after a steady condition is reached.

e = instantaneous value of the back electromotive force.

E = effective value of impressed electromotive force.

E_b = effective value of back electromotive force.

E_o = effective value of impedance electromotive force.

ω = the angle by which the impressed electromotive force leads the current.

η = the angle by which the impedance electromotive force leads the current.

σ = the angle by which the back electromotive force leads the current.⁷

A = current sensibility.

V = electromotive force sensibility.

A' = direct current sensibility.

⁷ When the electromotive force developed in the winding of the instrument is in exact opposition to the current σ is taken as zero as shown in Fig. 1.

W = power sensibility.

D = ratio of back to impressed electromotive force.

T_i = time constant for current indications.

T_e = time constant for electromotive force indications.

R_e = resonance range for electromotive force indications.

R_i = resonance range for current indications.

X = the step-up ratio of the transformer.

r_1 = resistance of primary circuit.

V_x = electromotive force sensibility with transformer.

— signifies that the quantity in question is referred to the set of conditions given on page 359.

The motion of the moving system of a galvanometer is given * by the equation

$$\alpha \frac{d^2\theta}{dt^2} + \beta \frac{d\theta}{dt} + \gamma\theta = \psi i \quad (1)$$

The free period

$$1/f_0 = 2\pi\sqrt{\alpha/\gamma}$$

or

$$\gamma/\alpha = p_0^2$$

and if

$$i = I' \cos pt$$

$$\frac{d^2\theta}{dt^2} + \frac{\beta}{\alpha} \frac{d\theta}{dt} + p_0^2\theta = \frac{\psi}{\alpha} I' \cos pt. \quad (2)$$

Integrating (2) gives

$$\theta = \frac{\psi I' \sin(pt + \sigma)}{\sqrt{p^2\beta^2 + \alpha^2(p_0^2 - p^2)^2}} - C e^{-\frac{\beta}{\alpha}t} \sin[\sqrt{p_0^2 - \beta^2/4\alpha^2}t + C'] \quad (3)$$

where C and C' are constants of integration and

$$\sigma = \tan^{-1} \frac{\alpha(p_0^2 - p^2)}{\beta p}.$$

A short time after starting the current the second member of equation (3) becomes zero, hence

$$\phi = \frac{\sqrt{2}\psi I}{\sqrt{p^2\beta^2 + \alpha^2(p_0^2 - p^2)^2}} \quad (4)$$

where ϕ is the maximum value of θ or the amplitude of the vibration.

*Gray, Abs. Meas. in Elec. and Mag., Vol. II, Part II, pp. 392-3.

The current is determined by the components of the electromotive forces in phase with it and the resistance or

$$I = \frac{E \cos \omega - E_b \cos \sigma}{r}. \quad (5)$$

The rate at which mechanical energy is supplied to the moving system is equal to the rate at which electrical energy is converted.

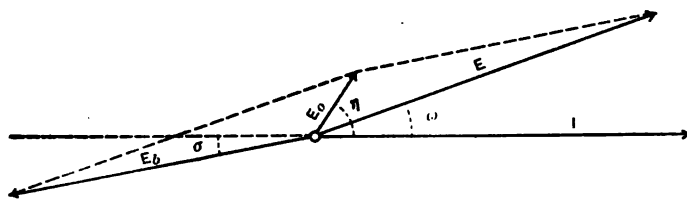


Fig. 1.—Vector Diagram showing the Various Phase Angles and Relations between the Electromotive Forces.

The former is the torque [due to the current and equal to ψi] times the angular speed [$\phi p \cos (pt + \sigma)$] and the latter is the current times the back electromotive force. Therefore

$$i\psi\phi p \cos (pt + \sigma) = ie$$

or

$$E_b = \psi\phi p / \sqrt{2}. \quad (6)$$

The substitution of the values of I from (5) and E_b from (6) in (4) gives

$$\phi = \frac{\sqrt{2}\psi E \cos \omega}{r \sqrt{p^2\beta^2 + \alpha^2(p_0^2 - p^2)^2} + p\psi^2 \cos \sigma} \quad (7)$$

and the multiplication of (4) by (7) gives

$$\phi^2 = \frac{2\psi^3 I E \cos \omega}{r[p^2\beta^2 + \alpha^2(p_0^2 - p^2)^2] + \sqrt{p^2\beta^2 + \alpha^2(p_0^2 - p^2)^2} p\psi^2 \cos \sigma}. \quad (8)$$

Or if $p = p_0$, (4), (7), and (8) become, since σ is then zero,

$$\phi = \frac{\sqrt{2}\psi I}{p\beta}, \quad \phi = \frac{\sqrt{2}\psi E \cos \omega}{p(r\beta + \psi^2)}, \quad \text{and} \quad \phi^2 = \frac{2\psi^2 I E \cos \omega}{p^2\beta(r\beta + \psi^2)}. \quad (9)$$

Defining the current sensibility as the ratio of the amplitude of the vibration to the effective value of the current, the electromotive force sensibility as the ratio of the amplitude to the effective value of the electromotive force at the terminals of the instrument, and the power sensibility as the ratio of the square of the amplitude to the power supplied gives

$$A = \frac{\sqrt{2}\psi}{p\beta}, \quad V = \frac{\sqrt{2}\psi}{p(\rho\beta + \psi^2)}, \quad \text{and } W = \frac{2\psi^2}{p^2\beta(\rho\beta + \psi^2)}. \quad (10)$$

In the second part of (10) $\cos \omega$ is taken as unity. This may generally be done without introducing an appreciable error, since in most instruments the impedance electromotive force, between terminals, is small in comparison with and very nearly in phase with the impressed electromotive force.

The time constant is given for current indications by equation (3) or

$$T_1 = \frac{2\alpha}{\beta}. \quad (11)$$

For electromotive force indications the back electromotive force is zero when t is zero and increases directly with the amplitude of the vibration. The current therefore starts with a certain value and decreases as the amplitude increases, which causes the amplitude to reach a certain fraction of its final value in less time than with a constant current. If the instrument is tuned so that the back electromotive force is in phase with the current it may be shown

that the retarding torque is $\frac{r\beta + \psi^2}{r} \frac{d\theta}{dt}$ instead of $\beta \frac{d\theta}{dt}$ as given by

equation (1) for current indications. It therefore follows that

$$T_e = \frac{2\alpha r}{r\beta + \psi^2}. \quad (12)$$

Since the resonance range is the change, in proportional parts, in the impressed frequency from resonance, necessary to reduce

the deflection by half, it may be derived from equations (7) and (9) by equating the second member of the first to half the second member of the latter, or

$$\frac{\sqrt{2}E\psi \cos \omega}{\sqrt{\beta^2 p^2 + \alpha^2 (p_o^2 - p^2)^2 + p\psi^2 \cos \sigma}} = \frac{\sqrt{2}E\psi}{2p_o(r\beta + \psi^2)}. \quad (13)$$

Then if we write $p_o(1+R)$ for p , we have, with a close approximation, $4p_o^4 R^2 = (p_o^2 - p^2)^2$, which when substituted in (13), and writing p_o for p , where it makes practically no difference, gives

$$R_e = \frac{\sqrt{[2 \cos \omega (r\beta + \psi^2) - \psi^2 \cos \sigma]^2 - \beta^2 r^2}}{2\alpha p_o r}. \quad (14)$$

Here, since there is a decided lack of tuning, $\cos \sigma$ is small and in comparison with 2 may be neglected while $\cos \omega$ has a value between .8 and 1. Taking the latter gives

$$R_e = \frac{\sqrt{3\beta^2 r^2 + 8\beta r\psi^2 + 4\psi^4}}{2\alpha r p_o}. \quad (15)$$

a value about 20 per cent too large if the resistance is small. If the resistance is very large, the back electromotive force has no effect on the value of the current, or only those terms of (15) involving the resistance are of importance, so that

$$R_t = \frac{\sqrt{3}\beta}{2\alpha p_o}. \quad (16)$$

3. EFFECT OF HARMONICS.

In those measurements in which the frequency enters and null methods are used an exact balance can not be obtained except when the electromotive force and current waves are of a pure sine form. In practice this condition is seldom if ever fulfilled, so the vibrations caused by the harmonics may limit the precision attainable. One of the chief characteristics of the vibration galvanometer is that the amplitude of the various harmonic vibrations is small. This is because there is no resonance, and the turning moment changes direction at intervals short in comparison with the half free period of the moving system. The amplitude of the vibration produced by any particular harmonic in the electromo-

tive force wave is given by equation (7) when the corresponding value of p is substituted.

The substitution of $3p_0$ for p gives

$$\phi_3 = \frac{\sqrt{2}\psi E_s \cos \omega_s}{r\sqrt{9\beta^2 p_0^2 + 64\alpha^2 p_0^4 + 3p_0\psi^2 \cos \sigma_s}}.$$

Here, unless the reactance is large, $(\cos \omega_s)$ may be taken as unity, while the terms in the denominator are all small in comparison with $8r\alpha p_0^2$, so we may write

$$\phi_3 = \frac{\sqrt{2}\psi E_s}{8r\alpha p_0^2}. \quad (17)$$

Defining the sensibility in the usual way

$$V_s = \frac{\sqrt{2}\psi}{8r\alpha p_0^2}.$$

Likewise

$$V_6 = \frac{\sqrt{2}\psi}{24r\alpha p_0^2},$$

$$V_7 = \frac{\sqrt{2}\psi}{48r\alpha p_0^2}, \quad (18)$$

$$V_8 = \frac{\sqrt{2}\psi}{80r\alpha p_0^2}.$$

Corresponding substitutions in (4) give

$$A_3 = \frac{\sqrt{2}\psi}{8\alpha p_0^2},$$

$$A_6 = \frac{\sqrt{2}\psi}{24\alpha p_0^2},$$

$$A_7 = \frac{\sqrt{2}\psi}{48\alpha p_0^2},$$

$$A_8 = \frac{\sqrt{2}\psi}{80\alpha p_0^2}. \quad (19)$$

4. EFFECT OF INDUCTANCE.

By reference to equation (7) it will be seen that the denominator can be decreased (if ψ is large in comparison with $r\beta$) by a decrease in $\cos \sigma$, even though in so doing the first term is increased, as it necessarily is when the instrument is thrown slightly out of resonance. If at the same time $\cos \omega$ can be kept near unity, the deflection will be increased. The analytical expression for $\cos \omega$ is complicated, since it involves the back electromotive force and consequently the amplitude of the vibration. However, referring to the vector diagram, Fig. 1, it will be seen that if a large inductance is placed in the galvanometer circuit, so as to make the angle η nearly 90° , and the frequency of the impressed electromotive force is increased slightly, so that the angle σ assumes a negative value somewhat less than 90° , it is possible for the back electromotive force to be much larger than the impressed electromotive force and the latter still have the larger component in phase with the current. The combination of the inductance and the slightly high frequency of the impressed electromotive force result in bringing the back and impressed electromotive force somewhere near quadrature and the latter somewhere near the phase of the current. If the frequency of the impressed electromotive force is not too much above the resonating frequency, the result may be an increase of the component of the current in phase with the impressed electromotive force. This means an increase in the power supplied to and consequently an increase in the amplitude of the vibration of the moving system. A condenser in series with the galvanometer and the lack of tuning in the opposite sense should give corresponding results.

5. THE USE OF A TRANSFORMER.

If the terminals of the galvanometer are connected to one of the windings of a transformer and the other winding is connected to the source of electromotive force, the equation for the current in the galvanometer is

$$I_2 = \frac{XE - I_1 X r_1 - E_b}{\rho} = \frac{XE - E_b}{\rho + X^2 r_1} \quad (20)$$

Here it is assumed that the currents in the transformer windings are in opposition and have the inverse ratio of the number of turns in the coils. The requirements of a transformer to meet these conditions will be considered later. A comparison of equation (20) with equations (5) and (7) gives

$$\phi = \frac{\sqrt{2}X\psi E \cos \omega}{(\rho + X^2 r_1) \sqrt{\beta^2 p^2 + \alpha^2 (p_0^2 - p^2)^2 + p\psi^2 \cos \sigma}} \quad (21)$$

or if $p = p_0$ and taking $\cos \omega = 1$

$$\phi = \frac{\sqrt{2}X\psi E}{p_0[(\rho + X^2 r_1)\beta + \psi^2]}$$

and

$$V_z = \frac{\sqrt{2}X\psi}{p_0[(\rho + X^2 r_1)\beta + \psi^2]} \quad (22)$$

$$V_z \text{ is a maximum when } X^2 = \frac{\rho\beta + \psi^2}{r_1\beta} \quad (22a)$$

6. THE SYSTEM OF UNITS.

In expressing the sensibility or deflection it will be convenient to use as the unit of current the microampere, of electromotive force the microvolt, of power the micro-microwatt, of deflection a millimeter broadening of a line image one meter in front of a plane mirror ($= 25 \times 10^{-5}$ radians), and cgs units for the other quantities.

The resistance must then be expressed either in ohms or electromagnetic units to correspond with the quantities with which it is associated.

In the following work when the sensibility is expressed in the arbitrary units or the resistance in ohms, it is indicated by a line over the symbol representing the quantity.

Referred to these conditions and writing $2\pi f_0$ for p_0 , we have from equations (10), (11), (12), and (22), respectively,

$$\bar{A} = \frac{A \times 10^{-7}}{25 \times 10^{-5}} = A \times 4 \times 10^{-4} = 9.0 \times 10^{-5} \frac{\psi}{f_0 \beta} \quad (23)$$

$$V = V \times 4 \times 10^5 = 9.0 \times 10^4 \frac{\psi}{f_0 (\rho\beta + \psi^2)} \quad (23a)$$

$$\bar{A}' = A' 4 \times 10^{-4} = 4 \times 10^{-4} \frac{\psi}{\gamma} \quad (24)^{\circ}$$

$$\bar{W} = W \times 160 = \frac{8.1 \psi^2}{f_0^2 \beta (\rho \beta + \psi^2)} \quad (25)$$

$$\bar{V}_z = \frac{9.0 \times 10^4 X \psi}{f_0 [(\rho + X^2 r_1) \beta + \psi^2]} \quad (26)$$

7. METHOD OF DETERMINING CONSTANTS.

If it is desired to make a change in one or more of the working constants, it is well to determine the intrinsic constants first; then a comparison with the above equations will suggest the changes to make and give the effect of any particular change. The intrinsic constants can all be determined from five independent measurements. Measurements which are easily made and from which the constants can readily be calculated are: (a) the resistance, (b) the direct-current sensibility, (c) the current sensibility, (d) the electromotive force sensibility, and (e) the resonating frequency.

(a) The resistance can be compared directly with a standard by means of a Wheatstone bridge or by other convenient methods.

(b) The direct-current sensibility can be obtained by passing a small known direct current through the coil and observing the change in deflection when the current is reversed.

(c) The galvanometer is then connected through a high resistance to the terminals of a suitable low resistance forming a part of the external circuit of an alternating-current generator. Means should be provided for regulating the speed of the generator. The frequency of the generator or the free period of the instrument is then adjusted so the tuning is good. The resistance shunted, the current through the shunt, and the resistance in series with the galvanometer gives the current, which, with the observed deflection, gives the current sensibility.

(d) The electromotive force sensibility is obtained by shunting a smaller resistance, and the removal of the resistance in series with the galvanometer. The electromotive force is then known

⁹ Defining the direct-current sensibility in the usual way, the relation follows from equation (1), since when the system has come to rest $\theta\gamma$ or $\phi\gamma = \psi i$. The deflection used is that produced on reversing the current or the double deflection to correspond to the broadening of the image produced by an alternating current

in terms of the shunt and the current through it. This, with the observed deflection, gives the electromotive force sensibility.

(e) The resonating frequency can be obtained from the speed of the generator, considering, of course, the number of pairs of poles.

Reference to equations (23), (23a), and (24) and remembering that $p_0^2 = \gamma/\alpha$, then gives, in terms of the five quantities given above, the intrinsic constants, as follows:

$$\begin{aligned}\alpha &= \frac{.91(\bar{A} - \bar{V}\rho)}{f_0^2 \bar{A} \bar{A}' \bar{V}}, & \beta &= \frac{8.1(\bar{A} - \bar{V}\rho)}{f_0^2 \bar{A}^2 \bar{V}}, \\ \gamma &= \frac{36(\bar{A} - \bar{V}\rho)}{f_0 \bar{A} \bar{A}' \bar{V}}, & \psi &= \frac{90000(\bar{A} - \bar{V}\rho)}{f_0 \bar{A} \bar{V}}.\end{aligned}\tag{27}$$

8. BASIS OF THE EQUATIONS.

In the derivation of the equations, it has been assumed that the moving system experiences an angular acceleration proportional to the resultant torque, a retarding torque proportional to the angular speed, a restoring torque proportional to the angular displacement, and a displacing torque proportional to the current and independent of the displacement and the rate of change of the current. That is, (a) the moment of inertia, (b) the moment of damping, (c) the moment of restoration, and (d) the moment of displacement are all constant.

It is further assumed that harmonics in the electromotive force or current wave produce no effect on the behavior of the instrument toward the fundamental, but cause an additional vibration depending upon their magnitude and frequency. It should be mentioned that the effect of the time constant of the circuit, which is generally small in comparison with the time constant of the instrument, has been neglected. If the time constant of the circuit is large it may materially effect the rate of change of the amplitude of the vibration.

The moment of inertia and the moment of restoration can better be considered together. If the vibration takes place about the center of the mass the moment of inertia is constant, and for small displacements the restoring moment is at least very nearly proportional to the displacement. When, however, the system is unsymmetrical the vibration, because of an unbalanced lateral

force due to the tension of the suspensions, takes place about some point or line which does not coincide with or pass through the center of mass and whose position is a function of the frequency of the driving force. Under these conditions the moment of inertia and the moment of restoration are both functions of the frequency of the driving force or current and the system has two free periods. In the article ¹⁰ referred to above, Rosa and Grover found that as the frequency of the current was changed, one of the galvanometers used by them gave maximum deflections at about 110.5 and 120 cycles per second, and a very small deflection at 115 cycles. This shows that the instrument had two free periods differing from each other by about 10 per cent. This large difference in the periods was caused by a lack of symmetry introduced on the repair of a broken suspension.

If care has been taken to keep the system symmetrical, or if it is carefully balanced, the two free periods may be very nearly the same, but when a few hundredths of a per cent change in the frequency produces a material effect on the deflection, it is possible that the difference in the periods may still be sufficient to produce an appreciable reduction in the sensibility and flattening of the curve, showing the relation between the deflection and the frequency. A little experience enables us to tell, in most cases, from the vibrations produced on tapping the instrument, if the two periods are nearly the same, and, if not, what changes should be made. If, then, the suspensions are bifilar and provided with a means of independent fine adjustment, the two periods can usually be adjusted in a few minutes to very nearly the same value.

The moment of damping is made up of a number of different factors, among which the air pressure and friction, eddy currents and hysteresis are probably the more important. The retarding torque is therefore only approximately proportional to the angular speed, so the moment of damping may depart appreciably from a constant.

The displacing torque is, to some extent, a function of the displacement and is effected by hysteresis and eddy currents. The latter makes the value smaller for alternating than for direct currents, so the moment of displacement is not quite a constant.

¹⁰ This Bulletin, 1, p. 291; 1905.

The equations given above, derived on the supposition of these quantities being constant, therefore should not be expected to give exact relations. They should, however, indicate fairly well how an instrument of good design and careful construction will behave under any ordinary set of conditions. A comparison of observed and calculated values in the following work gives in many cases differences no greater than the experimental errors.

9. METHOD OF TUNING.

The method of tuning first used consists of applying to the terminals of the galvanometer a small alternating electromotive force and adjusting its frequency or the free period of the moving system until a maximum deflection is obtained. This is the method usually used and for some purposes it is very satisfactory. In general, however, the change in amplitude of the vibration with a change in tuning is small, especially near resonance. If the tuning is made in this way, and observations taken for the electromotive force and current sensibility, a correct value is obtained for the former, but there may be an error of 50 per cent or more in the latter.

If, however, a high resistance (of the order of 100,000 ohms) is placed in series with the galvanometer, an electromotive force of suitable value used, and the adjustment made for a maximum deflection, the tuning is very much better. This gives a sufficient accuracy in the tuning for most purposes, including the determination of the constants.

However, even with the high resistance a sufficient precision in the tuning could not be obtained for some of the work. It was therefore necessary to use some other method, and the one chosen makes use of the phase angle between the displacement of the moving system and the driving force. On the shaft of the generator is mounted a large disk back of and slightly above which is placed the line source of light. The disk cuts off about half of that portion of the line which it is desired to use except when exposed by notches, one for each pair of poles, cut in the edge. These notches are so placed with respect to the windings that they expose the lower half of the line only at the instant when the electromotive force is a maximum. Relatively large resist-

ances are used, especially in the galvanometer circuit, so that the resonance curve is sharp and the current may be assumed to be in phase with the electromotive force. One observes then a band of light above and a line of light below the edge of the disk. When the tuning is good, the line appears below the center of the band. When it is disturbed, the line moves toward one side or the other, depending upon the sign of the disturbance, and the displacement is considerable before the shortening of the band is appreciable. A glance at the image tells at once the sign and gives an indication of the magnitude of the lack of tuning so that the adjustment can easily and quickly be made. By using a large resistance in series with a galvanometer having a small moment of damping, this method indicates a lack of resonance when the difference in the frequencies is of the order of 0.01 per cent of the resonating frequency.

10. GALVANOMETERS USED AND THEIR CONSTANTS.

The galvanometers used were one of the Rubens type made by W. Oemke, one of the Campbell type made by Robert W. Paul, and one built in the Bureau of Standards' instrument shop. In what follows the galvanometers are referred to as Nos. 1, 2, and 3 in the order here given. The latter was designed with the idea of getting an accurately balanced system of known moment of inertia and of investigating the effect of air pressure. The general construction is somewhat like the Rubens type, except that a single magnet is used. The pole pieces are of transformer iron, and the winding consists of four coils. The moving system consists of a disk of transformer iron 10.5 mm in diameter and 0.36 mm in thickness, a circular mirror 10 mm in diameter and 0.58 mm in thickness, and bifilar suspensions of fine steel wire, the latter provided with a means for the independent adjustment of their tension so that any lack of symmetry may be compensated. As arranged, the suspensions form a coil of a single turn which, vibrating in a strong magnetic field, causes a considerable damping. This, however, is of little consequence, since for the present purpose we are not concerned particularly with the sensibility. The case is arranged so that it may be closed tightly and the air exhausted. The fine adjustment of the tuning is made by a soft iron shunt to the magnet. This shunt is on the outside of the case, so that a considerable

change in the period can easily be made even while the moving system is in a vacuum. Unless otherwise stated it will be understood that the case is open.

The constants of these galvanometers are tabulated in Table I.

TABLE I.
Constants of the Galvanometers.¹¹

Galvanometer No.	1	2	3
Alternating current sensibility.....	1.5	7.3	4.2
Direct current sensibility.....	.018	.010	5.5×10^{-4}
EMF sensibility.....	.0014	.0061	.0075
Resonating frequency.....	100	100	100
Resistance.....	234	30	74
3d harmonic current sensibility.....	.0030	.0017	9.2×10^{-4}
3d harmonic EMF sensibility.....	13×10^{-4}	5.6×10^{-4}	9.6×10^{-4}
5th harmonic current sensibility.....	.0010	5.6×10^{-4}	2.2×10^{-4}
5th harmonic EMF sensibility.....	4.3×10^{-4}	19×10^{-4}	2.8×10^{-4}
Power sensibility.....	.0021	.045	.032
Resonance effect.....	39	116	2700
Ratio of back to impressed EMF.....	78%	97.5%	87%
Resonance range (current).....	1.1%	.14%	.10%
Resonance range (EMF).....	2.3%	9%	4.6%
Time constant (current).....	.2	1.7	2.5
Time constant (EMF).....	.09	.045	.23
Moment of inertia.....	.012	.014	.024
Moment of damping.....	.12	.018	.016
Moment of restoration.....	4500	5700	7700
Moment of displacement.....	20×10^4	14×10^4	11×10^4

Those values obtained from measurements of the current sensibility, the direct current sensibility, the electromotive force sensibility, the resonating frequency, and the resistance are placed in the second column, while those values obtained by direct measurement or by an independent method are placed in the first column. Where there are values in both columns, the first serves as a check on the theory. Additional data for galvanometer No. 2 are given by Figs. 3 and 5, and for galvanometer No. 3 by Fig. 4.

¹¹ The units used in this table are the same as given on page 359, that is, the unit of amplitude of the vibration is a millimeter broadening of the image of a source of light 1 meter in front of the mirror; the unit of current is the micro-ampere and the unit of electromotive force is the micro-volt.

The generator used in obtaining most of the data given was directly connected to a motor operating in synchronism¹² with a tuning fork whose frequency could be varied over a wide range by weights and continuously over a range of about 1.5 per cent by a spring. This gave the necessary control and regulation for the work at and near 100 cycles. The electromotive force and current of three times this frequency was obtained from another generator whose speed was neither very constant nor definitely known.

11. CHARACTERISTICS OF THE INSTRUMENTS.

An inspection of the curves or a comparison of the constants will show that resonance has a marked effect upon the amplitude

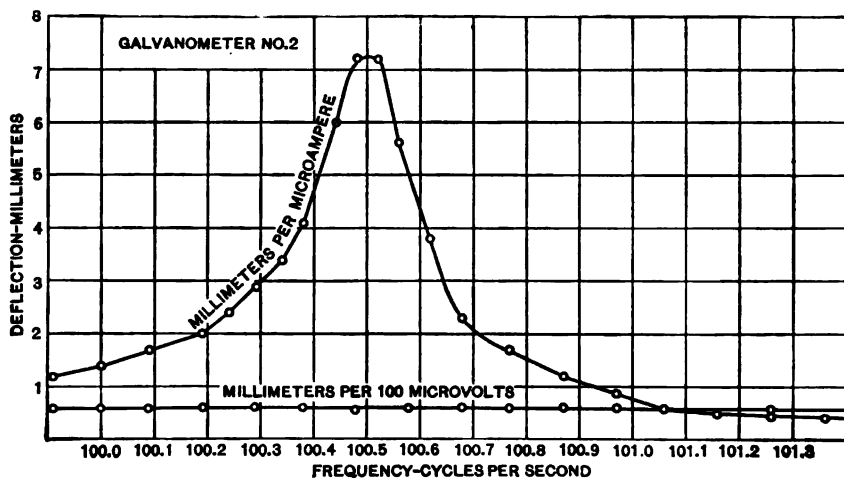


Fig. 2.—Showing the Effect of Tuning on Amplitude of Vibration with Constant Current and Constant Electromotive Force.

of the vibration. It seems to have been the idea that the effect of resonance¹³ was to increase the sensibility by about 100 times.

Taking the ratio of the alternating to the direct current sensibility gives: for No. 1, 83; for No. 2, 730; and for No. 3, 7600. If, however, by the effect of resonance we mean the ratio of the amplitude produced by a certain amount of power supplied at the resonating frequency to the deflection produced by the same amount

¹² Hough and Wenner, *Physical Review*, **24**, p. 535; 1907.

¹³ M. Wien: *Ann. der Physik*, **309**, p. 443; 1901.

Campbell *Phil. Mag.*, **14**, p. 497; 1907.

of power supplied by direct current, these figures are too large, since no account has been taken of the back electromotive force.

A matter closely associated with the resonance effect is the change in the ratio of the amplitude of the vibration to the current, with changes in the frequency of the latter. This ratio in the neighborhood of the resonating frequency is shown by Fig. 2, for galvanometer No. 2, and it will be seen that the ratio has a sharp maximum at 100.5 cycles per second and falls to one-half for a change of 0.14 per cent in the frequency.

It will also be noticed that a change of 0.5 per cent from the resonating frequency results in no noticeable reduction in the ratio of the deflection to the impressed electromotive force. Fig. 3 shows the ratio of the deflection to the electromotive force at

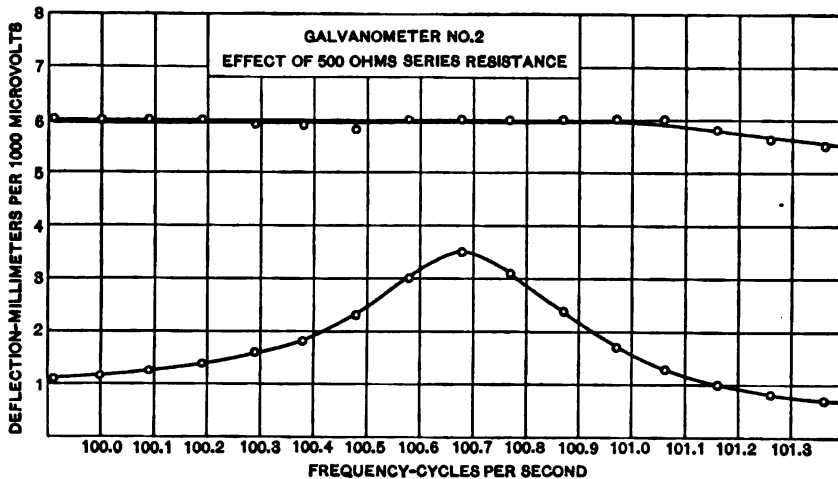


Fig. 3.—Showing Effect of Resistance in Decreasing Amplitude of Vibration with Constant Electromotive Force in the Circuit.

the terminals of the instrument and when 500 ohms are placed in series. The 500 ohms in series makes the resistance of the circuit 18 times larger, yet at the resonating frequency the ratio of the deflection to the electromotive force is reduced by only about 45 per cent. The reduction calculated from the constants given in the table is about 32 per cent. It is evident from the behavior of this instrument that when it is used in bridge work with a constant impressed electromotive force the resistance of the galvanometer circuit, if under 300 ohms, has but little effect upon the sensibility.

The effect of air damping on galvanometer No. 3, is shown by Fig. 4. It will be noticed that the air reduces the resonating frequency by about 0.75 per cent. This change in frequency is accounted for if we assume that the system drags with it a quantity of air whose moment of inertia is equal to that of a sphere of air having a radius slightly larger than the radius of the disk. It is also to be noticed that the removal of the air makes the resonance curve much sharper. The effect which is of real importance is the reduction of the sensibility by the air friction.

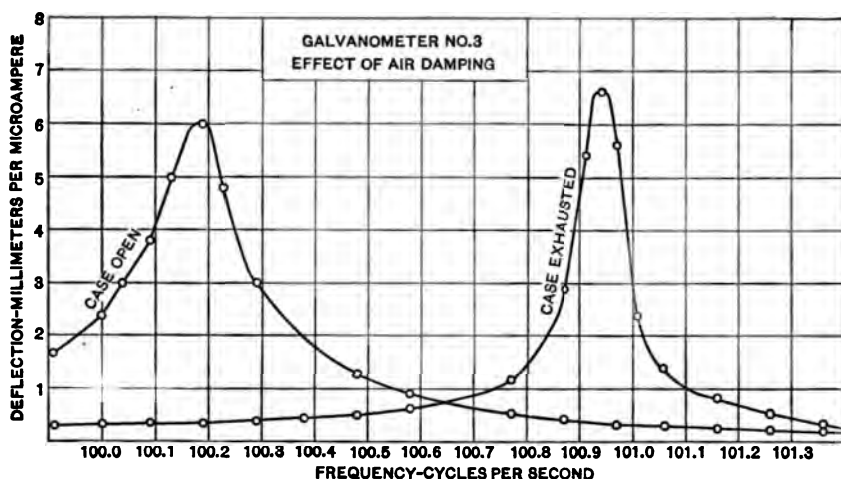


Fig. 4.—Showing Changes due to Air Damping, in Resonating Frequency, Sharpness of Tuning, and Current Sensibility.

Since the back electromotive force is proportional to the amplitude the power supplied per microampere is, according to the curves, about 9 per cent less when air damped. For the same deflection, then, the power supplied, when air damped, is about 11 per cent greater. While the adjustment of the instrument was changed between the time when the constants were determined and when the data for the curves were taken yet we may use, without appreciable error, the power sensibility given. This shows that the power in micro-microwatts necessary to maintain the vibration is 31 times the square of the amplitude. Taking 11 per cent of this gives as the power dissipated in air damping 3.5 times the square of the amplitude.

Fig. 5 shows for galvanometer No. 2 the effect of a large inductance on the ratio of the amplitude of the vibration to the electromotive force. The inductance used was the 480-volt winding of a 600-watt transformer and the increase in the ratio was 125 per cent. While the resonating frequency was 100.52, as shown in Fig. 2, yet the maximum amplitude of vibration occurred with a frequency of 101.06 cycles per second. The effect observed was much more marked with an inductance about a quarter as large. No effort was made to get an inductance of the most suitable value.

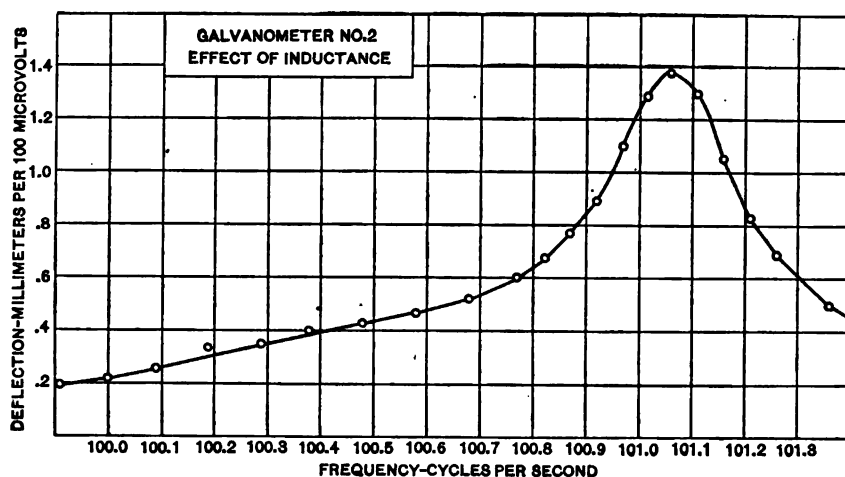


Fig. 5.—Showing Shift of Apparent Resonating Frequency and Increase of Amplitude of Vibration due to Inductive Circuit. Resonating Frequency 100.52, Apparent Resonating Frequency 101.06. Amplitude with Practically Noninductive Circuit .6 mm per 100 Microvolts.

12. AN ILLUSTRATIVE PROBLEM.

The consideration of the precision which may be attained in the balance of any one of the alternating current bridges will show the bearing of some of the above work on problems of this kind. On account of the simplicity of the equations, the problem chosen is the comparison of the product of a capacity and self-inductance with a frequency by the "series" method. Of course here only those factors which have to do with the precision of the balance will be considered. We are therefore concerned only with the case where the bridge is very nearly balanced and the galvanometer tuned to the frequency of the test current to be used. Most

of the equations following are only approximate and can not be used except under these conditions.

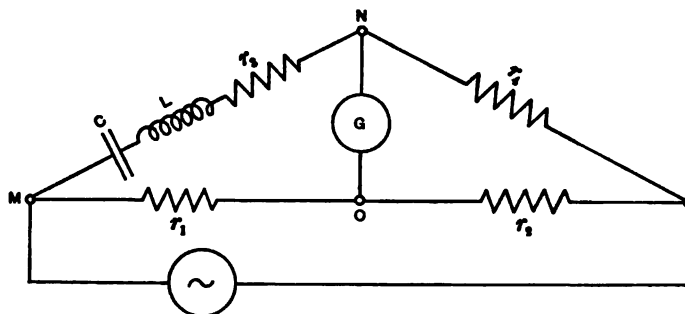


Fig. 6.—Series-Inductance Bridge.

The arrangement of the circuits is shown in Fig. 6, and the conditions for a balance are

$$\frac{r_1}{r_2} = \frac{r_3}{r_4} \text{ and } Lp = \frac{I}{Cp}. \quad (28)$$

For simplicity we shall take

$$r_3 = r'(1 + a) \text{ and } r' = r_1 = r_2 = r_4,$$

also

$$L' = L(1 + b),$$

where a is the lack, in proportional parts, in the resistance balance, and b is the lack, in proportional parts, of the inductance balance. Then if E^1 is the electromotive force impressed on the bridge the electromotive force between M and N and between M and O when the bridge is balanced is $\frac{1}{2} E^1$. When the bridge is unbalanced by a resistance r_3 in place of r' and an inductance L' in place of L the electromotive force between M and N is, if the galvanometer circuit is open,

$$\frac{E^1 \left[r_3 + \sqrt{-1} \left(L'p - \frac{I}{Cp} \right) \right]}{r_3 + r' + \sqrt{-1} \left(L'p - \frac{I}{Cp} \right)}$$

and since $L'p - \frac{I}{Cp} = Lbp$ this reduces to

$$\frac{E^1 (r' + r'a + \sqrt{-1} Lbp)}{2r' + r'a + \sqrt{-1} Lbp}.$$

Subtracting $\frac{1}{2} E^1$ gives as the electromotive force of the galvanometer circuit

$$E = \frac{E^1 (r'a + \sqrt{-1} Lbp)}{4r' + 2r'a + 2\sqrt{-1} Lbp},$$

or since for an approximate balance $4r'$ is the only significant term in the denominator,

$$E = \frac{E^1}{4r'} (r'a + \sqrt{-1} Lbp). \quad (29)$$

If the points N and O are connected by a winding of any electrical instrument in which the ratio of the impressed electromotive force to the current is m or simply by a conductor whose resistance is m and if the circuit is practically noninductive, the current through the instrument is given by the equation

$$I = \frac{E}{r' + m}, \quad (30)$$

and the power delivered to the instrument is given by the equation

$$w = \frac{E^2 m}{(r' + m)^2}. \quad (31)$$

w is a maximum when $m = r'$ or

$$w_m = \frac{E^2}{4r'}.$$

If the points N and O are connected by a vibration galvanometer having a resistance approximately equal to its reactance, the current is given by the equation

$$I = \frac{E - E_b}{r' + \rho} \quad (32)$$

where

$$E_b = D(E - r'I) \quad (34)$$

and

$$E_b = DE_1. \quad (35)$$

E_1 is the electromotive force impressed at the terminals of the

galvanometer. These three equations give

$$I = \frac{E(1-D)}{r'(1-D) + \rho} \quad (36)$$

and

$$E_2 = \frac{E\rho}{r'(1-D) + \rho}. \quad (37)$$

The power delivered to the terminals of the galvanometer is IE_2 , therefore

$$w = \frac{E^2\rho(1-D)}{[r'(1-D) + \rho]^2}. \quad (38)$$

To get the power converted w' , we may multiply w by D or

$$w' = \frac{E^2\rho(D-D^2)}{[r'(1-D) + \rho]^2}. \quad (39)$$

The condition for the maximum value of w' is that

$$D = \frac{\rho + r'}{2\rho + r'}, \quad (40)$$

or

$$\rho = r'(1-D). \quad (41)$$

Of these equations (29) gives the electromotive force of the galvanometer circuit, and this value substituted in (36) gives the current through the galvanometer or in (38) gives the power delivered to the terminals of the galvanometer. Equation (39) gives the power converted by the galvanometer and equations (40) and (41) the condition under which this power is a maximum. Where a transformer is to be used, equation (41) serves to determine the ratio which makes the sensibility of the whole combination a maximum. Since in well-designed and constructed instruments the power necessary to maintain a vibration of any specified amplitude depends only slightly upon the resistance or the ratio of the back to the impressed electromotive force (provided this ratio is over 75 per cent), the condition for maximum power converted is at least very nearly the condition for a maximum sensibility of the combination. Equations (40) and (41) will therefore be of service in determining conditions which will give the best results with a particular instrument.

Let us now consider the case where the resistances of the bridge arms are each 100 ohms, the self-inductance .25 henry, the frequency 100 cycles per second, and the electromotive force impressed on the bridge 20 volts.

In order that the bridge may be balanced the capacity required is a little over 10 microfarads. If the bridge is out of balance by 1 part in 100,000, both in the resistance and in the self-inductance, the electromotive force of the galvanometer circuit is, according to equation (29), 93 microvolts and the maximum power which the galvanometer may receive is, according to equation (32), 22 micro-microwatts. If galvanometer No. 2 is used, it will be seen from its power sensibility that unit amplitude of vibration is the maximum which can be obtained.

According to equation (38), the power actually received by the galvanometer is 6.1 micro-microwatts, and this gives .52 as the amplitude of the vibration. From equation (36) we get as the current through the galvanometer .071 microampere, and this multiplied by the current sensibility gives .52 as the amplitude of the

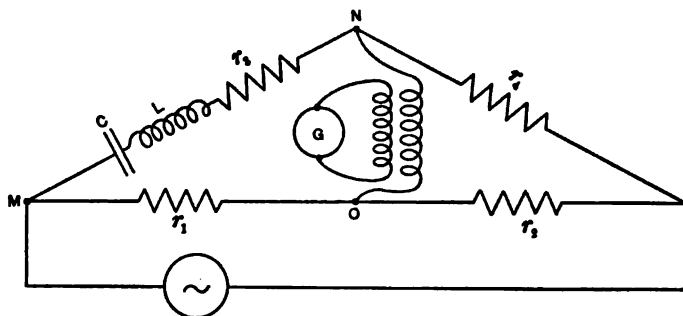


Fig. 7.—Series-Inductance Bridge showing transformer in Connection with the Galvanometer.

vibration. Again, using the electromotive force sensibility given by equation (23a), when ρ is taken as the total resistance of the galvanometer circuit (130 ohms), and multiplying by the electromotive force, 93, we get the same deflection.

If, however, we use a transformer connected as shown in Fig. 7, we see from equation (22a) that the step-up ratio of the transformer should be a little less than 3.5. Using 3.5 as the value of X in equation (26) gives .0105 as the electromotive force sensibility

of the combination and this multiplied by the electromotive force 93 gives .98 as the amplitude of the vibration. This, we have just seen, is about the maximum value attainable with this galvanometer.

The use of a transformer in this way, in effect, reduces the resistance of the galvanometer inversely as the square of the step-up ratio without changing its power sensibility or the ratio of the back to the impressed electromotive force. Reference to equation (41) shows that in effect the resistance of the combination of transformer and galvanometer should be 2.5 ohms to receive the maximum power available. Since the resistance of the galvanometer is 30 ohms (that of the transformer is assumed to be negligible), the step-up ratio should be $\sqrt{12}$. Taking this as 3.5 or 2.45 as the effective value of ρ , gives from equation (38) 22 micro-micro-watts as the power received by the galvanometer, and from the power sensibility we get again .98 as the amplitude of the vibration.

For a frequency of 300 the impedance of the arm containing the self-inductance and capacity is large in comparison with that of the other arms. In calculating the current through the galvanometer, we may consider this arm to have an infinite impedance, and, further, since the back electromotive force is so small in comparison with the impressed electromotive force, it need not be considered. The current, then, through the galvanometer is approximately the same as would flow through a resistance of 130 ohms in parallel with 100 ohms and with the combination in series with 100 ohms when the electromotive force is equal to that impressed on the bridge. This gives

$$I_3 = \frac{E_3}{350}.$$

If, then, the electromotive force impressed on the bridge has a 3d harmonic amounting to only 1 per cent, I_3 will be 570 micro-amperes. Multiplying this by the current sensibility for the third harmonic we see that we have a third harmonic vibration of about the same amplitude as produced by a lack of balance of one part in one hundred thousand for the fundamental. Under these conditions, then, the use of a test voltage having a 3d harmonic of 1 per cent limits the precision to about one part in one hundred thousand.

If a transformer having a step up ratio of three and one-half is used, the 3d harmonic current is reduced to about one-third of the former value, while as we have seen the sensibility of the combination to the fundamental is about doubled. This extends the precision of the balance to two parts in one million. It is thus seen that in certain cases the use of a transformer not only increases the sensibility of the combination but also permits the use of a test electromotive force having larger harmonics. If the relations given above for the use of a transformer are to hold with a fair approximation, it is necessary that the secondary have a reactance large in comparison with $1 \div (1 - D)$ times the resistance of the galvanometer, and that the resistance of the windings be small in comparison with the resistance of the circuits. In calculating the reactance it should be remembered that the permeability of the iron core is probably not over 200 for the very low magnetizing force used. For a frequency of 100, a 2-kilowatt 60-cycle "potential" transformer having the secondary wound for 15 times as many volts as the galvanometer has ohms will generally fulfill the conditions sufficiently well to give results within 20 per cent of those calculated from the equations. If bridges of different resistances are to be used the primary winding should be in sections to give the different ratios needed.

13. DESIGN.

While it is not our purpose to discuss fully, in this paper, the problem of the design of vibration galvanometers, yet a brief consideration, from this standpoint, of the theory and data given above will bring out a few of the principles involved. The consideration will be limited to instruments to be used in circuits of fairly low resistance, generally under 500 ohms and low electromotive force. This is the condition met with in alternating current bridge work which at the present constitutes the main field in which the vibration galvanometer is used. The instrument may be connected directly in series with the circuit, or to the secondary winding of a transformer the primary of which is connected in series with the circuit.

When the galvanometer is connected directly in series with the circuit, the relation between the amplitude of the vibration, the

impressed electromotive force, and the various intrinsic constants of the instrument is given by equation (9). It should be noticed that the amplitude is a maximum for a particular value of the moment of displacement, that is when

$$\psi^2 = r\beta. \quad (43)$$

This is really the same condition as is given by equation (40) and both signify that the mechanical power received by the moving system is a maximum when the back electromotive force is equal to one-half the electromotive force impressed on the circuit. With galvanometer No. 2 used on a bridge having a resistance of 100 ohms in each arm it will be seen that ψ^2 is about eight times as large as $r\beta$. As ψ is proportional to the field strength of the magnet, it follows that a reduction of 60 per cent in the field strength results in an increase in the sensibility of the combination of about 70 per cent. A comparison with equation (10) shows that this change results in a decrease of about 40 per cent in the power sensibility. Where $\psi^2 = r\beta$ equation (9) takes the form

$$\phi = \frac{E}{p\sqrt{2r\beta}}. \quad (44)$$

As r is made up of the resistance of the galvanometer and the arms of the bridge in a series parallel combination but little is to be gained by making the resistance of the galvanometer excessively small. In inductance and capacity bridges p is generally fixed from other considerations, so we have β as the only factor under this condition, which materially affects the sensibility of the combination. The power necessary to maintain a vibration of a given amplitude is proportional to the moment of damping. *The design then should be such that the back electromotive is approximately equal to one-half of the electromotive force impressed on the total galvanometer circuit, and the mechanical power necessary to maintain at the chosen frequency a vibration of any particular amplitude should be as small as possible.* It is necessary also to take into consideration the resolving power of the optical system to be used, as this often limits the amplitude of vibration which may be detected. Where the galvanometer is to be used regularly with a transformer the ratio of the back electromotive force to that impressed at the

terminals should be large, though there is little to be gained by making it over 90 per cent. Here, as before, the moment of damping, or the mechanical power necessary to maintain at the chosen frequency, a vibration of any particular amplitude, when the circuit formed by the winding is open, should be as small as possible.

It will be seen that the moment of inertia of the moving system does not affect directly the sensibility. It does, however, affect very materially the resonance range and the sensibility to the various harmonics. Where the balance depends upon the frequency or the wave-form the moment of inertia should be relatively large. On the other hand where the balance is independent of the frequency and especially if no special device is available for controlling the frequency, the moment of inertia should be very small to make the resonance range large.

We have seen (page 386) that the power lost by air friction by galvanometer No. 3 for a vibration of unit amplitude is 3.5 microwatts. If we assume that 75 per cent of this is lost by friction and pressure against the mirror we see that the power sensibility of a galvanometer having a frequency of 100, a mirror 1 cm in diameter and operating in air is limited to .4, or about 10 times the sensibility of galvanometer No. 2.

The sensibility may be made higher than this by changing the shape and reducing the surface of the moving system, yet if a very marked improvement is to be made in the sensibility it may be necessary to do away with the air damping by placing the instrument or at least its moving system in a vacuum.

14. SUMMARY.

1. The importance of the electromotive force developed by the relative motion of the magnet and the winding of a galvanometer is pointed out and the general theory of the vibration galvanometer is developed.

2. The fundamental equation is stated and, under certain conditions, a solution derived giving the various working constants in terms of the intrinsic constants.

3. The advantage of using a transformer with an instrument developing a relatively large back electromotive force is shown.

4. The double period sometimes observed is shown to be due to an unsymmetrical system.

5. A method of tuning is described which is more sensitive than the method generally used and which is applicable in some other cases where the vibration is forced. This method consists in adjusting the frequency of the driving force or the moving system so that the phase angle between the driving force and the displacement produced is 90° .

6. Equations are derived giving the intrinsic constants in terms of five easily measured quantities, viz, the resistance, the direct-current sensibility, the current sensibility, the electromotive-force sensibility, and the resonating frequency.

7. Intrinsic and working constants for three instruments are given, some of which were determined by two independent methods and thus serve as checks on the theory.

8. The principles to be followed in the design of a high-sensibility instrument for bridge work are developed, viz: (a) On account of the back electromotive force, heretofore overlooked, the resistance of the galvanometer should be very much less than the resistance of the bridge; (b) the back electromotive force should be one-half of the impressed electromotive force; (c) the mechanical power necessary to maintain at the desired frequency a vibration of a unit amplitude should be as small as possible.

WASHINGTON, May 25, 1909.

SPECIFIC HEAT OF SOME CALCIUM CHLORIDE SOLUTIONS BETWEEN -35°C AND $+20^{\circ}\text{C}$.

By H. C. Dickinson, E. F. Mueller, and E. B. George.

CONTENTS.

	Page.
I. THE CONTINUOUS-FLOW CALORIMETER.....	379
Calorimeter.....	380
Auxiliary apparatus.....	381
Observations.....	383
Computation.....	383
Results.....	384
II. THE VACUUM-JACKETED CALORIMETER.....	386
Calorimeter.....	386
Auxiliary apparatus.....	388
Calibration of the calorimeter.....	392
Radiation rate.....	394
Method of observation.....	396
Sources of error.....	399
Samples of calcium chloride tested.....	401
III. COMPARISON OF THE TWO METHODS.....	406
IV. SUMMARY.....	407

I. THE CONTINUOUS-FLOW CALORIMETER.

(H. C. Dickinson and E. F. Mueller.)

The circulating medium now most generally used in refrigerating plants is calcium chloride brine. The specific heat of this solution seems never before to have been determined with any considerable degree of accuracy for temperatures below 0°C , so that the values found are not applicable to the ordinary conditions of use. The present investigation was undertaken with a view to determining, with an accuracy of 0.1 or 0.2 per cent, the values of the specific heat at temperatures generally employed in practice.

The determination of the specific heat of a liquid which can not be mixed with water, at temperatures 60° below that of the surrounding atmosphere, presented some difficulties, particularly as in this case it was desirable to make determinations over temperature ranges of only 5° or 10° , e. g., -35° to -25°C . The con-

tinuous-flow calorimeter seemed to be adapted to the requirements of this problem.

Calorimeter.—To fulfill these special requirements, a calorimeter was constructed on the following general principle. A constant stream of cooled brine is made to flow through a coil immersed in a calorimeter, while the temperature of the calorimeter is maintained constant, and the same as that of its surroundings by heating electrically. The temperature of the brine at the inlet and outlet of the coil in the calorimeter is measured, and the amount of brine flowing in a given time is determined by weighing.

The details of this calorimeter are shown somewhat diagrammatically in figure 1. The calorimeter proper (*C*) is made of nickel-plated brass about 2 millimeters thick, in such a way that the cup can be withdrawn and the cover left in place supporting the brine coil (*S*), the heating coil (*M*), and the stirrer (*P*). The brine coil consists of about 2 meters of 6-millimeter copper tubing, which was first wound with cotton-covered constantan wire, then bent into a spiral and covered with a thick layer of shellac, to reduce the heat conductivity between the brine and the liquid (kerosene) in the calorimeter. The second heating coil (*M*) is also made of constantan, wound on an open frame and immersed directly in the kerosene. Each of the heating coils has a resistance of about 22 ohms. The three leads required for the two resistance coils are led out through fiber bushings in the cover. The interchangeable inlet and outlet for the brine coil are enlarged glass tubes, which contain thin copper thermometer sheaths (*T*), fitted with copper vanes.

The inner vessel or calorimeter proper is supported from the cover of the larger outer vessel, but has no metallic connection with it, fiber blocks being used for all supports to avoid conduction of heat from one vessel to the other. Particular care is taken to avoid transfer of heat between the incoming brine and the cover of the outer vessel. All the openings through the outer cover, except that for the stirrer shaft, are sealed, so that the whole outer vessel may be immersed in ice and water or in a stirred water bath, to keep its temperature constant. The air space within is kept dry by anhydrous CaCl_2 , to avoid any possible heat transfer by precipitation of moisture.

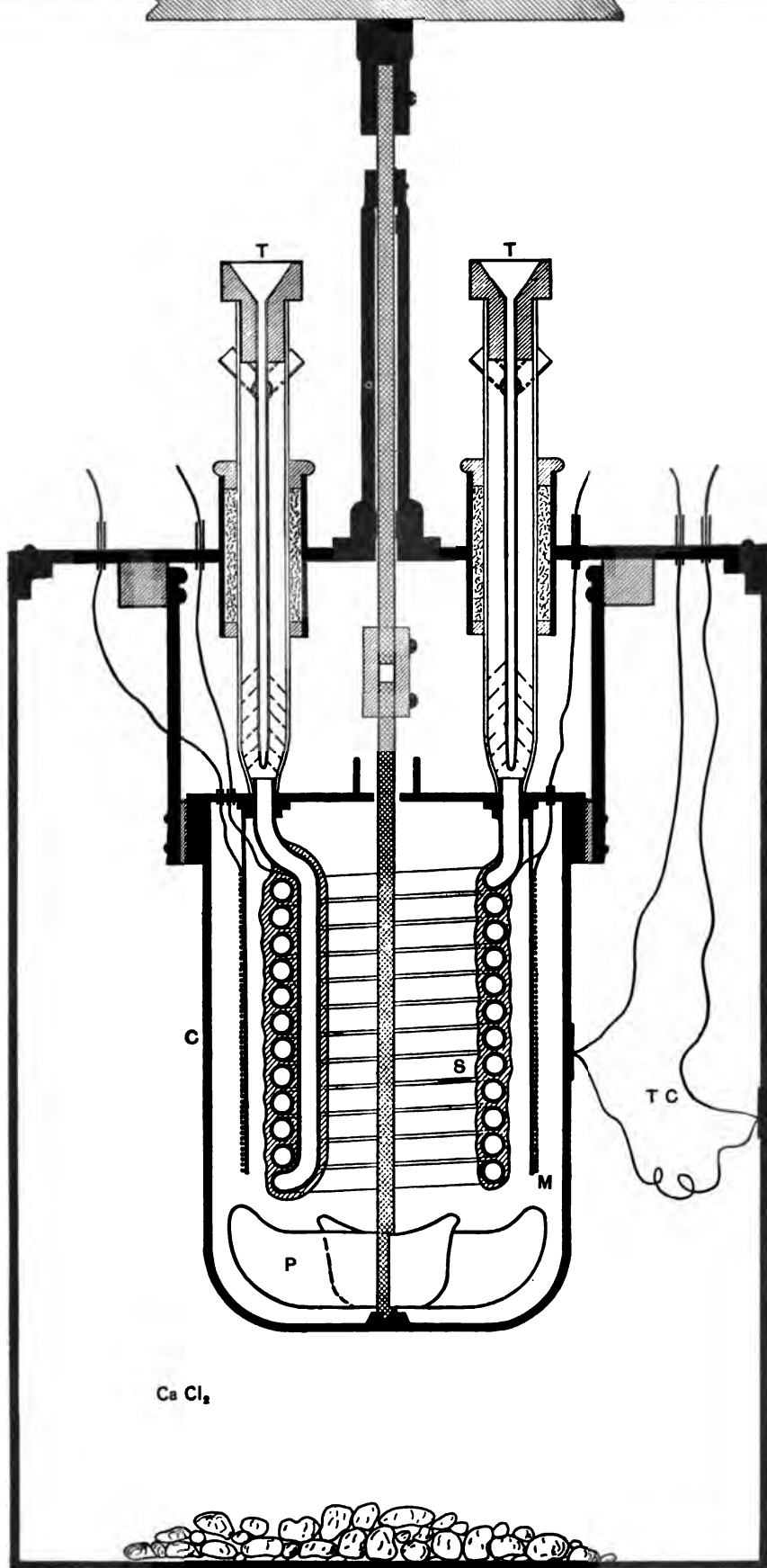


Fig. 1.—Continuous-flow calorimeter for determining the specific heat of brine. C, calorimeter

In operation, the calorimeter (*C*) is kept at the same temperature as the surrounding vessel, equality being shown by means of the copper-constantan thermocouple (*TC*) connected directly to a sensitive galvanometer. Differences of temperature not exceeding one or two tenths of a degree may occur, and a calibration of the thermocouple enabled a correction to be made for them.

The centrifugal propeller (*P*) is driven by a belt from a small motor and the speed kept constant by a friction governor on the motor.

The use of two heating coils, as mentioned above, one of which is in intimate contact with the brine coil but not with the calorimeter liquid, while the other is immersed directly in the calorimeter liquid, makes it possible to choose both the rate of flow of brine and the difference of temperature between inlet and outlet within rather wide limits, by simply varying the ratio of the amounts of energy supplied by the two coils. In this way, also, observations could be made with the same solution over the same temperature range, but with either a small or a very large difference between the temperature of the outer bath and that of the brine. This was done in a number of instances as a check on the accuracy of the method, with satisfactory results.

Auxiliary apparatus.—Measurement of the temperature difference between the inlet and outlet is made by means of two sensitive platinum resistance thermometers, which, together with the bridge used in making the resistance measurements, have been previously described.¹ The two thermometers are inserted in the

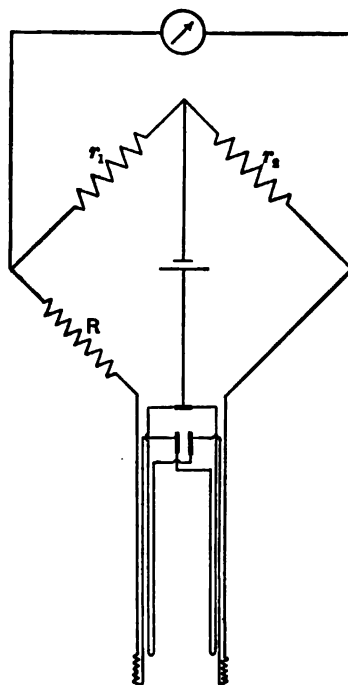


Fig. 2.—Wheatstone bridge and thermometer circuit, showing the method of connecting platinum-resistance thermometers for obtaining temperature differences.

¹ This Bulletin, 8, p. 641; 1907 (Reprint No. 68).

sheaths previously mentioned and the leads connected so that the difference in the resistances is measured on the bridge. The arrangement of the thermometers in the bridge circuit is shown in figure 2. The resistances are set to the nearest 0.01 or 0.005 ohm on the bridge, corresponding to 0°1 or 0°05 in temperature and the additional figures to 0°001 are determined from galvanometer deflections. The galvanometer used gave a deflection of about 7 millimeters per 0°1. The measurement of this difference offers considerable difficulty on account of the imperfect mixing of the emergent stream of brine in passing the thermometer and because the difference is not quite constant, sometimes varying by as much as 1 per cent. Equalization of temperature throughout the stream was secured by adding the vanes to the thermometer sheaths, while the mean temperature difference was computed from readings taken at ten-second intervals. The temperature of the brine as it enters the calorimeter is measured occasionally by means of the thermometer in the inlet side. This temperature need not be known closer than a few tenths of a degree.

In addition to the temperature measurement, the determination of specific heat with this calorimeter required measurements of time, mass, and electrical energy. The time (usually a ten-minute interval) was taken by means of a carefully rated watch, read with a telescope, and this measurement is reliable to about 0.05 per cent. An outlet tube is provided, which can be quickly turned at the beginning and end of a run to let the stream flow either into the weighing bottles or the waste bottle. The weight of brine which flowed through the calorimeter in ten minutes ranged from 3 to 5 kilograms and was weighed to tenths of a gram in the large glass bottle which received it.

The electrical energy supplied (at the rate of 100 to 300 watts) during the time the brine runs into the weighing bottle is computed from the time, the current flowing, and the difference of potential across the terminals of the heating coils. These measurements are made with a potentiometer in connection with a 0.1 ohm standard resistance and a volt box. The measuring system is protected against leakage from the high potential used on the heating coils by an equipotential shield.² When the two heating

² W. P. White, Potentiometer Installation, *Phys. Rev.* **25**, p. 340; 1907.

coils are used in parallel, each with its separate rheostat, the volt box is switched from the one to the other while the standard resistance is used to measure the total current. If the resistances of the two coils under working conditions are known the total energy can be computed. The electrical instruments have been frequently calibrated and measurements are all corrected to 0.01 per cent.

The supply of brine for these experiments is contained in an iron tank of about 100 liters capacity, immersed in a larger tank of brine containing an ammonia expansion coil for cooling to the desired temperature. The whole is so well packed with ground cork that the temperature of the inner tank changes only 0°2 or 0°3 per hour at -35°. The brine tank is arranged to give a constant head.

Observations.—Two observers are required and the observations are carried out in the following manner. After the supply of brine has been brought to the desired temperature, the brine is allowed to flow through the calorimeter, and the rate of flow adjusted to give about 400 grams per minute, and at the same time the switches are closed to send current through the heating coils. The current in the inner coil is then adjusted to give the desired rise of temperature (usually 10°), and finally the current in the outer coil is adjusted to maintain the calorimeter temperature the same as that of the outer vessel as shown by the thermocouple. When all conditions are steady the flow of brine is switched into the weighing bottle and allowed to run for exactly ten minutes, while one observer takes temperature readings every ten seconds and the other reads current and voltage and observes the temperature difference between the inner and outer vessels, and the time. Finally, the brine is weighed and its density determined.

Computation.—If the change of specific heat between temperatures T_1 and T_2 is proportional to change in temperature, then the specific heat σ at the temperature $\frac{T_1 + T_2}{2}$ may be computed from the following formula:

$$\sigma = \frac{(E_1 I_1 + E_2 I_2)t + m}{JW(T_2 - T_1)}$$

Where E_1 and E_2 are the potential differences at the terminals of the inner and outer coils, respectively; I_1 and I_2 the currents in these coils; t the time; and m a correction term (seldom amounting to 1 per cent of the total energy) to account for the energy due to stirring and change in the temperature of the calorimeter; W is the mass of brine; T_1 and T_2 the inlet and outlet temperatures; and J the number of joules per calorie, on the basis of the electrical units used. In these computations J has been taken as 4.196. The electrical units are the international ohm, and volt equal to $\frac{1000}{1434}$ of the emf. of the Clark cell at 15° C.

The following Table I is a sheet taken from the observations to serve as an illustration:

Date 2-24-08. Ten-minute run. Brine density 1.203. Two heating coils in parallel. Inlet temperature -20°0 C. Calorimeter temperature 0°00 (ice bath outside). All figures given have been corrected for instrumental errors and are the means of several readings.

	Inner coil.	Outer coil.	
Potential difference.....	60.459 volts	29.067 volts	
Resistance (under working conditions).....	22.757 ohms	22.475 ohms	
Observed sum of currents.....			3.9501 amp.
Computed currents.....	2.6568 amp.	1.2933 amp.	
Computed sum of currents.....			3.9501 amp.
Power.....	160.63 watts	37.59 watts	
Energy ($E_1I_1 + E_2I_2$) t			118930 joules
Energy supplied by stirrer (261 R. P. M.).....			+409
Energy for change in temp. of calorimeter (-0°08).....			+209
Total correction to energy, (m) = +409 + 209.....			+618
Energy supplied to brine.....			119550 joules
Weight of brine.....			3999.6 grams
$T_2 - T_1$ (from 50 readings).....			10°155
Specific heat at +14°9 C. = $\frac{119550}{3999.6 \times 10.155 \times 4.196}$			.7021

Results.—Results have been obtained for solutions of one of the purest commercial calcium chlorides at densities 1.259, 1.203, 1.140, 1.071, and for pure water. The latter was used as a check on the apparatus and method. The analysis of this sample of calcium chloride is given under No. 4, in Table VI.

The following Table contains the results of all the observations on a solution of density 1.203:

TABLE II.

Density 1.203

No. of Obs.	$\frac{T_1+T_2}{2}$	T_2-T_1 approximate	σ	Obs.—comp.
	° C.	° C.		
4	+ 3.8	9.7	.715	+ .002
4	+ 4.9	10.0	.716	+ .001
4	+13.7	10.3	.720	.000
1	+11.3	5.4	.717	— .001
3	—14.9	10.0	.703	+ .001
2	— 7.2	5.0	.706	.000
2	— 4.8	9.1	.708	+ .001
4	—16.5	10.2	.699	— .002
4	— 2.5	10.3	.708	— .001
4	+14.6	10.1	.720	— .001

The results of all the observations on this sample are summarized in the following formulas:

Density 1.260 $\sigma = 0.666 + .00064 t$ (from -35° to $+15^\circ$ C)

" 1.200 $\sigma = .708 + .00064 t$ (" -20° " $+15^\circ$ C)

" 1.140 $\sigma = .772 + .00064 t$ (" -10° " $+15^\circ$ C)

" 1.070 $\sigma = .869 + .00057 t$ (" 0° " $+15^\circ$ C)

where σ is the specific heat and t is the temperature. All densities are referred to a temperature of 20° C in terms of water at 4° . It will be noted that the relation between specific heat and temperature is linear within the limits of observational error. The following table is computed from these formulas:

TABLE III.

Temp.	Density 1.26	Density 1.20	Density 1.14	Density 1.07	Water .998
° C.					
—35	.644				
—30	.647				
—20	.653	.695			
—10	.660	.702	.766		
0	.666	.708	.772	.869	
+10	.672	.714	.778	.875	
+15	.676	.718	.782	.878	.9995

II. THE VACUUM-JACKETED CALORIMETER.

(H. C. Dickinson and E. B. George.)

Since an important part of this investigation is the comparison of the specific heats of different samples of commercial calcium chloride, a method was sought which would be more convenient than the flow calorimeter had proved to be, which would require a smaller amount of each sample of brine, and at the same time would serve as a check on the accuracy of the previous results. The somewhat familiar device of using a Dewar flask to reduce the heat transfer between the calorimeter and its surroundings seemed, with some modifications, to offer a basis for the construction of a calorimeter which could be filled with a weighed amount of cold liquid, as much as 50° below room temperature, and used in the ordinary manner, by measuring the heat supplied and the rise of temperature. After some preliminary experiments, a spherical Dewar bulb of about 6 liters capacity and with an excellent vacuum was selected as a basis of construction.

The method requires that the bulb be filled to a given level with the sample of solution at the lowest temperature to be observed and that the temperature be raised in steps (usually 5 degrees) by supplying energy electrically, alternating with accurate measurements of temperature by means of a platinum resistance thermometer, and finally that the amount of solution used be determined by weighing in the calorimeter. The calorimeter and auxiliary apparatus as finally used are described in the following pages.

Calorimeter.—The calorimeter proper, as shown by the accompanying figure 3, is a 6-liter, silvered, spherical Dewar flask *D* fitted with a brass top *B*, firmly fastened in place by means of plaster of Paris. A hard rubber cover *C* is fitted to this top and held in place by two thumb nuts. This removable cover supports rigidly the following: First, a heating coil *H* built up of two layers of "Advance" resistance ribbon, wound over mica on a thin brass tube and closed in by covering it with a sheet of very thin copper. The coil is thus hermetically sealed, and the brass and copper covering is protected from the action of the salt solution by a coating of elastic varnish. Current and potential leads from

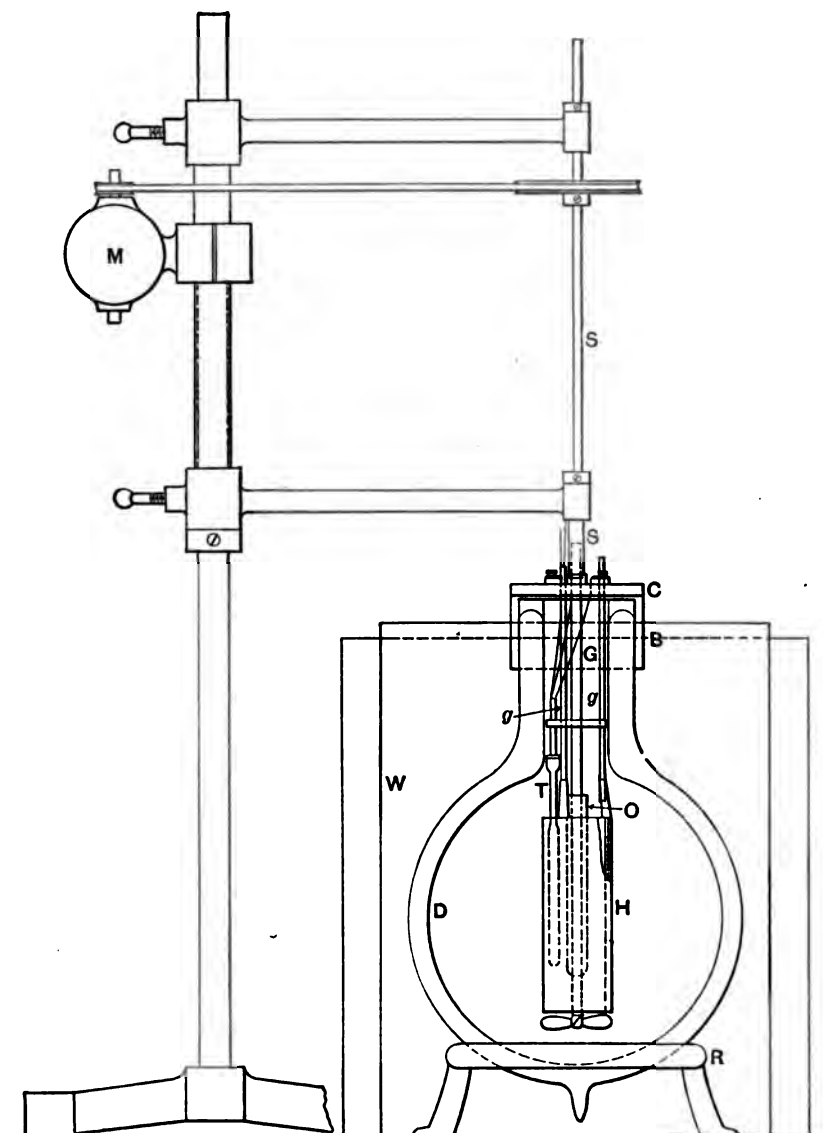


Fig. 3.—Dewar flask calorimeter and accessories.

this coil are led out through two of the three glass tubes *g* which support the coil from the cover. The third of these glass tubes has a small opening at the level of the top of the liquid in the calorimeter, and serves to indicate when the calorimeter is filled to the proper height and to draw off any excess of liquid. While filling, this tube is kept connected to the vacuum system, and the sound indicates when the liquid reaches the small inlet hole. Second, a very sensitive resistance thermometer *T* inclosed in a thin copper sheath and constructed in a manner similar to the two thermometers used in the first part of this investigation, except that three leads are used instead of four and the sensibility is somewhat greater. Third, an overflow tube *O*, which is continuously immersed in the liquid and serves to take up the amount which expands on heating. This device keeps the level of the liquid constant during an experiment, thus avoiding change in the amount of glass surface submerged and consequent change in the water equivalent of the calorimeter. The stirring device is also loosely supported from the cover in such a way that during operation there is no contact between the two. This stirrer consists of a glass tube *G* fitted at one end with a screw propeller and at the other end with a tapered brass sleeve fitting rigidly within the stirrer shaft *S*, which runs in bearings above the calorimeter and supports the glass tube and propeller free from contact with the calorimeter cover. The glass tube serves for filling or emptying the calorimeter, so that either operation can be carried out with the calorimeter fixed in position. The propeller is placed just below the hollow cylindrical heating coil so that a positive circulation is produced.

When in use the calorimeter rests on a rubber-covered ring *R* within a double-walled galvanized-iron tank *w*, and is immersed in a mixture of ice and water, or, for higher temperatures, in water only, at constant temperature.

Auxiliary apparatus.—The auxiliary apparatus used may be considered under three heads, mechanical, electrical, and apparatus for measurement of time. The mechanical arrangements for cooling the brine and filling the calorimeter are shown in figure 4. The solution to be used is prepared and cooled by immersing earthen jugs of the same in the brine reservoir of the

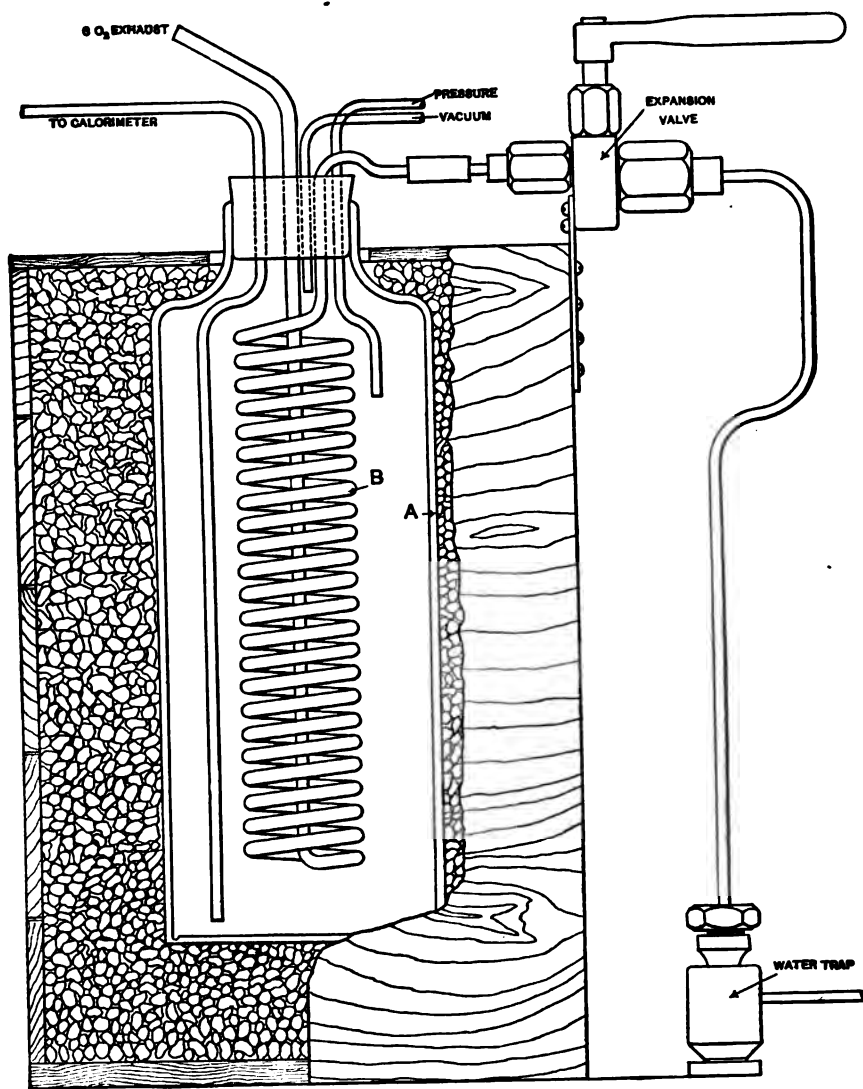


Fig. 4.—Apparatus for cooling brine and filling calorimeter.

bureau refrigerating system. When cooled the solution is drawn from the jugs into the refrigerating bottle *A* and there cooled by expansion of carbon dioxide in the coil *B* until crystals of ice begin to show. Enough of the solution is then forced from the bottle *A* into the calorimeter by means of compressed air to bring the surface of the solution in the calorimeter to the outlet previously described. At the end of a set of observations when the temperature within the calorimeter has been raised to 15° or 20° , the calorimeter is removed from the containing tank, carefully dried, and weighed by suspending it directly from the arm of a balance. From this weighing and the known weight of the calorimeter the weight of solution is found.

The electrical measurements consist of a determination of tem-

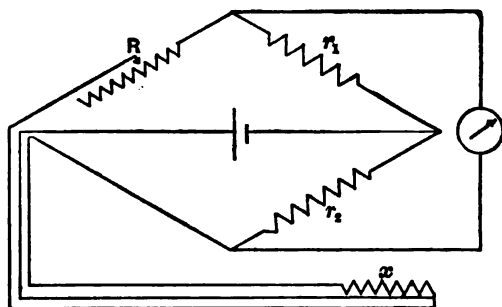


Fig. 5.—Wheatstone bridge and thermometer circuit.

perature by means of an accurate resistance thermometer used in connection with the special resistance bridge referred to in the previous part of this paper, and a determination of current and voltage during the time when the heating current is flowing through

the calorimeter heating coil. The arrangement of the thermometer and bridge circuits is shown in figure 5. Since measurements must be taken with a changing temperature the method used is to set the bridge resistance to a given value and note the time when the galvanometer deflection becomes zero. Several such observations are taken at temperature intervals corresponding to from 0.01 to 0.001 , according to the rate of temperature change. The arrangement of circuits for measurement of current and voltage and for recording the time is shown in figure 6. The adjustable resistance R is made equal to the resistance of the heating coil in the calorimeter, so that on throwing the quick-break switch S , no change is made in the current taken from the storage battery, and by using a battery which is partly discharged it is possible to keep the current constant to

well within 0.01 per cent for the time (usually five minutes) during which it is flowing through the heating coil. Measurements of current and voltage are taken alternately by throwing the switch *K*, which connects the potentiometer with either the standard 0.1 ohm or the volt box. All the resistances in the bridge, potentiometer, and volt box, and the standard 0.1 ohm were often checked, as in the first part of the work.

For the measurement of time a tape chronograph is used in connection with the Riefler clock kept by the Division of Weights and Measures. By the arrangement of switches shown in figure

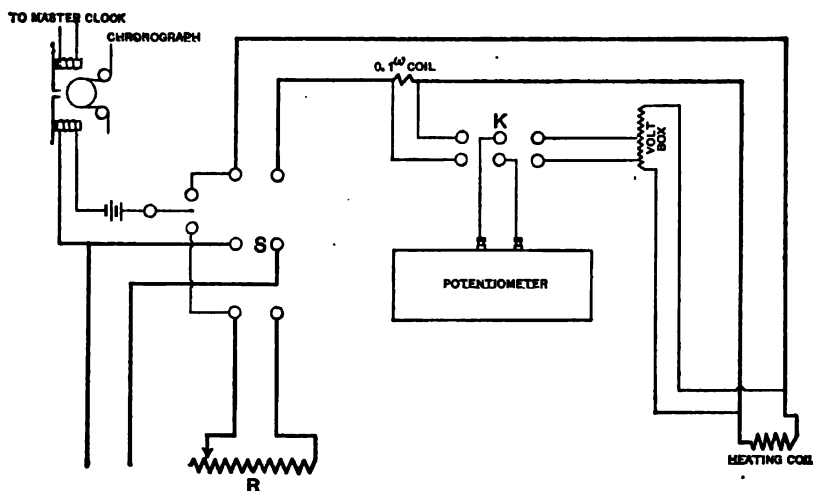


Fig. 6.—Diagram of circuits used for measuring energy and operating chronograph.

6 the operation of throwing the switch *S* automatically closes the chronograph circuit. The chronograph tape runs about 1 centimeter per second, so that times can be read easily to 0.02 or 0.03 second. The quick-break switch *S*, which operates the main current circuit and the chronograph circuit, is so made that there is not more than 1 or 2 millimeters of motion between the opening of the circuit on one side and the closing of the circuit on the other side. The time required to move this distance evidently enters as an error in the chronograph time interval, but since this error could not exceed a few thousandths of a second or a part in 100,000 it is neglected.

Calibration of the calorimeter.—The first step in the calibration of a calorimeter is usually the determination of its water equivalent. In the present case the water equivalent of the glass forming the inner wall of the Dewar bulb, up to the level of the surface of the liquid used, amounts to only about 1 per cent of the water equivalent of the solution, and the water equivalent of the heating coil, thermometer, stirrer, and overflow tube amounts to about 1 per cent more. Though the total water equivalent is thus only 2 per cent, its determination offers considerable difficulty. To determine this constant, observations were made with the calorimeter filled with pure water up to the designated level, measuring the energy supplied and the rise of temperature in the usual manner. Then three light rubber balloons having a volume of about 700 cubic centimeters were fastened down within the calorimeter and observations were taken with the water level the same as before but with the total mass of water some 700 grams less. By a comparison of these two sets of observations it is evident that the water equivalent may be computed independently of the value of the calorie in electrical units, but the accuracy of the water equivalent obtainable by this method is so far below the accuracy of the observations themselves that the method was considered impracticable.³ The only way to improve the relative

³ Assuming two observations made under similar conditions, except that the amount of water in the calorimeter is changed and the corresponding energy supply is so regulated that the same temperature rise is measured in the two cases, the equations

$$(W_1+x)\theta J=E_1 \qquad (W_2+x)\theta J=E_2$$

represent the two observations.

W_1 and W_2 are the amounts of water used

E_1 and E_2 are the total energy supplied

x is the water equivalent to be found

θ and J are respectively temperature rise and the mechanical equivalent.

Combining the two above equations to eliminate J , and differentiating with respect to E and x

$$(W_1+x)dE_2+E_2dx=(W_2+x)dE_1+E_1dx$$

Let K be the ratio $\frac{W_2+x}{W_1+x}$ then $W_2+x=K(W_1+x)$ and $E_2=KE_1$.

Substituting to eliminate E_2 and W_2 and making $dE_2=-dE_1=dE$ and writing $W_1=W$ and $E_1=E$.

$$(W+x)dE+KEdx=-K(W+x)dE+Edx$$

(Note continued on page 393.)

accuracy would have been to increase the amount of water displaced by the rubber balloons and a further enlargement of them would have changed the conditions of the experiment without greatly increasing the accuracy. Since the object aimed at was a determination of the specific heat of calcium chloride solutions in terms of water, and since the volume specific heat is not very different for these solutions and for water, it is evident that a computation of the water equivalent based upon an assumed value of the specific heat of water in terms of the electrical units, may be used throughout the series of experiments without introducing in them more than a small fraction of any error in the assumed specific heat of water.⁴ The calorie is taken as 4.196

(Note continued from page 392.)

$$dE(W+x)(1+K)=Edx(1-K)$$

$$\frac{dx}{W+x} = \frac{dE}{E} \frac{1+K}{1-K}$$

Or the percentage error introduced by using a value of x found in this way is equal to $\frac{1+K}{1-K}$ times the percentage error in the energy measurement used to determine x . The same would be true if θ were varied instead of E . In the present work $K=.8$, so that $\frac{1+K}{1-K}=9$, i. e., an error of 0.1 per cent in energy measurements or temperature differences used for determining the water equivalent might introduce an enormous error in the value found for the water equivalent and a systematic error of 0.9 per cent in the final value of the specific heats.

⁴ Suppose that the water equivalent x is determined by assuming the value of J and that the same values of x and J are used in a subsequent specific heat observation, the two equations

$$\begin{array}{ll} (1) & (2) \\ (W+x)\theta J = E_1 & (M\sigma+x)\theta J = E_2 \end{array}$$

(using the notation of the previous note with $M\sigma$ the product of mass of brine and its specific heat) represent the observations.

Substituting $x = \frac{E_1}{\theta J} - W$ from (1) in (2)

$$\left(M\sigma + \frac{E_1}{\theta J} - W\right)\theta J = E_2$$

differentiating with respect to J and σ and dividing by θJ

$$M\sigma dJ + MJd\sigma - WdJ = 0$$

$$\frac{d\sigma}{\sigma} = -\frac{dJ}{J} + \frac{W}{M\sigma} \frac{dJ}{J} = \frac{dJ}{J} \left(\frac{W}{M\sigma} - 1\right)$$

(Note continued on page 394.)

joules throughout this investigation. The method of observing and of computing the results for determination of the water equivalent is the same as for determinations of specific heat, described later, except that for observations with water the space outside the calorimeter was filled with water kept at a constant temperature and stirred during the observations.

*Radiation rate.*⁵—The radiation rate for the calorimeter was found to be, as might be expected, proportional to the temperature difference between the inner and outer walls, to within the limits of observational error. This being the case, it seemed probable that the most reliable results would be obtained by the use of a radiation constant deduced from a series of observations rather than the rate obtained from each observation. For the determination of this constant, careful observations were made of the rate of temperature rise of the calorimeter with the stirrer running at constant speed, when the temperature within the calorimeter ranged all the way between -35° and $+20^{\circ}$, with the outside temperature at 0°C . The stirrer is so mounted that there is no possibility of any friction within the calorimeter except fluid friction, and the rate of stirring was regulated by means of a friction governor so that the energy supplied by the stirrer, only a few tenths of a watt, could be taken as constant. The temperature at which the calorimeter, with the stirrer running, was in equilibrium with its surroundings was determined from radiation rates taken near this equilibrium temperature, and the loss (or gain) of heat by the calorimeter was determined in joules per second per degree difference between this equilibrium temperature and the observed temperature of the calorimeter.

(Note continued from page 393.)

Therefore a percentage error in the assumed value of J produces in the final result, a percentage error only $\left(\frac{W}{M\sigma} - 1\right)$ times the error in J . In the present case

$$0.7 \left(\frac{W}{M\sigma} - 1 \right) = -0.2$$

i. e., an error of 1 per cent in the assumed value of J would introduce an error of from 0.0 per cent to 0.2 per cent in the results.

Evidently if $W = M\sigma$ (that is if the total water equivalent of the liquid in the two cases is the same) J is entirely eliminated.

⁵ "Radiation rate" here refers to the change of temperature due to all outside causes.

Observations of the radiation constant for different temperatures between -35° and 20° C indicate that it changes by a small amount, perhaps 5 per cent, but this change can not be readily determined and the use of different values is inconvenient. For this reason a method of computation was adopted which allows the use of the mean radiation constant over the whole range without introducing any error greater than that in the determination of the mean constant itself. The method is as follows:

The total correction in joules for an observation is $j = cT(\theta - \theta_0)$ where c is the radiation constant, T is a time interval, θ and θ_0 are respectively the mean temperature of the calorimeter and the equilibrium temperature.* If the radiation rate is observed at two temperatures, say 20° below and 20° above θ_0 and if the values of c as computed from these two rates and the observed θ_0 , differ slightly, a small change in θ_0 will serve to make the two values of c equal, leaving the total observed corrections the same, so that no error is introduced at $+20^{\circ}$ and -20° by using these new values of c and θ_0 . At other temperatures nearer θ_0 a small error is introduced which is greatest near θ_0 where $\theta - \theta_0$ is small.

In practice the value of c taken from a series of observations is $.197 \pm .002$; T is about 300 seconds, θ_0 assumed, never differs by more than 1° from the mean observed value, the total supply of energy in any one determination is about 100,000 joules, therefore the maximum error is 60 joules in 100,000, or .06 per cent.

The following table gives the series of values found for this constant and the mean which was used in all the computations.

TABLE IV.

$\theta - \theta_0$	c	$\theta - \theta_0$	c
$+20^{\circ}$	0.190	-25°	0.205
-19°	0.191	-23°	0.188
$+20^{\circ}$	0.198	-22°	0.195
$+21^{\circ}$	0.204	$+20^{\circ}$	0.201
$+20^{\circ}$	0.193	-19°	0.206

Mean $c = 0.197$.

* That is, the temperature at which the "radiation rate" becomes 0.

The total correction to be applied for radiation seldom exceeds 2 per cent and that only at the lowest temperatures in the case of the more dense solutions.

Method of observation.—A series of observations on a single filling of the calorimeter generally consists of from five to eight independent parts, each of which gives a mean value of specific heat for a temperature range of 5 degrees. These observations range from the lowest temperature at which the solution can be used up to room temperature, and are carried out in the following manner. After the calorimeter is filled with cold brine, as previously described, the stirrer is connected up and started. As soon as temperature equilibrium is established within the liquid, a series of observations of temperature is begun. At the lowest temperatures there are sometimes differences of temperature in the mass of the liquid amounting to as much as a hundredth of a degree, as shown by irregular indications of the resistance thermometer, so that in order to get a satisfactory measurement of the temperature of the whole mass some ten observations are made, as follows. The bridge is set for a resistance a little higher than that of the thermometer and the time is recorded when the galvanometer shows that a balance is reached, then the bridge is again set for a higher resistance, and so on by steps of .001 to .0005 ohm, corresponding to 0°01 to 0°005 C.

The mean of these resistances is taken as the true resistance at the mean time observed, and the true temperature is thus determined.⁷ When a satisfactory temperature measurement has been made the heating current is switched on to the calorimeter coil at a given time, the exact time is automatically registered on the chronograph tape, which is only allowed to run for a few seconds to include the even minute on the clock and the time of throwing the switch. The observer's watch is relied upon for starting the chronograph and throwing the switch. As soon as the switch is thrown an observer begins to take alternate readings, on the potentiometer, of current and voltage as described in connection with the electrical apparatus. At the end of five minutes (usually) the

⁷ Sometimes two such sets of observations were made, separated by an interval of several minutes, in order to compute or check the value of the radiation constant c .

current is thrown off, the time being again noted automatically, and as soon as the temperature distribution in the calorimeter becomes uniform a second series of temperature measurements is begun, as previously described. When the temperature in the calorimeter approaches that of the outside the change due to radiation becomes less and observations are taken at smaller temperature intervals down to 0.001. In this way alternate observations are taken on the temperature and the energy supplied. The energy supply at each step is sufficient to raise the temperature about 5 degrees. A sample sheet (Table V), taken directly from the series of observations, shows the method of observing and computing results.

The formula used in the above computation is as follows:

$$\sigma = \frac{1}{W\theta} \left\{ \frac{EIT + j}{4.196} - \theta_x \right\}$$

in which

σ = Specific heat.

E = emf. at terminals of heating coil (volts).

I = Current in amperes.

T = Time during which the current passes.

j = Radiation correction in joules.

θ = Difference of temperature produced by the energy supplied electrically and by radiation.

x = Water equivalent of the calorimeter and accessories.

W = Weight of brine.

The temperature differences θ are the differences of resistance $R_3 - R_1$, $R_3 - R_2$, etc., each multiplied by a factor which depends upon the constants of the particular platinum thermometer, and upon the mean resistances $\frac{R_1 + R_2}{2}$, $\frac{R_2 + R_3}{2}$, etc.

The radiation correction j , which amounts to less than 2 per cent of the energy supply, is computed in terms of energy and difference in resistance of the resistance thermometer from its resistance at the equilibrium temperature of the calorimeter. The constant c (p. 395) is in practice modified to correspond with differences of resistance instead of differences of temperature.

A study of the temperature rise of the calorimeter during the time when energy is being supplied to raise its temperature about 1° per minute indicates that the effective radiating temperature of the outer surface of the calorimeter lags about ten seconds behind the temperature as computed from the energy supplied, except at the beginning and end of the rapid temperature rise. But the variations at the beginning and end were found to almost exactly balance each other, so that for the purpose of computation it is assumed that the radiation rate K_1 holds from T_1 to a time ten seconds later than the middle point of the time while energy is being supplied, and that the radiation rate K_2 holds from this point to the time T_2 .

Sources of error.—The errors in this determination of specific heats may be conveniently discussed under the heads of systematic errors and fortuitous errors. The chief sources of systematic errors are evidently in the determination of water equivalent, radiation constant, and calibration of the resistance thermometer and resistances. A constant error in the timing device might be considered, but in this case any error here has been shown to be too small to consider in comparison with the others. The water equivalent was determined with an apparent probable error of less than 2 per cent and the radiation constant with an error of less than 4 per cent. Since the water equivalent itself is only 2 per cent the probable error due to this determination is under 0.04 per cent. Since the radiation correction is less than 1 per cent for all but the lowest temperatures the probable error due to radiation is of about the same magnitude. Errors in the electrical measurements were always kept smaller than 0.02 per cent. The results at 0° as compared with those obtained by means of the continuous-flow method described in the earlier part of this paper show no difference as great as 0.05 per cent, except for brine of a density 1.26 which was so viscous that the results by the continuous-flow method were always somewhat in doubt.

Since the calibration of the calorimeter was carried out with the surrounding bath at a temperature of about 20° C while the observations on brine were made with this bath at 0° , it is evident that an error in the method of correcting for radiation, as described above, would enter systematically into the results. With this fact

in view, a series of observations were taken on the same sample of brine of density 1.07, first with the outer bath at 0° , then at 25° . The mean results of these two series were identical, showing that whatever error exists is too small to be observed.

Errors which are purely accidental may occur in so many parts of the observations that a full discussion of them can not be undertaken here. The weighing of the mass of brine used in the calorimeter and determination of density were subject to the most serious error and an error in either of these measurements affects a series of from five to eight specific heats. Weighings were made to 0.01 per cent, but because of an unexplained steady change in the weight of the empty calorimeter amounting to nearly 0.1 per cent of the weight of brine, the weighings, taken at frequent intervals, are reduced to correspond to individual weighings of the filled calorimeter, and on this account the final weights may be in error by 0.02 or 0.03 per cent. The density determinations have about the same order of accuracy as the weighings of brine. Densities were determined by weighing samples of the brine used in each series of determinations in a calibrated liter flask. Uncertainties in the true mean temperature of the whole mass of brine in the calorimeter, because of imperfect stirring, seem to have affected the results for the less dense solutions.* The final collection of data shows that the accidental errors are much larger in the case of lower densities, and since all other causes of error are the same or less for these solutions and the different sets of observations are interspersed so that not all the observations on the denser brine came together, it seems to be shown conclusively that the lack of uniformity in temperature is almost entirely responsible for the larger range of observational errors for the less dense solutions. This defect could have been readily eliminated if it had been suspected, but the apparent regularity of the temperature changes at all temperatures except the lowest, indicated that equilibrium was established to within $0^{\circ}.001$ or $0^{\circ}.002$.

Considering all the sources of error, the accuracy of the final results should exceed 0.1 per cent at all points. In the case of

*At first thought it appeared that the mixing of the less dense solutions must be better than that of the more dense, but it is evident upon further thought that the apparent constancy of the thermometer in the less dense solutions may indicate merely uniformity of the stream lines, while an unstirred layer remains next the walls. With greater viscosity this layer is set in motion.

the more dense solutions, the variation of observed values from a smooth curve does not exceed this amount at any point. From the nature of the case an accuracy greater than this is not of sufficient importance to warrant further observations. The apparatus in its present condition, however, can be relied upon to give an accuracy of better than 0.1 per cent for any single observation provided sufficient care is taken in making the individual measurements and provided the rate of stirring is increased for solutions of low density.

Samples of calcium chloride tested.—The present investigation comprises observations on solutions of C. P. calcium chloride at densities of 1.07, 1.14, 1.20, and 1.26, on solutions of calcium chloride from the same lot as that used in the flow calorimeter and on samples from three other sources. The samples differ largely in composition, as is shown by the following chemical analysis (Table VI) made by the Chemical Division of the Bureau of Standards on samples of the solution after the specific heat determinations.

TABLE VI.
Chemical Analyses of Commercial Samples.

	No. 1	No. 2	No. 3	No. 4
CaCl ₂	17.72	19.09	16.33	24.75
MgCl ₂	6.53	3.92	5.98	0.00
NaCl	.62	0.67	0.55	0.63
Total solids	24.87	23.68	22.86	25.38
Water	75.13	76.32	77.14	74.62

Recalculated to per cent Total Solids.

	No. 1	No. 2	No. 3	No. 4
CaCl ₂	71.25	80.62	71.44	97.52
MgCl ₂	26.26	16.55	26.16	0.00
NaCl	2.49	2.83	2.41	2.48
	100.00	100.00	100.00	100.00

NOTE.—NaCl determined by weighing as Na₂SO₄.

TABLE VII.

Observed Specific Heats of Chemically Pure Calcium Chloride Solutions.

Temperatures		Densities			
		1.07	1.14	1.20	1.26
-25° C	-13° F				0.648
-20°	- 4°			0.695	.651
-15°	5°		0.764	.700	.654
-10°	14°		.768	.705	.657
- 5°	23°	0.873	.772	.709	.660
0°	32°	.877	.775	.712	.663
5°	41°	.880	.778	.715	.667
10°	50°	.882	.781	.719	.670
15°	59°	.884	.784	.722	.673
20°	68°	.887	.787	.725	.676

NOTE.—The following empirical formula gives a relation between density and specific heat at 0° C., satisfying the above observations to within 0.1 per cent.

$$D = 2.8821 - 3.6272\sigma + 1.7794\sigma^2.$$

D=Density.

σ =Specific heat.

The results of experiment on solutions of C. P. chloride are summarized in Table VII, and the results of individual observations are shown on the curves of figure 7.

From the curves given in figure 8 it may be seen that the specific heats of the various commercial samples differ from that of chemically pure brine by less than 0.5 per cent and are in some cases higher and in others lower.

Table VIII is made out for conveniently determining the heat capacity of brines of different densities. This table is computed by means of the formula given in Table VII.

To use the table it is only necessary to look up the specific heats for the desired density of brine at the upper and lower limits of the temperature range to be covered. Since the specific heat varies almost linearly, the mean of these two values will be the mean specific heat over the range of temperature to be used. For ex-

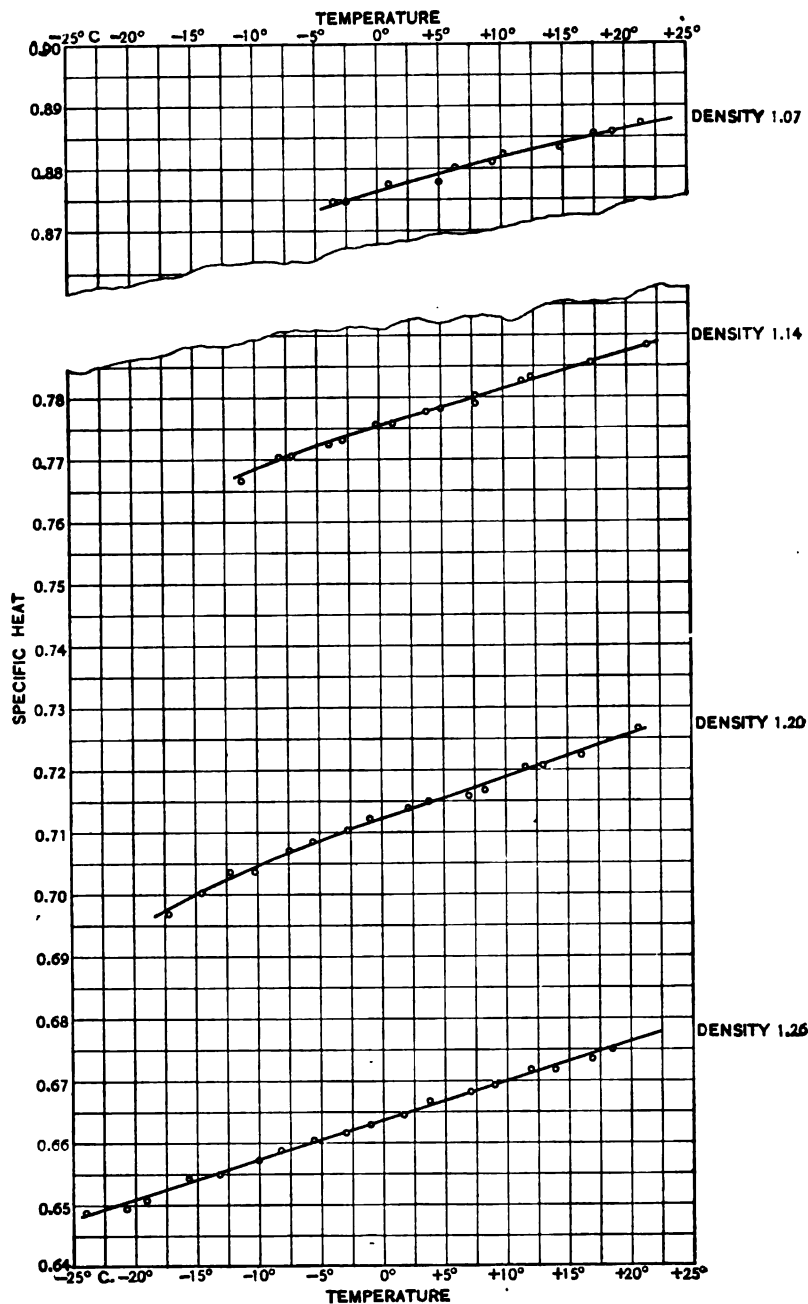


Fig. 7.—Temperature-specific-heat curves, showing observations for densities 1.070, 1.140, 1.200, 1.260.

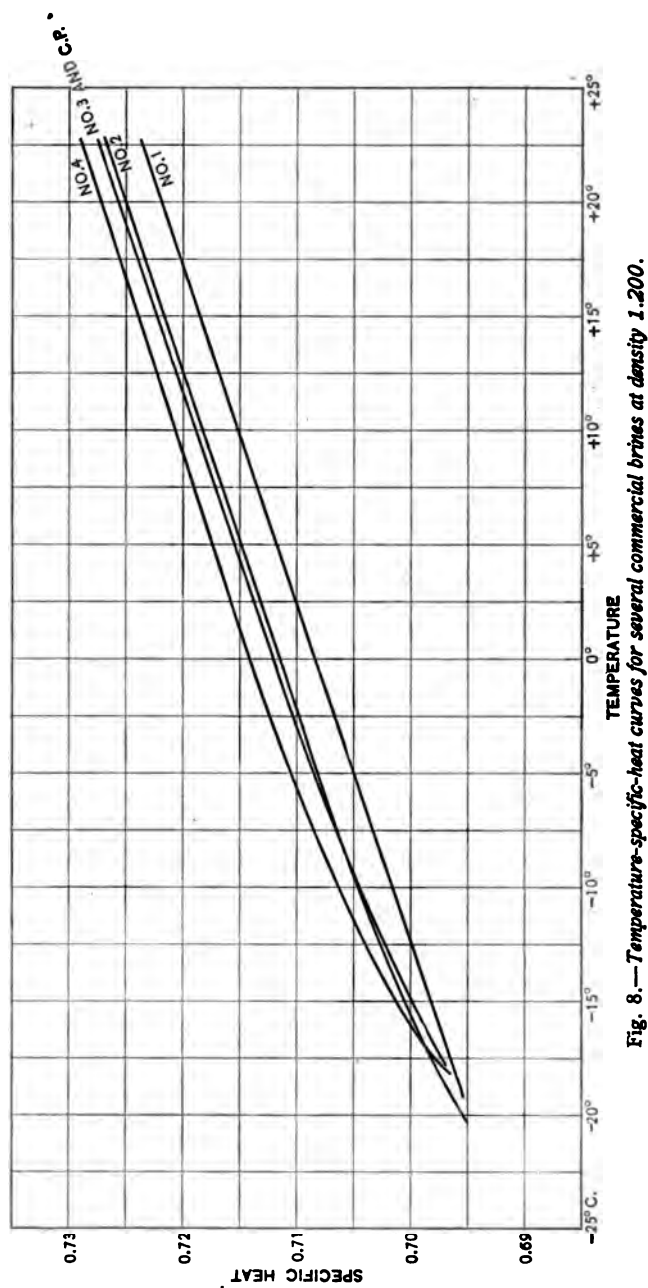


Fig. 8.—Temperature-specific-heat curves for several commercial brines at density 1.200.

ample, brine of a density 1.20 is to be used between temperatures -20°C and -10°C . The specific heat for brine of this density at -20° is found to be 0.695 and at -10°C , 0.705; the mean of these is 0.700. Thus, if brine of this density be cooled to -20° and allowed to warm up to -10° the heat absorbed is 0.700 times that for water or $0.700 \times 10 = 7.00$ Calories per kilo. Or in B. T. U., since the temperature range is 18°F ., the amount of heat is $.700 \times 18 = 12.60$ B. T. U. per pound.

TABLE VIII.

Showing Variation of Specific Heat with Temperature for Chemically Pure Brine of Several Densities in Common Use.

Temperature	Densities			
	1.175	1.200	1.225	1.250
-25°C				0.653
-20°		0.695	0.674	.657
-15°	0.725	.700	.679	.661
-10°	.730	.705	.683	.665
-5°	.734	.709	.687	.668
0°	.737	.712	.690	.671
$+5^{\circ}$	.740	.715	.693	.674
$+10^{\circ}$	.743	.719	.697	.677
$+15^{\circ}$	.746	.722	.700	.680
$+20^{\circ}$	.749	.725	.703	.683
-10°F			.670	.654
0°	.722	.697	.676	.659
$+10^{\circ}$	.728	.703	.681	.663
$+20^{\circ}$	.733	.708	.685	.667
$+30^{\circ}$	.736	.711	.689	.670
$+40^{\circ}$	.740	.715	.693	.674
$+50^{\circ}$	.743	.719	.697	.677
$+60^{\circ}$	.746	.722	.700	.680
$+70^{\circ}$	.750	.726	.704	.684

Several observations were made on the freezing points of the solutions tested and the results are given for solutions of C. P. calcium chloride in the following table:

TABLE IX.

Density	Freezing temperatures
1.12	— 9° C
1.14	—13°
1.16	—16°
1.18	—20°
1.20	—24°
1.22	—29°
1.24	—34°
1.26	—40°

The freezing point is taken as the temperature at which crystals begin to form. It is not definite to nearer than 1° and the above results are not reliable to better than 1° or 2°. The solutions of commercial calcium chloride of the same density differ only by 2° or less from the above values, and are in general lower.

III. COMPARISON OF THE TWO METHODS.

By the use of the flow calorimeter an accurate determination of the water equivalent is avoided, so that it is adapted to the fundamental determination of specific heats of fluids, especially the specific heat of water, in terms of the electrical units. Further, there is no loss by radiation, and these two advantages appeared at first to be very important. With the vacuum calorimeter, on the other hand, a radiation constant and a water equivalent must be determined, and as shown in the discussion, the latter is difficult to determine accurately. When, however, the vacuum calorimeter is used to determine the heat capacity of a substance as compared with that of water, the error in determination of the water equivalent becomes of very small importance, as may be seen from the note on page 393. In the present case the total correction for water equivalent and for radiation was reduced to less than 4 per cent. The flow calorimeter is subject to a large error in the measurement of the temperature difference, particularly when the liquid has high viscosity, on account of the necessity of measuring the temperature at certain points in the flowing stream. Any system of mixing which tends to eliminate this error increases the difficulty of obtaining constant flow.

In actual working, the vacuum calorimeter is distinguished by greater simplicity, convenience, and accuracy. The amount of solution required for working is from one-tenth to one-twelfth that necessary for the flow calorimeter, and the time required for a set of observations is in almost the same ratio.

The greater accuracy of the results obtained with the vacuum calorimeter is best seen from the fact that the curvature of the specific heat temperature lines was found from these observations, while the best that could be done with the flow calorimeter observations was to establish a linear relation. In this connection it is to be noted that the vacuum calorimeter is better adapted to giving the variation of specific heat with temperature, since the results of a single series of observations give the complete curve for one solution.

IV. SUMMARY.

Two calorimeters—a continuous-flow calorimeter and a Dewar flask calorimeter—have been adapted to the determination of specific heats of liquids over a temperature range of a few degrees in the interval between -35°C to $+20^{\circ}\text{C}$.

Measurements of temperature were, in both cases, made with platinum resistance thermometers.

Heat was supplied electrically and accurately measured by the potentiometer method.

The flow calorimeter was used for determinations of the specific heat of a single sample of commercial calcium chloride. The results obtained by this method showed individual variations as large as 1 per cent. The final results are probably accurate to 0.2 per cent or 0.3 per cent.

The inconvenience of the flow calorimeter and the time required for observations led to the adoption of another method. A calorimeter was built using a 6-liter Dewar flask containing a heating coil, thermometer, and stirring device. The individual results obtained by this calorimeter are in agreement to better than 0.2 per cent. Determinations of specific heat have been made with this calorimeter for chemically pure calcium chloride solutions of densities 1.07, 1.14, 1.20, and 1.26, and for three samples

of commercial calcium chloride of density 1.20. The solution used in the flow calorimeter was also used at densities 1.20 and 1.26. The results by the two methods are in agreement to within the accuracy of the observations with the flow calorimeter.

Freezing points of several solutions between densities 1.12 and 1.26 were also determined.

In conclusion, the authors wish to acknowledge their indebtedness for many valuable suggestions to Dr. C. W. Waidner, at whose suggestion the work was undertaken.

WASHINGTON, November 4, 1909.

ON THE DEFINITION OF THE IDEAL GAS.

By Edgar Buckingham.

1. *Nature and purpose of the definition.*—The notion of the ideal gas is that of a gas having particularly simple physical properties to which the properties of the real gases may be considered as approximations; or of a standard to which the real gases may be referred, the properties of the ideal gas being simply defined and the properties of the real gases being then expressible as the properties of the standard plus certain corrections which pertain to the individual gases. The smaller these corrections the more nearly the real gas approaches to being in the "ideal state." This conception grew naturally from the fact that the earlier experiments on gases showed that they did not differ much in their physical properties, so that it was possible to define an ideal standard in such a way that the corrections above referred to should in fact all be "small" in terms of the unavoidable errors of experiment. Such a conception would hardly arise to-day, or if it did, would not be so simple as that which has come down to us from earlier times when the art of experimenting upon gases was less advanced.

It is evident that a quantitative definition of the ideal standard gas needs to be more or less complete and precise according to the nature of the problem under immediate consideration. If changes of temperature and of internal energy play no part, all that is usually needed is a standard relation between pressure and volume. Boyle's law is universally adopted as this standard relation, so that the equation

$$(pv)_t = \text{constant} \quad (1)$$

is always a part of the definition of the ideal gas. In problems relating to the behavior of gases at either constant volume or

constant pressure, the p, v relation is sometimes of no importance, so that Boyle's law might be omitted from the definition with no sacrifice of the utility of the standard; but in practice, Boyle's law is always included, either directly or by implication.

Now, two gases might, conceivably, obey Boyle's law within any desired limit of precision and yet differ sensibly in relation to changes of temperature or of internal energy. Hence, if such changes enter into the problem at hand, the properties of our standard or ideal gas need further specification. For many physical purposes one further specification suffices. If, however, we go further and enter upon the consideration of chemical reactions, the properties of the standard may, with advantage, be still further specified.

2. *Absolute temperature.*—The absolute temperature by a gas thermometer is defined, on the constant pressure scale, by the equation

$$T_p = \text{const} \times v \quad (2)$$

or, on the constant volume scale, by the equation

$$T_v = \text{const} \times p \quad (3)$$

If the gas obeys Boyle's law, these two independent scales become identical,¹ if the size of the degree is the same for both, as, for instance, when the ice and steam points are made 100° apart on both scales. Hence we have the equation

$$pv = \text{const} \times T \quad (4)$$

as the definition of *the* absolute temperature on the scale of a thermometer filled with a gas that follows Boyle's law.

The thermodynamic temperature θ which constantly appears in the equations of thermodynamics is defined in quite another way. By Carnot's theorem the efficiency, E , of any thermal engine working periodically and reversibly by taking in heat from a hot body and giving out heat to a cold body, is the same for all such engines and depends only on the temperatures of the hot and cold bodies. By the principle of the equivalence of work and heat, the work W done by the engine during one cycle is equivalent

¹ See Note I at the end of this paper.

to the excess of the heat Q_1 taken in from the hot body over the heat Q_2 given out to the cold body, so that we must have

$$E = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} \quad (5)$$

Carnot's theorem makes no reference to the numerical values by which the temperatures of the two bodies may happen to be designated, nor even to the possibility of numbering temperatures at all; it does not involve the conception of temperature as a measurable quantity. Nevertheless, Carnot's theorem furnishes us with a convenient and simple scheme for numbering temperatures.

Let equation (5) be put in the form

$$\frac{Q_1}{Q_2} = \frac{1}{1 - E} \quad (6)$$

Then, since E depends only on the two temperatures, the same is true of Q_1/Q_2 , so that for any two temperatures Carnot's theorem determines a number—the value of this ratio Q_1/Q_2 . Now, since it is convenient in reasoning about temperatures to have the temperatures numbered, for identification, in accordance with some system, i. e., with some "thermometric scale," and since we are at liberty to select whatever system we find convenient, let us assign to any two temperatures such numbers θ_1 and θ_2 , as shall have the same ratio as Q_1 and Q_2 , θ_1 being the higher of the two temperatures. We then have the equation

$$\frac{\theta_1}{\theta_2} = \frac{Q_1}{Q_2} \quad (6)$$

as one element in the definition of a scale of temperature. The definition may be completed and the numerical value of every temperature fixed, by specifying that the numbers pertaining to the ice and steam points shall differ by 100, as in the common centigrade scales.

This system of numbering temperatures was devised by Kelvin in 1854 and is known as Kelvin's scale or the thermodynamic absolute scale. It may be simply defined, except as to size of the

degree, by saying: Any two temperatures are to each other as the quantities of heat taken in and given out at these temperatures by a reversible thermal engine working between them, i. e., with its source at the higher and its refrigerator at the lower.

Carnot's theorem may then be put in the convenient and familiar form

$$E = \frac{Q_1 - Q_2}{Q_1} = \frac{\theta_1 - \theta_2}{\theta_1} \quad (7)$$

the general form being

$$E = \frac{Q_1 - Q_2}{Q_1} = f(t_1, t_2) \quad (8)$$

when the scale of temperature, t , has not yet been specified.

Suppose, now, that we imagine the cylinder of the Carnot engine to be filled with a substance of which the mechanical and thermal properties are well known; it may be possible to find from these properties the value of W/Q_1 or E , in terms of the scale of temperature t , in which these properties are expressed. Since all reversible engines have the same efficiency, the value of E thus found will give us the value of $f(t_1, t_2)$ in equation (8) and so furnish a relation between t_1 and t_2 on the one hand, and θ_1 and θ_2 on the other, through the intermediation of equation (7). Suppose, in particular, that the cylinder of the engine is filled with a gas that follows Boyle's law so that we may use equation (4). When we try to find the value of W/Q_1 by using equation (4), it turns out that the problem can be solved if, and only if, we assume that the heat taken in during an isothermal expansion is all converted into outside work, or, in other words, that the internal energy of the gas does not change. The very simple result is, in this case,

$$\frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1} \quad (9)$$

or

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \quad (10)$$

The proof of this proposition is given in all text-books of thermodynamics and need not be repeated here.

We see, then, that if a thermometer is filled with a gas which (a) obeys Boyle's law, and (b) suffers no change of internal energy during change of volume at constant temperature, the ratio T_1/T_2 of any two temperatures measured on this thermometer is the same as the ratio Q_1/Q_2 of the quantities of heat taken in and given out by a reversible engine working between these temperatures. And since by definition θ_1/θ_2 is equal to this same ratio, it follows that

$$\frac{T_1}{T_2} = \frac{\theta_1}{\theta_2} \quad (11)$$

and the two scales are proportional and may be made identical by proper choice of the degrees. For such a gas we therefore have

$$T = \text{const} \times \theta \quad (12)$$

and equation (4) may be written in the form

$$pv = R\theta \quad (13)$$

R being a constant depending on the mass of gas and the units of p , v , θ .

In quantitative reasoning about gases, the absolute temperatures T_p and T_v occur very frequently. But in all thermodynamic equations, Kelvin's thermodynamic absolute temperature θ is also continually appearing. Our work is therefore much simplified if these three numerical values for any given temperature are identical, so that a single symbol for temperature is enough. The two gas-scale temperatures, T_p and T_v , become identical if the gas obeys Boyle's law; and, as has just been said, their common value T becomes, with the proper choice of degrees, identical with θ , if in addition, the internal energy of the gas is independent of the volume at constant temperature, i. e., if

$$\left(\frac{\partial \epsilon}{\partial v} \right)_\theta = 0 \quad (14)$$

Evidently, therefore, it is highly desirable that our ideal standard shall be so defined as to insure the validity of equation (13), if this can be done without making the ideal gas differ too much from the real gases. We have, then, to consider whether $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta$ is nearly zero for the real gases. If it is, equation (14) may be made a part of the definition of the ideal gas and the properties of the real gases still be regarded as differing from those of the ideal standard by quantities small enough to be considered as mere correction terms and disregarded in a first approximation.

3. *The Gay-Lussac effect.*—If we let a gas expand “freely,” i. e., without doing any outside work, any change of its internal energy must be compensated by a gain or loss of heat. And, conversely, any gain or loss of heat involves a corresponding change of the internal energy. Suppose that the process is made isothermal; then unless equation (14) is satisfied there will be an absorption or development of heat. If, on the other hand, the expansion is so sudden as to be nearly adiabatic, the temperature of the gas will fall, if the isothermal process required the addition of heat and rise in the opposite case. Hence, if in a sudden free expansion the mean temperature of the gas neither rises nor falls, there has been no tendency to take in or give out heat, and the internal energy has therefore not changed. Hence equation (14) is satisfied for this gas within the limits of observational error of the method.

In 1807 Gay-Lussac published the results of experiments of this sort on air, hydrogen, carbonic acid, and oxygen. He found that when the gas expanded by rushing from one vessel into another of the same size, previously exhausted, its mean temperature did not change by an amount which he could detect with certainty, the temperature in the first vessel falling as much as that in the second rose. Later experiments of a somewhat similar nature by Joule gave a like result. We may therefore conclude that for these gases the quantity $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta$, or the “Gay-Lussac effect,” is small and equation (14) nearly satisfied. Hence, if we make it a part of our definition of the ideal gas that equation (14) shall be exactly satisfied, or that the Gay-Lussac effect shall be exactly

zero, the gas thus defined will fulfill the requirement of representing the real gases with errors small enough to be negligible in a first approximation. The statement that the internal energy of a gas is independent of the volume during isothermal changes, which has here been put more briefly in the statement that the Gay-Lussac effect is zero, is commonly known as Joule's law.

Our results so far may then be shortly stated as follows: If a gas obeys Boyle's law, its constant pressure and constant volume scales agree; and if, in addition, it obeys Joule's law, its scale agrees with Kelvin's scale. In other words, equations (1) and (14) suffice for the validity of equation (13); and furthermore, equations (1) and (14) are nearly enough fulfilled by the ordinary real gases under ordinary conditions that they are suitable elements in the mathematical definition of the properties of an ideal standard gas for which we may then consider equation (13) as satisfied, this equation being of the highest utility in thermodynamic reasoning.

Having shown that equations (1) and (14) lead to equation (13), it is well also to show that they may be deduced from it or that (1) and (14) taken together are in all respects equivalent to (13). It is at once evident, if θ is set constant, that a gas which satisfies equation (13) must obey Boyle's law, so that we need only consider the Gay-Lussac effect, for which we shall deduce a general equation.

If ϵ represents the internal energy of any body, Q heat added to it, and W work done on it, the first law of thermodynamics states that in any infinitesimal change of state

$$\delta\epsilon = \delta Q + \delta W \quad (15)$$

The second law, in one of its various forms, states that if the change is reversible, $\delta Q/\theta$ is a complete differential, and that we may therefore write, for a reversible change,

$$\delta Q = \theta \delta \eta \quad (16)$$

where η , known as the entropy, is the quantity, completely determined by the state of the body, of which $\delta Q/\theta$ is the differential. If the only outside force on the body is a uniform normal pressure on its surface, the work done on the body may be written

$$\delta W = -p \delta v \quad (17)$$

Hence, for any such change of state, the two laws of thermodynamics may be summarized in the equation

$$\delta\epsilon = \theta\delta\eta - p\delta v \quad (18)$$

If the change is isothermal, this gives us

$$\left(\frac{\partial\epsilon}{\partial v}\right)_\theta = \theta\left(\frac{\partial\eta}{\partial v}\right)_\theta - p \quad (19)$$

Subtracting $\delta(\theta\eta)$ from both members of (18) we have

$$\delta(\epsilon - \theta\eta) = -\eta\delta\theta - p\delta v \quad (20)$$

and since changes of energy and entropy can occur only when the state of the body changes, $(\epsilon - \theta\eta)$ must be treated as a function of θ and v only, and the second member of equation (20) is a complete differential. It follows that

$$\left(\frac{\partial\eta}{\partial v}\right)_\theta = \left(\frac{\partial p}{\partial \theta}\right)_v \quad (21)$$

so that (19) may be put in the more intelligible form

$$\left(\frac{\partial\epsilon}{\partial v}\right)_\theta = \theta\left(\frac{\partial p}{\partial \theta}\right)_v - p \quad (22)$$

This perfectly general expression for the Gay-Lussac effect in any fluid was given in sensibly this form by Kelvin² in 1851.

From equation (22) we can compute the value of the Gay-Lussac effect for any fluid for which we know the p, θ relation. In particular, let the fluid be a gas which satisfies equation (13). We have at once, by differentiation and substitution in equation (22),

$$\left(\frac{\partial\epsilon}{\partial v}\right)_\theta = \theta\frac{R}{v} - p = 0 \quad (23)$$

so that equation (14) is necessary as well as, with (1), sufficient for the validity of (13). It is evident, however, that (14) may be satisfied by a substance that does not obey Boyle's law. It is

² Math. Phys. Papers 1, p. 227, equ. (4')

merely necessary that the constant volume scale of the given gas agree with the thermodynamic scale or that

$$p = A\theta \quad (v \text{ constant}) \quad (24)$$

for by differentiating and substituting in (22) we have

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \theta A - p = 0 \quad (25)$$

And furthermore, setting the second member of (22) equal to zero and integrating at constant volume leads directly to (24). Hence, if any gas obeys Joule's law, its constant volume scale agrees with Kelvin's scale, and vice versa, regardless of whether Boyle's law is satisfied. Evidently, then, for some purposes, in problems not concerned with changes in volume, a gas that obeys Joule's law exactly is sufficient as a standard and might be taken, in dealing with such problems, as the ideal gas.

4. *The usual definition of the ideal gas.*—The ideal gas is, then, usually defined as follows:

(a) It obeys Boyle's law; and either

(b) Its temperature scale agrees with the thermodynamic scale, or

(b') It obeys Joule's law, i. e., its Gay-Lussac effect is zero.

We have shown that these two definitions are equivalent in all respects, *b'* following from *a* and *b*, and *b* from *a* and *b'*. It may, however, be remarked that from an experimental standpoint neither *b* nor *b'* is altogether satisfactory. For it is desirable to have the ideal standard defined in terms of quantities that are easily accessible to experiment, so that any real gas may easily be tested to find how far it comes short of being in the ideal state. The statement of Boyle's law is perfectly satisfactory in this regard. But since we have no thermometer which reads directly in terms of the thermodynamic scale, we have no easy means of finding how the scale of any given gas thermometer deviates from that scale. The alternative specification that the ideal gas satisfies Joule's law suffers under a like disability. The Gay-Lussac experiment, or Joule's modification of it, is difficult or

impossible to carry out with the desired accuracy, as is shown by the fact that so able an experimenter as Joule got only negative results, although we are certain that the Gay-Lussac effect is not generally zero for any of the real gases.

The above definition of the ideal gas is therefore practically somewhat defective, though mathematically satisfactory, and we may ask whether the second part of the definition, b or b' , can not be so changed as to have a closer relation to practicable experimental work. The answer to this question was given by Kelvin³ in 1851, when, after the above-mentioned experiments of Joule, he invented the famous porous-plug experiment as a substitute for the free-expansion experiments.

5. *The Joule-Kelvin effect.*—Let the gas to be studied be forced uniformly, and so slowly that its kinetic energy is negligible, through a pipe which is stopped at one point by a plug of porous material. If the plug is sufficiently fine grained, the gas issues from it uniformly and quietly at a pressure somewhat lower than that at which it entered the plug. During the change of state of a given mass of gas which consists in passing through the plug from the higher to the lower pressure, the following equation must be fulfilled:

Heat put in = increase of energy + work given out.

The work done on 1 gram of gas, while it is entering the plug, by the gas behind it, is evidently the product of its specific volume and the constant pressure to which it is subject, or $p\nu$. As the gas issues from the plug, the work it does on the gas ahead of it is, similarly, the product of the new specific volume by the constant lower pressure. The whole work given out by the gram of gas is therefore the difference of these or the increase in the value of the product $p\nu$. Hence if the drop of pressure in the plug is infinitesimal we have

$$\delta Q = \delta e + \delta(p\nu) \quad (26)$$

³ 3 Math. Phys. Papers, 1, p. 220.

or, if δp is the numerical value of the drop in pressure,

$$\frac{\delta Q}{\delta p} = \frac{\delta \epsilon}{\delta p} + \frac{\delta(pv)}{\delta p} \quad (27)$$

We have as yet said nothing about possible changes of temperature, and equations (26) and (27) are quite general, being merely expressions of the first law as applied to this particular sort of change of state.

Let us now specialize by assuming the rate of supply or withdrawal of heat to be such as to prevent the gas from changing in temperature as it flows through the plug; then equation (27) reduces to

$$\left(\frac{\partial Q}{\partial p}\right)_\theta = \left(\frac{\partial \epsilon}{\partial p}\right)_\theta + \frac{\partial}{\partial p}(pv)_\theta \quad (28)$$

Let the heat supplied to unit mass of gas per unit *fall* of pressure be denoted by ρ , so that $\delta Q = \rho (-\delta p)$ or

$$\rho = -\left(\frac{\partial Q}{\partial p}\right)_\theta \quad (29)$$

The quantity ρ is known as the "Joule-Thomson or Joule-Kelvin effect." Equation (28) may now be written

$$\rho = -\left(\frac{\partial \epsilon}{\partial p}\right)_\theta - \frac{\partial}{\partial p}(pv)_\theta \quad (30)$$

which is merely a mathematical way of saying that as the gas flows through the plug the rate of absorption of heat equals the rate of increase of internal energy plus the rate of doing outside work.

Returning now to our former definition of the ideal gas, we see that if Boyle's law holds, the last term of equation (30) vanishes, leaving

$$\rho = -\left(\frac{\partial \epsilon}{\partial p}\right)_\theta \quad (31)$$

But we have

$$\left(\frac{\partial \epsilon}{\partial p}\right)_\theta = \left(\frac{\partial \epsilon}{\partial v}\right)_\theta \left(\frac{\partial v}{\partial p}\right)_\theta \quad (32)$$

and since $\left(\frac{\partial v}{\partial p}\right)_\theta$ could vanish only for an absolutely incompressible substance, it follows that $\left(\frac{\partial \epsilon}{\partial p}\right)_\theta$ and $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta$ must vanish to-

gether, if at all. Hence, for a gas that obeys Boyle's law and therefore satisfies equation (31) it is indifferent whether we specify that the Gay-Lussac effect shall vanish or that the Joule-Kelvin effect shall do so, and the second specification of our former definition may be replaced by this new one, if this is otherwise satisfactory as a part of the definition. It is satisfactory; for the Joule-Kelvin effect may be measured more accurately than the Gay-Lussac effect, and is therefore, from a purely physical point of view, a better quantity than the Gay-Lussac effect to have appearing in the equations which embody the definition. Furthermore, though measurable, it is small for the ordinary gases under ordinary conditions; so that if the ideal gas satisfies the condition of showing no Joule-Kelvin effect, it still fulfills the requirement of not differing much from the real gases. We may then adopt the equation

$$\rho = 0 \quad (33)$$

as substitute for (14), when taken together with equation (1), in specifying the definition of an ideal gas which shall satisfy equation (13).

It is sometimes implied that expansion through a porous plug is equivalent to a free expansion.⁴ If the gas is one that obeys Boyle's law, this is true for an isothermal expansion, since pv does not change and no external work is done. But unless Boyle's law holds this is not the case, and a gas that showed no Gay-Lussac effect might still show a Joule-Kelvin effect. For though by

equation (32) the term $\left(\frac{\partial \epsilon}{\partial p}\right)_\theta$ of equation (30) vanishes if the gas

obeys Joule's law, equation (33) will not be satisfied unless the gas obeys Boyle's law. Hence, the specification that the gas shall

⁴ See Note II at the end of this paper.

show no Joule-Kelvin effect can not replace the specification that it shall obey Joule's law unless it is also specified that the gas shall obey Boyle's law.

It is well to show directly that equation (33) follows from equation (13) by deducing a general equation for the Joule-Kelvin effect as we did in section 3 for the Gay-Lussac effect. Let us write equation (30) in the form

$$\rho = -\left(\frac{\partial \epsilon}{\partial v}\right)_\theta \left(\frac{\partial v}{\partial p}\right)_\theta - p \left(\frac{\partial v}{\partial p}\right)_\theta - v \quad (34)$$

By taking the value of $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta$ from equation (22), and utilizing the relation⁵

$$\left(\frac{\partial v}{\partial p}\right)_\theta \left(\frac{\partial p}{\partial \theta}\right)_v \left(\frac{\partial \theta}{\partial v}\right)_p = -1 \quad (35)$$

equation (34) may at once be reduced to the form

$$\rho = \theta \left(\frac{\partial v}{\partial \theta}\right)_p - v \quad (36)$$

Like equation (22) for the Gay-Lussac effect, this equation is general and applies to any fluid. From it we may compute the value of ρ for any substance for which the v, θ relation is known. In particular, if the substance is a gas which satisfies equation (13), differentiation and substitution in (36) give us

$$\rho = \theta \frac{R}{p} - v = 0 \quad (37)$$

so that the Joule-Kelvin effect is zero if the gas is ideal by the definition contained in equation (13).

Since the p, v relation is not involved in equation (36), we may have $\rho = 0$ for a gas which does not obey Boyle's law. In fact, if we set the second member of (36) equal to zero and integrate at constant pressure, we get

$$v = B\theta \quad (p \text{ constant}) \quad (38)$$

⁵ See Note I at the end of this paper.

Hence, in order that the Joule-Kelvin effect shall vanish, it is necessary and sufficient that the constant-pressure scale of the gas in question shall agree with the thermodynamic scale and the gas need not obey Boyle's law.

As regards insuring the validity of equation (13), we might dispense with any explicit reference to Boyle's law and merely say that the Gay-Lussac and Joule-Kelvin effects shall both be zero.

For if, in equation (30), ρ is zero; and if in addition $\left(\frac{\partial \epsilon}{\partial p}\right)_\theta$ is zero, as it is by equation (32) when the Gay-Lussac effect is zero,

it follows that $\frac{\partial}{\partial p} (pv)_\theta$ is also zero, or that Boyle's law is ful-

filled. Since, however, Boyle's law is the simplest part of the definition of our ideal standard gas, it would, in general, be disadvantageous to replace it by something else that is less simple.

6. *The specific heats.*—For many purposes the definition of the standard or ideal gas as already given is sufficient. Whenever in comparing the behavior of different gases we find some property which is nearly the same for all, it may be convenient, as a first approximation, to treat this property as exactly the same for all, so that reasoning about it shall apply to all the gases indifferently. We may, in other words, assign some fixed average value of this property to the ideal gas, to which the real gases are then approximations. The notion of the ideal gas might thus evidently be greatly enlarged and its definition extended to include a large number of specifications. There is no logical reason for stopping at one point rather than another; but in practice, the definition is usually terminated at the point already reached, the only further extension that is at all common being a statement regarding the specific heats.

During a change of temperature at constant volume no outside work is done, so that any heat taken in goes to increasing the internal energy, and we have the equation

$$C_v = \left(\frac{\partial \epsilon}{\partial \theta}\right)_v \quad (39)$$

as one way of defining the specific heat at constant volume. By differentiating with regard to v and noting that since v and θ are independent, the order of differentiation may be inverted, we get

$$\left(\frac{\partial C_v}{\partial v}\right)_\theta = \left[\frac{\partial}{\partial v}\left(\frac{\partial \epsilon}{\partial \theta}\right)_v\right]_\theta = \left[\frac{\partial}{\partial \theta}\left(\frac{\partial \epsilon}{\partial v}\right)_\theta\right]_v \quad (40)$$

Hence, if the Gay-Lussac effect $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta$ is zero, as for the ideal gas already defined, it follows that $\left(\frac{\partial C_v}{\partial v}\right)_\theta = 0$, and the specific heat at constant volume does not change with the volume so long as the temperature is constant. Experiment shows that for the real gases under ordinary conditions C_v is nearly constant. Hence, it is frequently made a part of the definition of the ideal gas that C_v shall be exactly constant. As we have just seen, C_v must be independent of the volume for the ideal gas already defined, but the statement that it shall also be independent of the temperature can not be deduced from the former definition and constitutes a new and independent element in the definition.

It is easily shown that for a gas that satisfies equation (13) the specific heats at constant volume and at constant pressure differ only by an absolute constant. Hence, from a mathematical standpoint it is immaterial whether we make the new addition to our definition by saying that C_v shall be constant or that C_p shall be constant; the former is more usual. There is, however, a difference in one respect, namely, that for the real gases the variations of C_v with temperature are much smaller than those of C_p , thus making the specification

$$\left(\frac{\partial C_v}{\partial \theta}\right)_v = 0 \quad (41)$$

the best form in which to put the addition to our old definition, as it is the most usual.

7. *Recapitulation of the definition.*—We find in common use the following definitions of the ideal gas which we have shown to be in all respects mutually equivalent, each being necessary and sufficient for the others:

(A) The ideal gas obeys Boyle's law and its temperature scale agrees with the thermodynamic scale; i. e.:

$$(pv)_\theta = \text{const}; \quad T = \text{const} \times \theta$$

or

$$pv = R\theta$$

(B) The ideal gas obeys Boyle's law and Joule's law, i. e.:

$$(pv)_\theta = \text{const}; \quad \left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \theta \left(\frac{\partial p}{\partial \theta}\right)_v - p = 0$$

(C) The ideal gas obeys Boyle's law and its Joule-Kelvin effect is zero, i. e.:

$$(pv)_\theta = \text{const}; \quad \rho = \theta \left(\frac{\partial v}{\partial \theta}\right)_p - v = 0$$

To these there is frequently added the further independent statement that:

(D) The specific heat of the ideal gas at constant volume is independent of the temperature, i. e.:

$$\left(\frac{\partial C_v}{\partial \theta}\right)_v = 0$$

Many further additions to the definition might be, but in ordinary practice are not, made.

The equations of thermodynamics may, by mere mathematical transformation and without the introduction of new physical facts, be put into such a vast array of different forms that the same substance may often be expressed in a variety of different ways in either words or equations. For some purposes the above definition might be expressed more conveniently in other sets of equations or in other statements about physical properties; but the form of definition given under (A), (B), or (C), with or without the addition of (D), contains the substance of the notion of the ideal standard gas as that notion is most generally used, and the equations are convenient mathematical expressions of it.

8. *The cohesion pressure of a real gas.*—If a gas has any self-attraction or cohesion, expansion involves an increase of the potential

energy of the attraction by an amount equal to the work done against the attractive forces. In an isothermal expansion, this forms a part of the value of $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta$, but it need not form the whole.

From the molecular-kinetic point of view, on which the mean kinetic energy of molecular translation is determined solely by the temperature, an increase of internal energy during isothermal expansion goes in part to increasing the intermolecular potential energy, but it may also, in any but a simple monatomic gas, go in part to increasing interatomic intramolecular potential energy if, as is quite conceivable, the molecules change their forms or dimensions with changing mean distance of separation and changing frequency of collision. If by cohesion we mean the attraction between the molecules, which acts to retard any molecule approaching the bounding surface of the gas and so decreases the mean change of momentum upon reflection, and therefore the pressure on the bounding wall, then we must admit that an absence of cohesion does not suffice to insure that the internal energy shall not change during an isothermal expansion. Or if by the "cohesion pressure" we mean the extra pressure that would have to be applied from without to preserve the same volume if the intermolecular attractions could be annihilated without otherwise changing the nature of the gas, then saying that the cohesion pressure is zero is not equivalent to saying that the Gay-Lussac effect is zero or that the gas obeys Joule's law. To put this idea in more precise form we may proceed somewhat as follows:

Let π represent the cohesion pressure. Then the equation of state of any fluid may always be written in the form

$$(p + \pi)(v - b) = R\theta \quad (42)$$

where R is a constant. It must be supposed that in general the intermolecular attractions will depend on both temperature and density, i. e., π will be a function of θ and v . But whatever be the form of π , the remaining quantity b , known as the covolume, may always be adjusted so that the equation shall be satisfied. In general, b also will evidently be a function of θ and v .

From equation (42) we have

$$p = \frac{R\theta}{v-b} - \pi \quad (43)$$

$$\left(\frac{\partial p}{\partial \theta}\right)_v = \frac{R}{v-b} + \frac{R\theta}{(v-b)^2} \left(\frac{\partial b}{\partial \theta}\right)_v - \left(\frac{\partial \pi}{\partial \theta}\right)_v \quad (44)$$

whence by substituting in equation (22) we have, as an expression for the Gay-Lussac effect in terms of π and b

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \pi - \theta \left(\frac{\partial \pi}{\partial \theta}\right)_v + \frac{R\theta^2}{(v-b)^2} \left(\frac{\partial b}{\partial \theta}\right)_v \quad (45)$$

From this we see that in any isothermal expansion δv , the increase of internal energy $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta \delta v$ is equal to the work $\pi \delta v$ done against the cohesion pressure, plus two other terms which can not in general be set equal to zero. So far as the mathematical form of the equation is concerned, either π or b may be given any expression we please, made constant for example, and the other may still be so adjusted that equation (42) shall represent the behavior of the gas. But in general the last two terms of (45) will not *both* vanish, nor need their sum vanish.

To illustrate, suppose that the equation of state has the simple form given by van der Waals, namely,

$$\left(p + \frac{a}{v^2}\right)(v-b) = R\theta \quad (46)$$

with a , b , and R constant. The derivatives of π and b with regard to θ vanish, and we do have

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \frac{a}{v^2} = \pi \quad (47)$$

But if we take an equation only slightly more complicated, such as the first equation of Clausius, or

$$\left[p + \frac{a}{\theta(v+c)^2}\right](v-b) = R\theta \quad (48)$$

with a , b , c , and R constant, equation (45) gives us

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \frac{2a}{\theta(v+c)^2} = 2\pi \quad (49)$$

and only half the Gay-Lussac effect is to be attributed to work done against forces represented by $\pi = \frac{a}{\theta(v+c)^2}$. In more complicated equations, with the covolume dependent on the temperature, the last term of (45) may also have a finite value. The physical interpretation of these two terms is impossible until we have some clear physical reason for assigning a particular mathematical form, $\pi = f(v, \theta)$, to the cohesion pressure π . But it is evident that we can not, offhand, say that the Gay-Lussac effect is a measure of the cohesion; and we can not, in the definition of the ideal gas, replace the specification that it shall obey Joule's law by the specification that its cohesion pressure shall vanish unless we *define* the cohesion pressure as in this case equal to the Gay-Lussac effect, and so to zero. The cohesion pressure is, in the nature of things, not accessible to direct measurement, whereas the Gay-Lussac effect may, in principle at least, be determined by simple experiment. Hence it is, in itself, altogether proper to adopt the equation

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \pi \quad (50)$$

as a definition of the cohesion pressure.⁶ But if we do this we meet another difficulty, for we find that the π thus defined will not fit into equation (42), which is a very useful equation. For since $\left(\frac{\partial \epsilon}{\partial v}\right)_\theta$ is a definite physical property of the gas, equation (50) fixes the form which π must have. But if (50) is to be satisfied and the π so defined is to be identical with the π of the equation (42), equation (45) gives us

$$\left(\frac{\partial \pi}{\partial \theta}\right)_v = \frac{R\theta}{(v-b)^2} \left(\frac{\partial b}{\partial \theta}\right)_v \quad (51)$$

⁶ See Webster and Rosanoff, *Phys. Rev.*, Sept., 1909.

And if π is already fixed by equation (50), equation (51) imposes a restriction on b , which can then no longer be adjusted at pleasure, so that, with the value of π already fixed by (50), equation (42) shall represent the behavior of the gas. In other words, the π of equation (50) can not *in general* be identified with the π of equation (42), and the cohesion pressure defined by (50) is not the same as what is usually meant by the cohesion pressure, the usual meaning of the term being that of the π in equation (42).

NOTE I.

(A) Let x , y , and z be any three quantities connected by a single relation which permits us to express any one in terms of the other two by an equation of the form

$$z = f(x, y)$$

Differentiating this we have

$$\delta z = \left(\frac{\partial z}{\partial x}\right)_y \delta x + \left(\frac{\partial z}{\partial y}\right)_x \delta y \quad (a)$$

Now let z be constant, i. e., $\delta z = 0$. This is an added condition, so that x and y can no longer change independently, and we must have

$$\delta x = -\left(\frac{\partial x}{\partial y}\right)_z \delta y$$

Introducing this value together with $\delta z = 0$ into equation (a) we have

$$0 = \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z \delta y + \left(\frac{\partial z}{\partial y}\right)_x \delta y$$

Dividing by $\left(\frac{\partial z}{\partial y}\right)_x \delta y$ and rearranging, we get this into the more compact form

$$\left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y = -1$$

a relation which must always hold between the derivatives of any three quantities of which any two determine the third.

(B) Let the three quantities be p , v , and t . Then we have

$$\left(\frac{\partial p}{\partial v}\right)_t \cdot \left(\frac{\partial v}{\partial t}\right)_p \cdot \left(\frac{\partial t}{\partial p}\right)_v = -1 \quad (b)$$

an equation which is frequently useful in transforming thermodynamic equations relating to fluids.

Let the substance considered be a gas that obeys Boyle's law so that

$$(pv)_t = A = f(t) \quad (c)$$

Then we have

$$p = \frac{A}{v} \quad \left(\frac{\partial p}{\partial v}\right)_t = -\frac{A}{v^2} = -\frac{p}{v}$$

Substituting in equation (b) and changing signs, we have

$$\frac{p}{v} \cdot \left(\frac{\partial v}{\partial t}\right)_p \cdot \left(\frac{\partial t}{\partial p}\right)_v = 1$$

Dividing this by $p \left(\frac{\partial t}{\partial p}\right)_v$, we have finally

$$\frac{1}{v} \left(\frac{\partial v}{\partial t}\right)_p = \frac{1}{p} \left(\frac{\partial p}{\partial t}\right)_v \quad (d)$$

This equation is merely a statement that as the temperature t (measured on any scale whatever) changes, the fractional change of volume at constant pressure is the same as the fractional change of pressure at constant volume; or that the volume and pressure coefficients are equal.

If we define two separate and independent scales of temperature by a thermometer filled with a gas, one of them T_p proportional to volume at constant pressure, and the other T_v proportional to pressure at constant volume, the two scales must evidently become proportional to each other if the volume and the pressure coefficients are equal, and identical if the degrees are made equal. Hence, for a gas which obeys Boyle's law the T_p and T_v scales agree.

NOTE II.

In Gay-Lussac's free expansion experiments, work was done by the part of the gas that finally remained in the original vessel on the rest of the gas, which passed over into the second vessel; but no work was done on anything outside. During the rush of gas which followed the opening of the stopcock between the two vessels some kinetic energy was produced, but this was soon dissipated by viscosity and the mean temperature of the gas was found to be unchanged, the temperature in the first vessel having fallen as much as that in the second vessel had risen.

In isothermal expansion through a porous plug by a gas that obeys Boyle's law, the phenomena are simpler. As in the Gay-Lussac experiment, no external work is done by the gas since $p\bar{v}$ does not change. But in addition to this, no one part of the gas does or has done on it more work than another, since the flow is steady and continuous. Furthermore, no sensible kinetic energy is generated, if the plug is sufficiently fine-grained, the dissipation corresponding to that of the eddy currents in Gay-Lussac's experiment taking place in the plug itself and during the expansion, whereas in Gay-Lussac's experiment the dissipation was, in part at all events, subsequent to the expansion. Hence, for a gas that obeys Boyle's law the Joule-Kelvin experiment, when performed isothermally, is equivalent to but somewhat simpler than the Gay-Lussac experiment.

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CONTENTS

	Page
Mica Condensers as Standards of Capacity <i>Harvey L. Curtis</i>	431
The Mutual Inductance of Two Parallel Coaxial Circles In Terms of Hypergeometrical Series <i>Frederick W. Grover</i>	489
A New Method for the Absolute Measurement of Electric Quantity <i>Burton McCullom</i>	503
The Comparative Sensitiveness of Some Common Detectors of Electric Oscillations <i>Louis W. Austin</i>	527
Photometric Units and Nomenclature <i>E. B. Rosa</i>	543
A Modified Method for the Determination of Relative Wave-Lengths <i>Irwin G. Priest</i>	573



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MICA CONDENSERS AS STANDARDS OF CAPACITY

By Harvey L. Curtis

CONTENTS

	Page
I. INTRODUCTION.....	432
II. METHODS OF MEASUREMENT OF CAPACITY.....	433
1. Absolute measurement of the capacity of a mica condenser.....	433
2. Alternating-current comparison of capacities.....	435
(a) Series-resistance method.....	435
(b) Series-inductance method.....	436
(c) Advantages and disadvantages of each method.....	438
(d) A direct-reading capacity bridge.....	439
(e) The choice of resistances.....	439
3. Alternating-current comparison of a capacity and an inductance...	440
4. Direct-current comparison of capacities.....	441
(a) The differential galvanometer method.....	441
(b) The method of mixtures.....	441
5. Methods of "stepping up".....	445
6. Measurement of absorption.....	446
III. EXTERNAL CONDITIONS AFFECTING THE CAPACITY.....	447
7. The temperature coefficient.....	447
8. The pressure coefficient.....	458
9. The effect of voltage.....	461
IV. THE STABILITY OF MICA CONDENSERS.....	463
V. THE RELATIONSHIP BETWEEN THE DIFFERENT APPARENT CAPACITIES..	467
10. Qualitative relations.....	468
11. Application of Maxwell's theory of absorption to direct-current measurements.....	468
12. Application of Maxwell's theory to alternating-current measurements.....	475
13. Determination of the geometric capacity.....	478
14. Effect of leakage.....	482
15. Possible sources of error in the measurement of the geometric capacity.....	483
16. Empirical formulas.....	485
17. Cases of practical importance.....	487
VI. SUMMARY.....	487

I. INTRODUCTION

In the choice of a condenser to be used as a standard of capacity, the following things need to be considered: Its stability, the effect of its absorption upon the measured capacity, and the influence of such external conditions as temperature, atmospheric pressure, and the applied voltage upon this capacity. The requirements necessary to satisfy the conditions thus imposed are only approximately fulfilled by air condensers, as has been shown by Giebe,¹ and it is the purpose of this investigation to show how nearly they are met in the case of mica condensers. Giebe found that with air condensers the effect of absorption and the influence of changes of voltage are not measurable, while the temperature and pressure coefficients may be readily determined. However, he found that, although his condensers were constructed with special reference to the attainment of stability, there sometimes occurred an unaccountable change of one part in ten thousand in the capacity within an interval of two or three days, and it is well known that ordinary air condensers often show much greater changes than this. Moreover, except for the smallest capacities, an air condenser is of necessity very bulky, so that it can be satisfactorily used in comparatively few cases.

A mica condenser is of much more convenient size, but the absorption produces a very marked effect upon the measured capacity. However, it has been shown by Zeleny,² Hill,³ and others that this effect is always the same under the same conditions, so that the apparent capacity by any method of measurement is perfectly definite. The effect of temperature upon the capacity has been investigated by Bedell,⁴ Terry,⁵ and others, but the author has not found any investigation of the effect of temperature and pressure upon the stability of mica condensers nor of the relation between the effects of absorption and temperature upon the capacity. This investigation was undertaken to determine the conditions under which a condenser is most stable, to find out as much as possible concerning the effect of absorption on the measured capacity, and to determine the effect of such external condi-

¹ *Zeit. für Instr.*, **29**, p. 301; 1909.

² *Phys. Rev.*, **22**, p. 65; 1906.

³ *Phys. Rev.*, **26**, p. 400; 1908.

⁴ *Phys. Rev.*, **1**, p. 81; 1893.

⁵ *Phys. Rev.*, **21**, p. 193; 1905.

tions as temperature, atmospheric pressure, and applied voltage upon the different apparant capacities. In this way it was hoped to determine to what extent a mica condenser can be considered a satisfactory standard of capacity.

II. METHODS OF MEASUREMENT OF CAPACITY

The methods of measurement of capacity used in this investigation can be classed under four heads: (1) Absolute measurement in terms of resistance and time; (2) alternating current comparison of one capacity with another; (3) alternating current comparison of a capacity with an inductance; (4) direct current comparison of two capacities. Beside these measurements, determinations of the phase difference, of the absorption, and of the insulation resistance have been frequently made. A complete discussion of all these methods is unnecessary, but in the following is given a description of the distinctive features of the methods which have been used in this investigation.

1. ABSOLUTE MEASUREMENT OF THE CAPACITY OF A MICA CONDENSER

The method most often used in the absolute measurement of a capacity is Maxwell's bridge method.⁶ A description of the way in which it is employed in this laboratory has already been published.⁷ In the case of condensers which show no absorption this is capable of a high degree of accuracy; but with condensers in which there is appreciable absorption the apparent capacity will not only depend on the number of charges and discharges per second, but will also depend upon the length of contact of the charge and discharge brushes, even though the number of charges and discharges per second remains constant. The importance of this last statement will be realized when one considers that the absorption current is relatively large during the first hundredth of a second after the charging begins, so that when the total length of charge is approximately three-thousandths of a second, a variation in the length of charge of

⁶ Maxwell, *Electricity and Magnetism*, § 776.

⁷ This Bulletin, 1, p. 153.

a thousandth of a second will produce an appreciable change in the quantity of electricity flowing into the condenser. Tables I and II indicate the magnitude of these effects in the case of good mica condensers.

TABLE I

To Show the Effect of the Number of Charges and Discharges per Second upon the Apparent Capacity of a Mica Condenser

Condenser	Capacity at 50 cycles	Capacity at 100 cycles	Change in parts in a hundred thousand
No. 6	0.500052 mf	0.499772 mf	56
No. 7	0.498666	0.498378	58

Both of these are condensers of moderate absorption.

TABLE II

To Show the Effect of Length of Charge and Discharge on the Apparent Capacity as Measured by Maxwell's Bridge

Condenser	Length of contact of charging brush, seconds	Length of contact of discharging brush, seconds	Relative change in capacity in parts in a hundred thousand
No. 6 A medium-grade mica	0.0019	0.0025	0
	0.0025	"	22
	0.0036	"	36
5471A0.1 A high-grade mica	0.0018	0.0025	0
	0.0021	"	8
	0.0025	"	11
5471A0.1	0.0025	0.0014	11
	"	0.0018	10
	"	0.0021	5
	"	0.0026	0
Air	0.0016	0.0014	0
	0.0025	0.0025	0

It follows that the apparent capacity of a mica condenser as determined by Maxwell's method may be made very definite by specifying the number of charges and discharges per second, and the length of time of charge and of discharge. Nevertheless, it is seldom that it will be of use in measurements of precision, as it is not often that the capacity under these conditions is desired. However, as will be shown later, the capacity determined by this method differs but little from the alternating-current capacity at the same frequency.

The method actually used in obtaining the apparent capacity of a mica condenser is to measure the capacity of an air condenser by Maxwell's method and then to compare it immediately with the mica condenser by whatever method is desired. This gives the absolute value of the apparent capacity. When the mica condenser has nearly the same capacity as the air condenser it will give very satisfactory results, but in case the mica condenser has a capacity more than 50 per cent greater than the capacity of the air condenser, it is more satisfactory to compare two or more mica condensers of the same nominal capacity as the air condenser with the air condenser and then combine these to give the desired value. This requires an investigation of the methods of "stepping up," which will be discussed later.

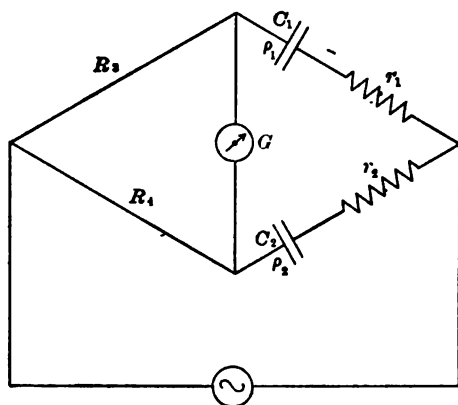
2. ALTERNATING-CURRENT COMPARISONS OF CAPACITY

Grover ⁸ has given a very complete description of the various bridge methods for comparing two condensers by alternating current. The two most valuable ones are (a) the series resistance method and (b) the series-inductance method.

(a) The Series-Resistance Method

In the series-resistance method the current in the two arms of the bridge is made to agree in phase by means of the resistances r_1 and r_2 in series with the condensers. This is necessary since, on account of the absorption, the phase angles of the condensers are not 90° , but they behave as though resistances ρ_1 and ρ_2 were in series with perfect condensers of capacities C_1 and C_2 . The quantities ρ_1 and ρ_2 are called fictitious resistances. In order that there may be a balance when a vibration galvanometer or a telephone

⁸ This Bulletin, 8, p. 371.



NOMENCLATURE

r_1 , r_2 , R_3 , and R_4 are resistances;
 C_1 and C_2 are capacities;
 ρ_1 and ρ_2 are the fictitious resistances of the condensers.

Fig. 1.—Diagram of the series-resistance method of comparing capacities

is used as an indicating instrument, the following relations must hold.

$$\frac{R_3}{R_4} = \frac{\rho_1 + r_1}{\rho_2 + r_2} = \frac{C_2}{C_1} \quad (1)$$

This shows that there are two independent adjustments which must be made to obtain a balance.

NOMENCLATURE

R_3 and R_4 are resistances;
 L_3 and L_4 are inductances;
 C_1 and C_2 are capacities;
 ρ_1 and ρ_2 are the fictitious resistances of the condensers.

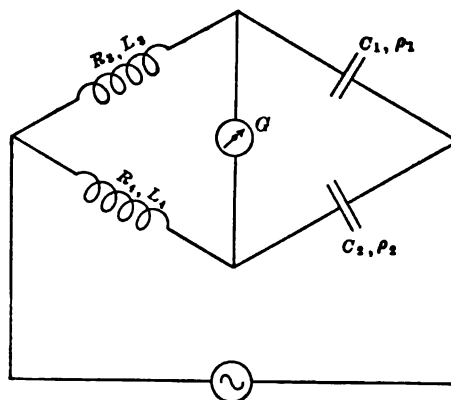


Fig. 2.—Diagram of the series-inductance method of comparing capacities

(b) The Series-Inductance Method

In the series-inductance method the currents are made to agree in phase by means of inductances placed in the resistance

arms of the bridge. It is necessary that one of these shall be a variable inductance.

A balance is obtained by successive variations of

R_3 or R_4 and L_3 or L_4 . Then

$$\frac{C_1}{C_2} = \frac{R_4}{R_3} + \frac{p^2 C_1}{R_3} (L_3 \rho_2 - L_4 \rho_1) \quad (2)$$

and

$$R_3 \rho_2 - R_4 \rho_1 = \frac{L_4}{C_1} - \frac{L_3}{C_2} \quad (3)$$

where $p = 2\pi$ times the frequency.

In the case of mica condensers, if L_3 and L_4 are kept as small as practicable, the correction term of (2) is negligible⁹ even at high frequencies. Hence

$$\frac{C_1}{C_2} = \frac{R_4}{R_3} \quad (4)$$

NOMENCLATURE
 ρ is the fictitious resistance;
 C is the capacity;
 p is 2π times the frequency;
 ϕ is the phase angle;
 θ is the phase displacement.

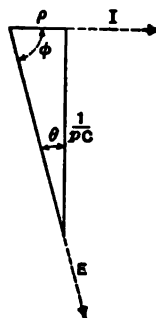


Fig. 3.

The relation between the current and electromotive force may be represented by the usual diagram (see Fig. 3). If ρ and therefore θ is small, $\tan \theta = \theta = pC\rho$. Hence by multiplying equation (3) by $\frac{pC_1C_2}{R_3R_4}$ and noticing that by equation (4) $\frac{C_1}{R_4} = \frac{C_2}{R_3}$

$$pC_2\rho_2 - pC_1\rho_1 = p \left(\frac{L_4}{R_4} - \frac{L_3}{R_3} \right)$$

$$\theta_2 - \theta_1 = p \left(\frac{L_4}{R_4} - \frac{L_3}{R_3} \right) \quad (5)$$

⁹ Grover, this Bulletin, 8, p. 391.

If C_1 is an air condenser having negligible internal resistance, θ_1 is zero, and this equation can be used to determine the phase difference θ_2 of a mica condenser with which it is being compared. It has been found more valuable to determine the phase difference than the fictitious resistance of a condenser, since the phase difference varies over a comparatively small range with change of frequency, while the variation of the fictitious resistance may be roughly expressed by saying that it varies inversely as the frequency. From equation (1) it follows that

$$\theta_2 - \theta_1 = p (C_1 r_1 - C_2 r_2) \quad (6)$$

which determines the phase difference for the series resistance method.

(c) *Advantages and Disadvantages of Each Method*

The series resistance method is very useful where two condensers of very different quality are to be compared, since one may use for the compensating resistance r a resistance box having a wide range of resistances. However, it has the disadvantage of introducing the capacity of this compensating resistance into the measured capacity. In the case of fairly large capacities, r may be kept small, and the error introduced is not serious; but in the case of very small capacities the error may be very appreciable.

The chief difficulties in connection with the series inductance method are the correction factor and the difficulty of finding a variable inductance having a suitable range for any set of measurements. As already indicated, the correction factor in the case of condensers such as we are considering can be neglected, provided suitable inductances are used. By making both L_3 and L_4 variable inductances, the second difficulty is partially overcome. In any case with condensers of good quality and a fair range of inductances from which to choose, the difficulty is more apparent than real.

It will be seen that for the kind of condensers used in this investigation the series inductance method is to be preferred, and it has been used almost exclusively. However, in either case a substitution method will give better results than a direct comparison of capacities. In this case C_2 is an auxiliary condenser,

preferably of about the same value as the condensers to be compared. The standard condenser C_1 is placed in the bridge and a reading taken. It is then replaced by the unknown condenser C_1' and a second reading taken. From these readings the capacity of C_1' , in terms of C_1 is readily computed.

(d) **A Direct-Reading Capacity Bridge**

In order to make the bridge direct reading, the following method of procedure was devised. With C_1 in the bridge, R_4 is set so that its resistance is some decimal multiple of the numerical value of C_1 ; then R_3 and L_4 are varied until a balance is obtained.

$$\text{Then } C_1 = R_4 \frac{C_2}{R_3}$$

Now replace C_1 by the unknown condenser C_1' and adjust R_4 and L_4 for a balance. Let this value of R_4 be R_4' . Then

$$C_1' = R_4' \frac{C_2}{R_3} = R_4' \frac{C_1}{R_4}$$

But since C_1 numerically equals R_4 , C_1' numerically equals R_4' .

(e) **The Choice of Resistance**

In selecting resistances for use in the arms R_3 and R_4 , three things must be considered, viz, the effect of errors in the adjustment of the resistance coils on the measured value of the capacity, the permissible current through the coils, and the effect of residual inductances and capacities in the coils. In using the substitution method the errors in the adjustment of R_3 do not enter, and those of R_4 enter only in so far as $\frac{R_4'}{R_4} = 1 + \frac{\Delta R_4}{R_4}$ is in error. With capacities of approximately the same value and a well adjusted resistance box, any correction on this account is negligible. The effect of residual inductances and capacities in the coils is to introduce an error in the value of $L_4 - L_4'$ by the amount that the effective inductance of R_4' differs from that of R_4 . In ordinary cases this is negligible, but at high frequencies it becomes so important that accurate measurements of the phase difference are often impossible.

The permissible current through any coil will depend on its resistance as well as on the readiness with which it conducts away the heat generated. With coils of the ordinary size, immersed in oil, the current should be such that the expenditure of power in any one coil is not over one watt. With even moderate frequencies and voltages and a medium-sized condenser it is not permissible to use a 1000-ohm coil, and at high frequencies it is often desirable to avoid the use of a 100-ohm coil. Hence, if measurements are to be made to a part in a hundred thousand, it is necessary to have some means of varying R_4 by steps of a thousandth of an ohm. For this purpose a specially designed resistance box was constructed. It consists of three coils in series permanently connected between the terminals of the box, around each of which there is arranged a variable shunt. These shunts are of such values that each step on the first dial varies the resistance between the terminals by a tenth of an ohm, each step on the second dial varies the resistance by a hundredth of an ohm, and on the third dial each step varies the resistance by a thousandth of an ohm. The resistance when all of the dials are on zero is 10 ohms. The advantage of this over a slide wire is that all the contacts are in the shunt circuits which have comparatively high resistance, so that any variation of contact resistance will produce a negligible effect.

3. A. C. COMPARISON OF A CAPACITY WITH AN INDUCTANCE

The Anderson bridge method has been used whenever it has been necessary to compare a capacity with an inductance by alternating current. This has been fully described,¹⁰ and needs only be mentioned here. Ordinarily an inductance is measured in terms of a capacity, but once the value of an inductance is determined, the process may be reversed. This may be desirable in investigating the constancy of a condenser, since an inductance wound on marble is a very permanent standard.

¹⁰ Rosa and Grover, this Bulletin, 1, p. 291.

4. DIRECT CURRENT COMPARISONS OF CAPACITY

(a) The Differential Galvanometer Method

For direct current comparisons of capacity two methods have been tried, the differential galvanometer method and the method of mixtures. In the differential galvanometer method two condensers are charged with approximately equal quantities of electricity, then discharged simultaneously through the two coils of a differential galvanometer. If the discharge does not begin at practically the same instant through both coils there will be a slight movement of the galvanometer mirror, so that it is very difficult to tell whether or not there is a true deflection. Moreover, in comparing two condensers having different absorptions, the rate at which the absorbed charge comes out of the two condensers may be quite different, so that there may be a slight deflection even though the total quantity which passes through each of the coils is the same. For these and other reasons the differential galvanometer method for the comparison of capacities has been abandoned.

(b) The Method of Mixtures

In the method of mixtures (see Fig. 4) the battery, B , maintains a current in the resistances R_1 and R_2 . On closing the key K_1 the condensers C_1 and C_2 are charged to the potential differences between the terminals of R_1 and R_2 , respectively. When the key K_1 is released, the condensers discharge against each other and any free charge which remains is discharged through the galvanometer on pressing the key K_2 . A small resistance, r , is inserted to prevent oscillatory charge or discharge. If C_1 and C_2 are air condensers, by adjusting R_1 or R_2 and repeating the operation, a point is soon reached where there is no deflection of the galvanometer. Then

$$R_1 C_1 = R_2 C_2 \quad (7)$$

By using the substitution method, this arrangement can be made direct reading in the same manner as with the alternating current bridges.

With condensers which show absorption the apparent capacity will depend upon the length of the charging period, the length of time of mixing before discharging through the galvanometer, and

on the previous history of the condenser. If a condenser is charged to a high voltage for some time, an appreciable absorbed charge will continue to come out of it for a long time thereafter. To completely discharge a condenser is often very difficult, but in the case of a mica condenser it is usually possible to discharge it to such a point that the residual charge which appears in the first second is negligible.

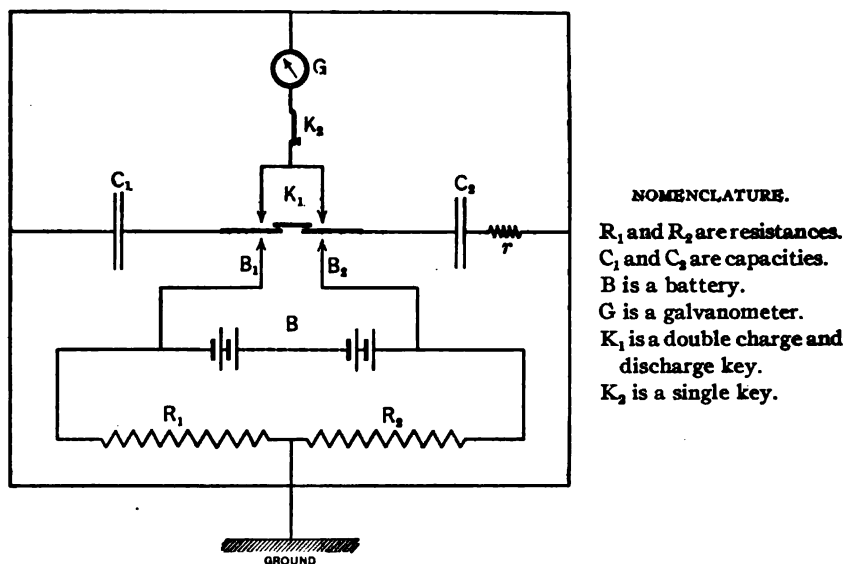


Fig. 4.—Diagram to show the method of comparing capacities by the method of mixtures

Having discharged the condensers, the method of procedure is to charge the condensers for a definite time by depressing the key K_1 , then to release K_1 , and after a certain definite time to close the key K_2 . If there is a deflection of the galvanometer mirror, adjust the resistances, thoroughly discharge the condensers, and repeat the process. Continue until there is no deflection of the galvanometer. This gives the **Acyclic Capacity** under the given conditions. It should be noticed that the time of contact of K_2 must be very short, since if the absorbed charges are coming out from the condensers at different rates, there will be a current through the galvanometer which may produce a deflection, even though there was no free charge at the instant of closing the key.

The above method will give very accurate results, but is very slow in operation. A method of improving it in this respect is to charge and discharge the condensers periodically. The capacity determined in this way we will call the **Cyclic Capacity**. Then in each discharge the total residual charge consists of portions from each of a number of previous charges. Hence the cyclic capacity is larger than the acyclic capacity, the time of charge and discharge being the same in both cases.

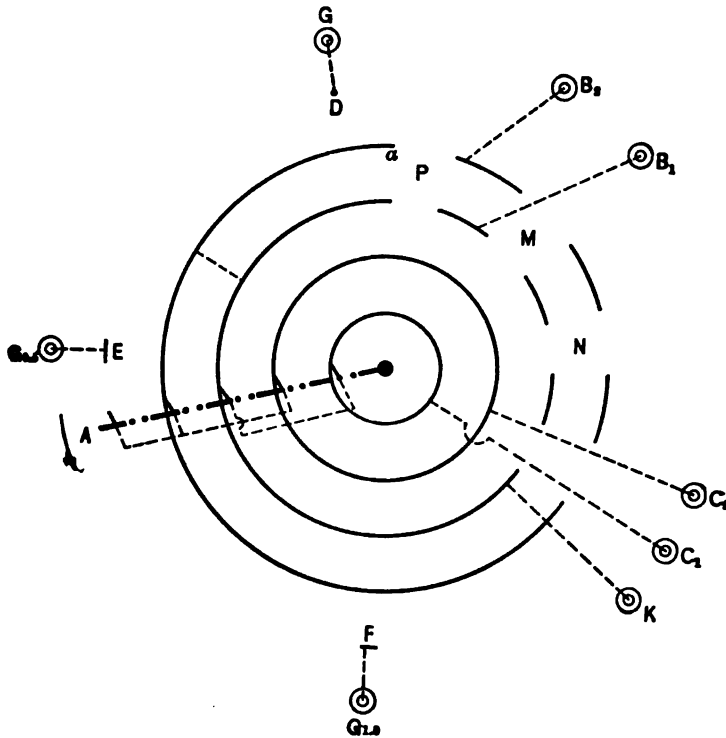


Fig. 5.—*Diagram of apparatus for charging and discharging condensers in the method of mixtures*

To accomplish the periodic charging and discharging of the condenser, as well as to make the time of charge and discharge of the condensers as definite as possible when the acyclic method is used, the apparatus shown in Figs. 5 and 6 was constructed. This apparatus merely replaces the keys K_1 and K_2 in the preceding diagram. It consists of a hard rubber base on which are mounted

portions of five brass rings. Above this rotates a hard rubber arm *A* which carries five brushes that bear upon the rings. This arm is driven by a motor so geared that the arm makes one revolution every two seconds. With a variable resistance in the armature circuit of the motor, and a clock ticking seconds, the adjustment of the speed of the motor can easily be made with the necessary accuracy. The inside condenser terminals are connected to the binding posts C_1 and C_2 , and the battery terminals to the posts B_1 and B_2 . If the acyclic capacity is desired it is necessary to insert single contact keys between the posts B_1 and B_2 on the one hand and the battery terminals on the other. The galvanometer may be connected to any of the posts marked *G*. If connected to the post $G_{0.5}$, the length of mixing is 0.5 second. If connected to the post $G_{1.0}$, the length of mixing is 1.0 second. The point *D* connected to the post, *G*, can be moved along the circumference of the circle, so that the time of mixing can be varied from 0.1 second to 0.01 second, or even less. When these short times are used, the time of mixing is determined electrically, as will be explained later. The method of connecting the brushes is indicated by the dotted lines in front of the arm *A*.

Let us suppose the galvanometer connected to $G_{1.0}$, and the arm *A* rotating in the direction indicated by the arrow. When the arm passes the opening *M* and the brushes touch the brass pieces beyond, the condensers are charged almost instantly, and remain charged while passing across *P*, thus making the total time of charge 0.2 second. The condensers are discharged when the brushes touch the pieces beyond *P*, there remaining the quantity by which the free charges of C_1 and C_2 differ. From this time until the galvanometer brush touches *F* which is connected to $G_{1.0}$, C_1 and C_2 are giving up their absorbed charges. If the algebraic sum of these absorbed charges is just equal to the residual of the free charge, there will be no discharge through the galvanometer. In this way the apparent capacity of C_1 under the given conditions is determined, provided C_2 is known.

Small brass pieces are provided which fill the gaps *M* and *N* in the third and fourth rings, so that the time of charge may be made 0.2 second, 0.4 second, or 0.6 second. A brass piece is also provided which may be connected to *D*, this extending the arc from *D* to beyond the diameter which passes through *a*. This is



Fig. 6 — *Apparatus for charging and discharging condensers in the method of mixtures*

for determining the time of mixing and is used in the following manner, when it is necessary to employ apparatus not regularly used in this set-up. A battery whose emf is E is connected to a condenser of capacity C , and the charge which collects on the condenser is discharged through a long-period ballistic galvanometer giving a deflection d . The same battery and galvanometer are then connected in a circuit with a high resistance, a single contact key, and the posts K - G of this apparatus. With the arm rotating, if the key is closed a current will flow from the time the fourth brush touches the ring beyond P until the fifth brush leaves D . This will give a ballistic throw to the galvanometer. Adjust R until there is the same deflection d as when the condenser was discharged. Then

$$EC = \frac{Et}{R} \text{ or } t = RC \quad (8)$$

If the third brush touches the ring beyond P after the fourth brush has touched its ring, the connections on the rotating arm must be changed so that the third brush is connected to the fifth. This is necessary, since the discharge of the condensers does not begin until the last of the two brushes touches its ring on passing P .

5. METHODS OF "STEPPING UP"

As stated previously, the apparent capacity of a mica condenser under any condition is determined by comparing it with an air condenser of the same nominal value, provided such a condenser is available. But as 0.1 mf is the largest air condenser in this laboratory, and as it is often necessary to measure values of 0.5 or 1.0 mf, some means of "stepping up" from lower to higher values is necessary.

If a mica condenser is constructed so that both of the outside sheets of tin foil are connected to the same terminal, and if this terminal is connected to earth during the measurement, then the capacity is practically independent of its surroundings. This terminal is called the outside or A terminal. In A. C. work, using an insulated system, the mean value of the potential of either terminal of the machine is sensibly the earth potential, so that in A. C. bridges the outside terminal of the condenser is connected directly to one terminal of the machine. Such con-

condensers may be used additively, and their capacity when in parallel is the sum of their individual capacities. They can not be connected in series where high precision is desired, since it is impossible to so arrange them that the outside plates of both condensers will be at the earth potential, though with alternating current the error is very small.

To determine the apparent capacity of a microfarad using a tenth microfarad air condenser, there are several possible procedures. One of the simplest is to compare five tenth-microfarad mica condensers with the air condenser, then compare their sum with a half-microfarad, and take the sum of the six condensers as the standard microfarad.

In the above discussion it has been assumed that each condenser has separate terminals, so that in connecting the condensers together no appreciable capacity will be introduced on this account. With condensers having large terminal blocks the capacity between these blocks often introduces a very troublesome correction.

6. MEASUREMENT OF ABSORPTION

It will be shown later that there are two factors which should be considered in determining the absorption of a condenser. One is the rate at which the absorbed charge reappears and the other is the value of the total absorbed charge. These quantities are very difficult to measure. Hence the following simple procedure has been adopted as giving some knowledge of the relative qualities of different condensers.

The condenser is charged for one second, insulated for one second, short-circuited for one second, then insulated for thirty seconds, at the end of which time it is instantaneously discharged through a ballistic galvanometer. It is then insulated for thirty seconds and again discharged through the galvanometer. This is continued until five residuals have been measured. The sum of these deflections multiplied by the constant of the ballistic galvanometer gives the measured portion of the absorbed charge. The product of the capacity of the condenser by the charging voltage gives the free charge. The absorption is taken as the ratio of the measured portion of the absorbed charge to the free charge.

III. EXTERNAL CONDITIONS AFFECTING THE CAPACITY

7. THE TEMPERATURE COEFFICIENT

Each apparent capacity of a condenser has its own temperature coefficient. In other words, the temperature coefficient of the capacity measured with 1000 cycles is different from that measured with 100 cycles, and both of these are different from that obtained by measurements with direct current. It will be shown later that the difference between the different apparent capacities is due to absorption, and that the true¹¹ or geometric capacity would be obtained either if the condenser could be instantly discharged when being measured by direct current, or if it could be measured with alternating current of infinite frequency. The geometric capacity can not be measured directly, so that the true temperature coefficient of the capacity is not easily determined. However, the capacity at 100 cycles is but little different from the geometric capacity, so that in the following it is assumed, unless otherwise stated, that the temperature coefficient at 100 cycles is the true temperature coefficient of the capacity.

There is a wide variation in the temperature coefficient of mica condensers. With silvered-mica condensers the temperature coefficient is small and positive. With the ordinary mica condensers, where paraffin forms a part of the dielectric, it is usually negative and may be as large at 0.1 per cent per degree, though in most cases it is very much smaller. In seeking the cause for this wide variation, attention was attracted to two 0.5-mf condensers in the same case, one of which had a temperature coefficient of 6 parts in a hundred-thousand per degree, while the other had a temperature coefficient of 32 parts. An examination showed that the condenser with the larger temperature coefficient was much thicker than the other condenser. At the suggestion of Dr. Rosa, this condenser was placed in molten paraffin, and when thoroughly warmed through, was compressed in a vise and left until cold. The result was that the capacity was increased over 50 per cent, while the temperature coefficient was reduced to only one part in a hundred thousand per degree. This same treatment was then tried on a number of condensers, and in all cases the temperature coefficient became less. In some it actually changed

¹¹ A Zeleny, *Phys. Rev.* **22**, p. 65; 1906.

sign and became positive. Of course following this treatment it is necessary to readjust the capacity if a definite value is desired. It should also be remarked that this treatment always affected the phase difference, sometimes favorably, sometimes unfavorably. Table III gives some of the results.

TABLE III

Condenser	Temperature coefficient in parts in 100,000		Phase difference	
	Before treatment	After treatment	Before treatment	After treatment
No. 1	-14.5	-3.7	4' 20"	6' 30"
2	- 9.1	+0.5	8' 50"	13' 25"
3	-14.0	-5.0	5' 20"	7' 10"
5	- 7.0	-5.0	3' 20"	2' 45"
10	-10.7	+1.8	5' 30"	4' 35"
6	- 6.4	-1.5	3' 25"	4' 10"
4460R	-31.0	-1.0	2' 20"	2' 0"
7763 ₁	-11.8	-4.5	2' 5"	1' 20"
7763 ₂	-12.6	-2.0	2' 40"	2' 00"
7763 ₃	- 3.6	-1.0	1' 40"	2' 5"

It is desirable to have the temperature coefficient of any standard as low as possible, but this is especially true with a condenser, since its thermal conductivity is so poor. With the best of these condensers an error of several tenths of a degree in the measurement of the temperature, will not introduce an error of a part in a hundred thousand in the capacity.

The experiments of Rosa and Grover show that for condensers with a high melting point paraffin the temperature coefficient is approximately a constant below 30° C. Experiments were now undertaken to determine the exact form of the capacity-temperature curve, and at the same time to see how the temperature coefficient is affected by the method employed in measuring the capacity. The condensers were placed in a case in which the temperature could be maintained constant at any desired temperature to 0.1° C by means of a vapor pressure thermostat. The reference condensers were in another case which was maintained at 25° C throughout the experiments by a similar thermostat. Each of the condensers had a capacity which did not

differ greatly from 0.1 mf. This value was chosen so that the condensers could be compared directly with an air condenser. At the beginning of the experiments a careful determination was made of the apparent capacity at 100 cycles of the two reference condensers. The mean of these condensers was then assumed constant, and the value of the air condenser at any time was determined by comparing it with the reference condensers by means of A. C. of 100 cycles. The relative values of the two reference condensers changed slightly during the course of the experiments, but a careful absolute measurement failed to show that the mean value had changed appreciably. The results are shown in the accompanying curves. (Figs. 7 to 12.) All of these curves are plotted to the same scale, so that the relative behavior of the different condensers is easily followed. Not only is there indicated the change with temperature of the capacity at different frequencies of A. C. and different times of discharge with D. C., but also the change with temperature of the phase difference, of the absorption, and in some cases of the insulation resistance.

Condensers No. 3 and No. 5471B are of American make, and have metal clamps. The 0.1 mf section of 5471B is a very satisfactory condenser. The temperature coefficient under all conditions is small and the change of capacity with frequency is not large; also the phase difference and absorption are small. The 20.1 mf section of 5471B behaves very well with A. C. Its change with frequency is but little more than is found in the best condensers and its phase difference is not large in any case. However, with direct current the change of capacity with the length of discharge is very marked, and the absorption is also large. The temperature coefficient of No. 3 is not large, but otherwise the behavior is not satisfactory on either A. C. or D. C. The curves are given as an example of what may be expected of a poor mica condenser.

Condenser 1038 is of German manufacture. It is one of the oldest condensers in this laboratory, and has wooden clamps. The section here shown is very similar to the other sections which have been examined. All of the sections have a large negative temperature coefficient on A. C., but otherwise behave very well on both A. C. and D. C. Condenser No. 20 was manufactured by Siemens and Halske according to specifications furnished by this

bureau. A high melting point paraffin was used in its construction, and it is compressed between heavy metal clamps. The behavior with A. C. is especially good, the phase difference at 1200 cycles being so small that it could not be satisfactorily measured. With D. C. it does not make quite so good a showing, but is still among the very best. There is also shown the curves for a silvered mica condenser. However, as will be pointed out later, there are factors influencing the behavior in this condenser which are not operative with the others, so that conclusions which may be drawn respecting the other condensers are not applicable to this one. However, among all the condensers examined, this is the only one that needs to be treated as an exception.

The apparent capacity at 1200 cycles is very little different from the geometric capacity, as will be shown later. Hence any conclusions drawn from the values of the capacity at 1200 cycles will be applicable to the geometric capacity. The figures show that the capacity-temperature curve at 1200 cycles is very nearly a straight line within the range investigated (15 to 30° C). In three cases the curvature, if any, is too small to be detected. In four cases the curves are convex, upward; i. e., there is an algebraic decrease of the temperature coefficient with increasing temperature. The magnitude of these changes will be seen from Table IV. In no case is the change sufficiently large to warrant the application of a parabolic formula.

TABLE IV

Temperature Coefficient at 1200 Cycles in Various Temperature Intervals

Condenser	Temperature coefficient in parts per hundred thousand per degree		
	15°-20°	20°-25°	25°-30°
5471B 0.1 section	- 3	- 8	-11
2' 0.1 section	- 5	- 5	- 5
1038 0.1 section	-15	-22	-22
1038 0.2 and 0.2 in series	-19	-23	-27
No. 20	- 5	- 5	- 5
Nos. 13 and 14 in series	+ 3	+ 3	+ 3
No. 3	- 5	-10	-16

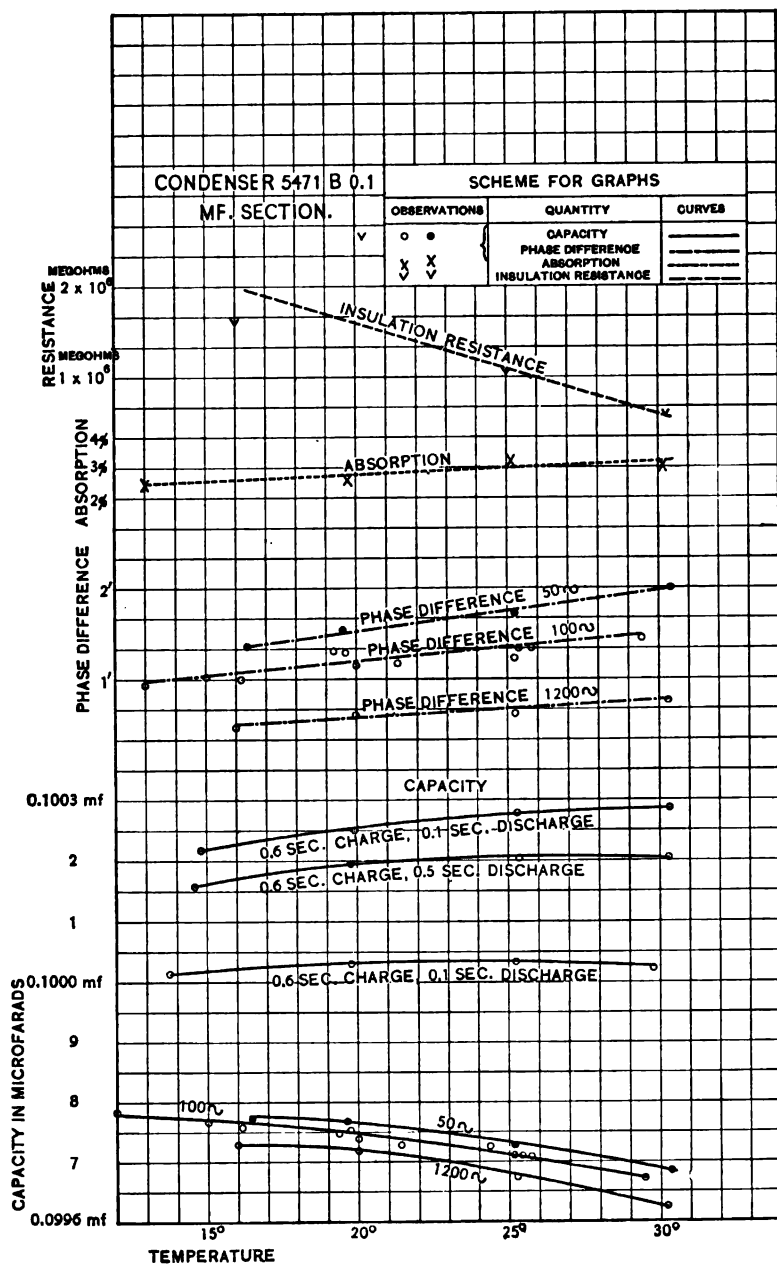


Fig. 7

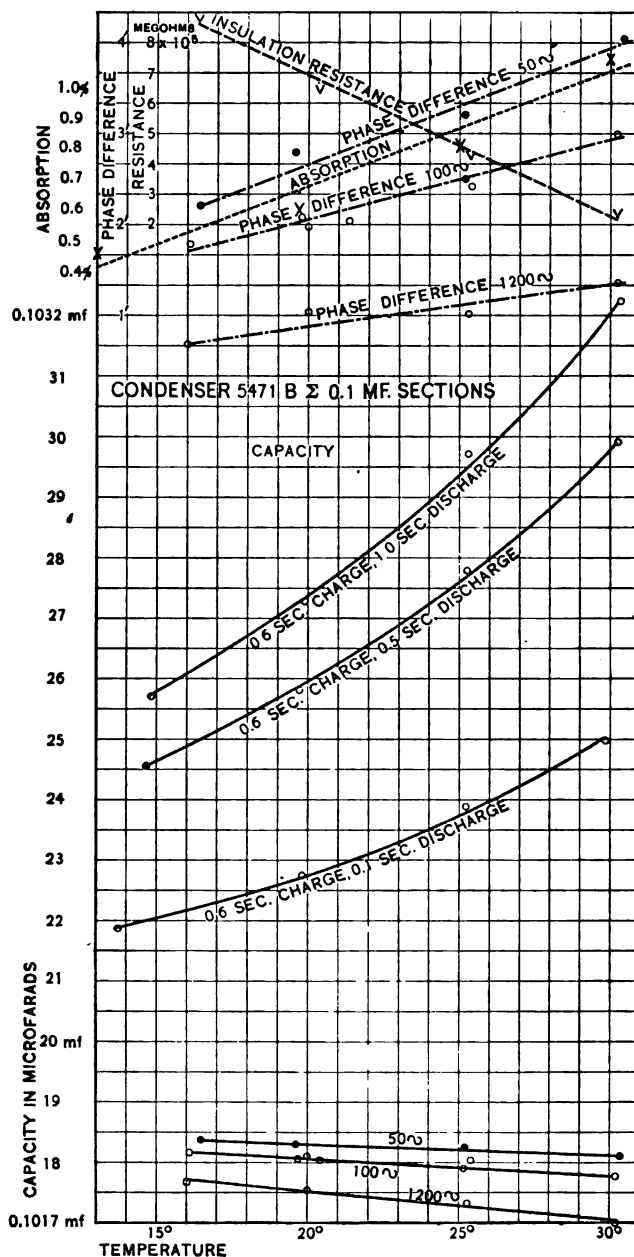


Fig. 8

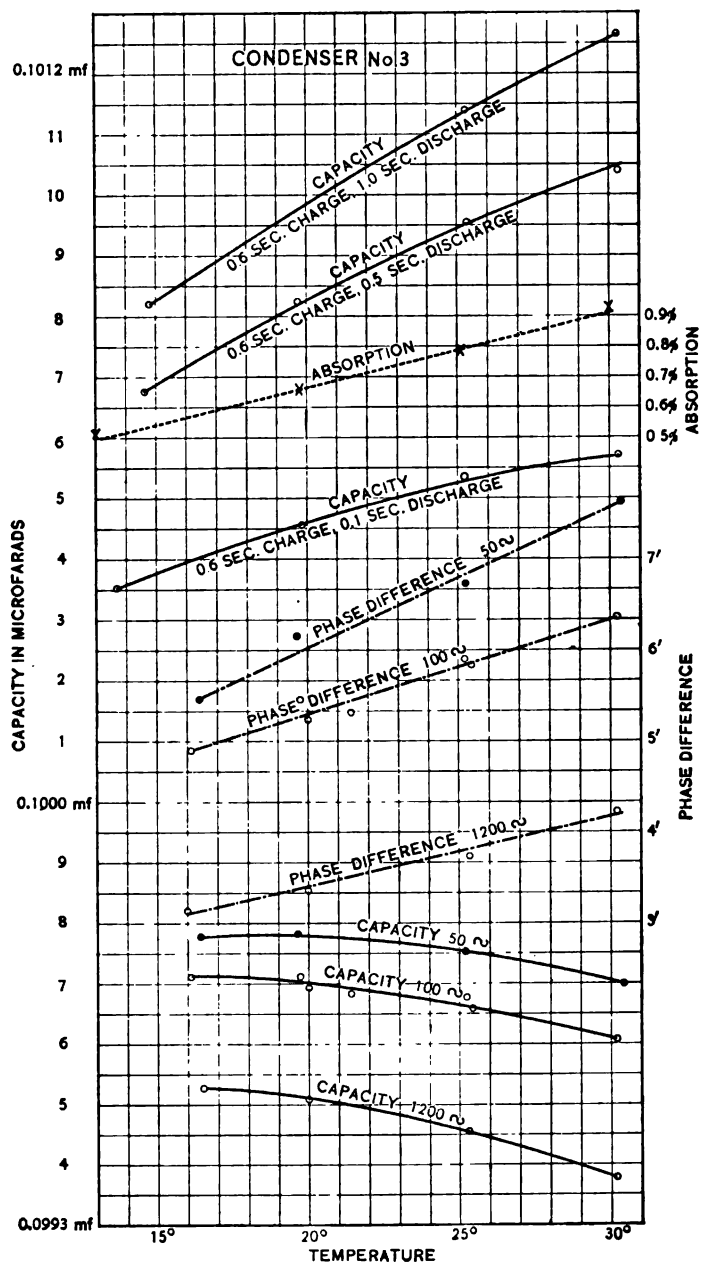


Fig. 9

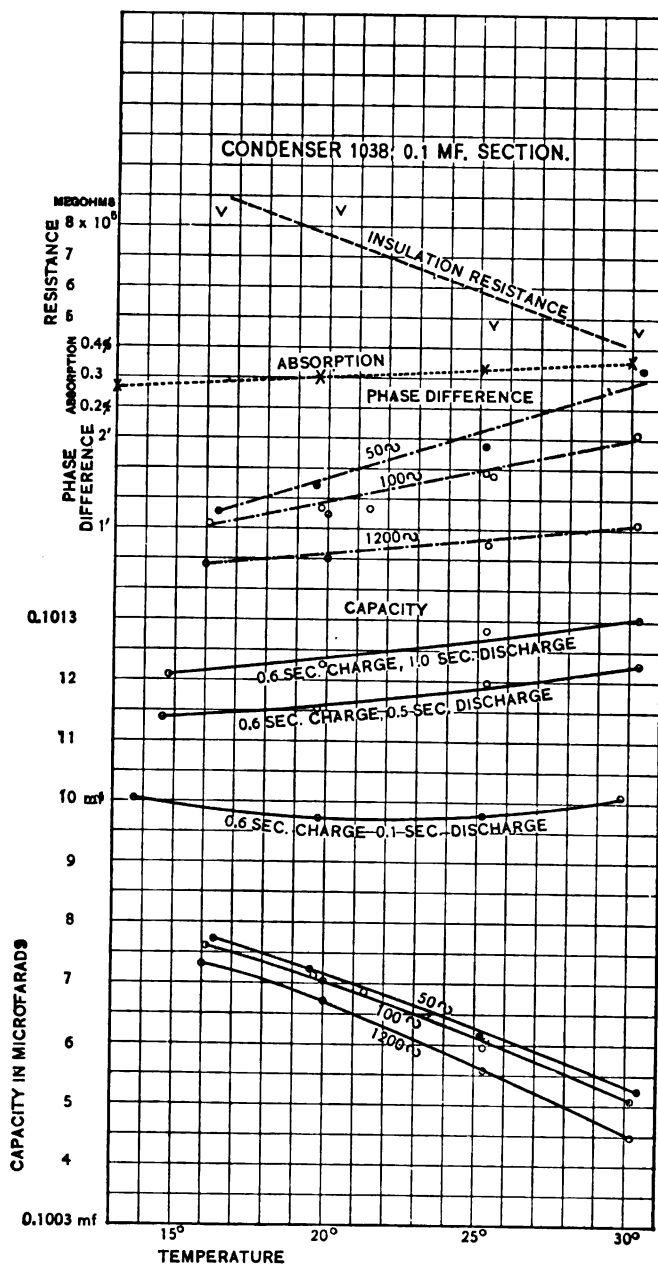


Fig. 10

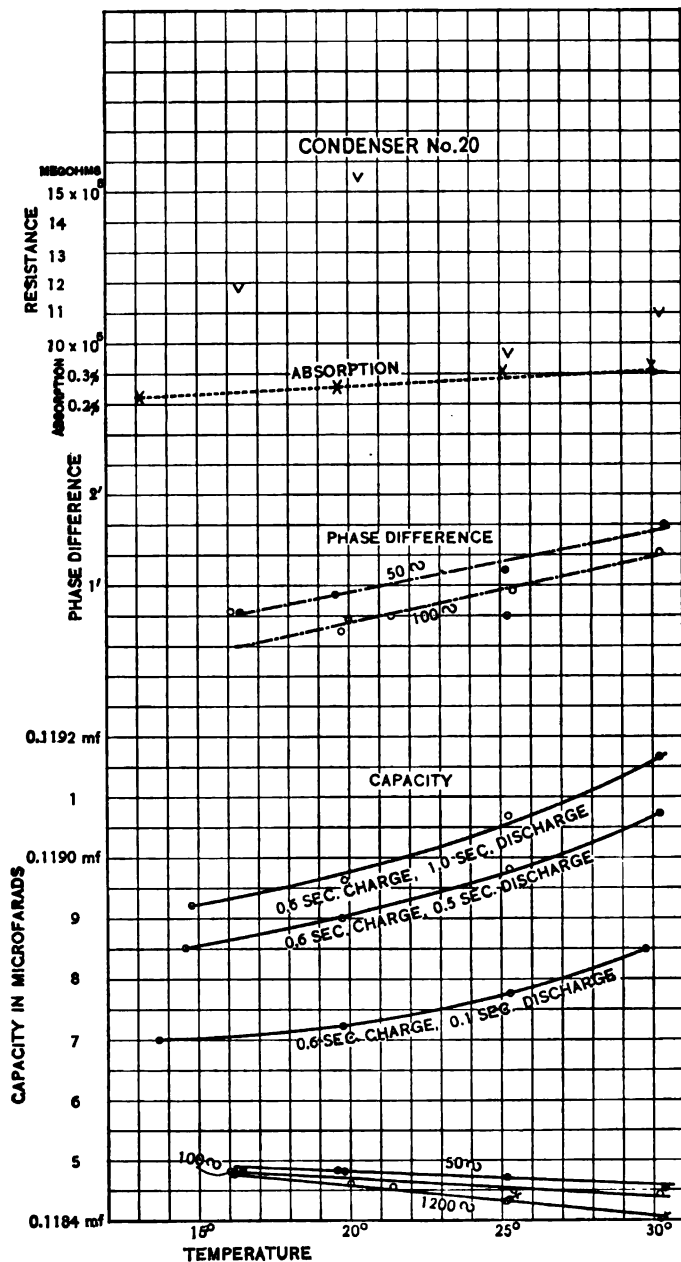


Fig. 11

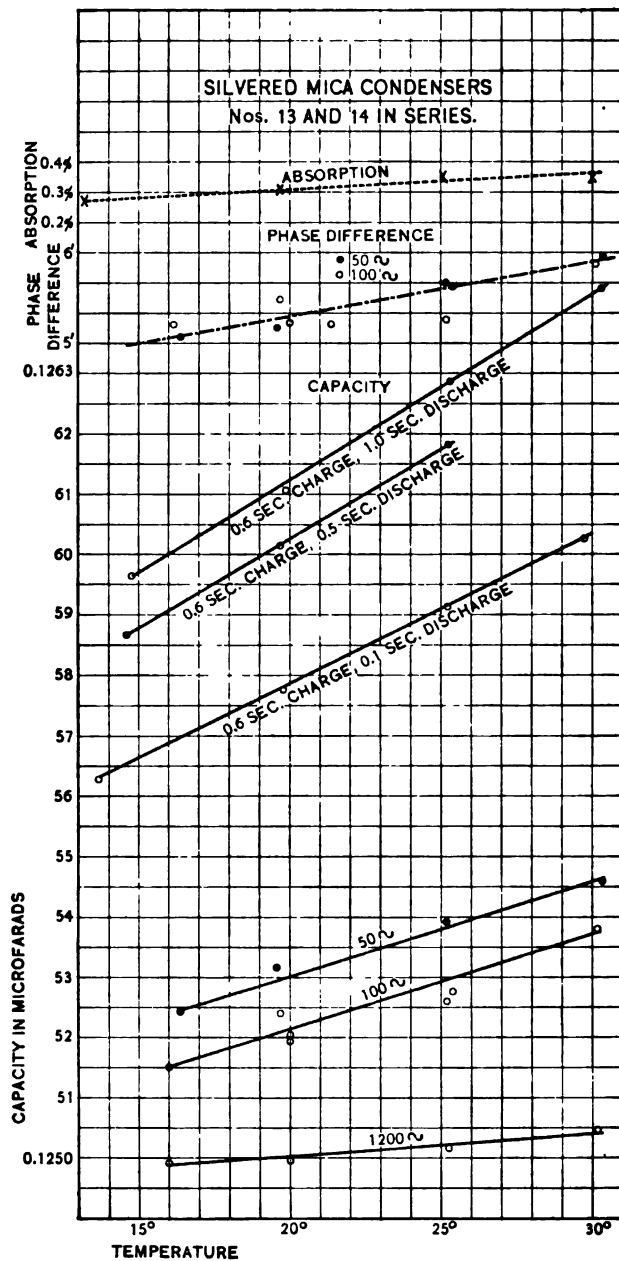


Fig. 12

In going to lower frequencies there is always an algebraic increase in the mean temperature coefficient of the apparent capacity, but over a wide range of frequencies this is not large. However, with the relatively long times of discharge used in the D. C. measurements there is often a very decided change in the temperature coefficient. The effect may be expressed qualitatively by saying that the temperature coefficient of the apparent capacity increases algebraically as the period increases. This is well shown in Table V:

TABLE V
Mean Temperature Coefficient at Different Frequencies

Condenser	A. C.			D. C. 0.6 Sec. Charge		
	1200~	100~	50~	0.1 Sec. Dis.	0.5 Sec. Dis.	1.0 Sec. Dis.
5471B 0.1 section	- 7	- 6	- 6	+ 1	+ 3	+ 4
Σ 0.1 "	- 5	- 3	- 2	+17	+35	+42
1038 0.1 "	-20	-18	-17	0	+ 6	+ 6
1038 0.2 and 0.2 in series	-23	-22	-21	- 2	+ 3	+ 5
No. 20	- 5	- 3	- 2	+10	+14	+16
Nos. 13 and 14 in series	+ 3	+15	+15	+25	+30	+31
No. 3	-10	- 7	- 5	+13	+23	+28

With the D. C. measurements the capacity-temperature curve is often not even approximately a straight line. However, as there seems to be no simple way of connecting the equations of the different curves, and as it is scarcely desirable to compute a parabolic form for each apparent capacity, it is better to refer directly to the curves when calculations from the mean temperature coefficient are not sufficiently accurate.

The explanation of the above phenomena is to be found in a consideration of the absorption of the condensers. The increase of the absorption with increasing temperature is very large, in some cases being as much as 100 per cent in the interval considered. Moreover there may be a change in the form of the absorption curve as the temperature is changed. Since it is due to absorption that there is a difference in the apparent capacities of a condenser, any increase in the absorption will increase this

difference. Hence we must expect that there will be an algebraic increase in the temperature coefficient with increasing period due to absorption, and this agrees with experiment. It is not so easy to draw conclusions as to the effect of the change in shape of the absorption curve. However, it will be noticed that the changes in the shape of the D. C. capacity curves are in all cases progressive changes as the length of discharge is increased. Any change of this sort can of course be explained by a change in the shape of the absorption curve.

It is well to notice that in all cases there is an increase in the phase difference with increasing temperature, and that the lower the frequency the greater is this increase. Also the insulation resistance decreases very markedly on going to higher temperatures. It is therefore very evident that a mica condenser gives the most satisfactory results at relatively low temperatures.

8. THE PRESSURE COEFFICIENT

Since a condenser is normally subjected to the changes of the atmospheric pressure, it seemed desirable to see if there is a change in capacity due to this cause. Accordingly a condenser was placed under a bell-jar through which leads were passed, and the air partially exhausted. Measurements of the capacity before and after exhaustion showed not only that there is a decrease in the capacity when the pressure is decreased, but also that some time is required after a change in pressure before an equilibrium condition is reached, during which time there is a continuous change in the capacity. An idea of the rate at which the capacity changes, in the case of a condenser which shows a very marked change with pressure, can be obtained from Table VI. In another experiment it was found that some very good condensers which were placed in a desiccator and the pressure reduced to 1 mm of mercury did not reach equilibrium for several days.

In order to see if this after effect is not negligible when there is only a small change in pressure, a number of condensers were measured, first at atmospheric pressure, then at a pressure 6 cm less than atmospheric pressure and finally at atmospheric pressure. Any after effect would be shown by a failure to return to the original capacity. In one or two cases there seemed to be a very slight effect, but in all other cases it could not be detected. Since this

TABLE VI

Lag of Capacity on Changing the Pressure

Condenser 4660R. April 7, 1906

Pressure	Capacity	Time
760 mm	0.49979 mf	11.30 a. m.
10	650	35
10	607	40
10	593	43
10	583	46
10	575	49
10	566	52
760	966	56
760	972	59
760	975	12.02

is the case, a pressure coefficient can be computed which will apply to small changes of pressure. This has been done, and the results are given in Table VII. It is noticeable that in one case there is an increase of capacity with a decrease of pressure.

TABLE VII

Change of Capacity with Pressure

Condenser	Pressure coefficient of the capacity in parts per hundred thousand per cm change in pressure
5471A 0.1 section	2.0
Σ 0.1 section	2.0
5471B 0.1 section	1.7
Σ 0.1 section	1.0
No. 1	0.5
7765L	2.0
7765R	-0.2
7766L	1.1
7766R	3.4
7763	0.3
No. 12	1.5

Measurements were made with 100 cycles A. C.

48848°—10—3

The cause of this change of capacity with change of pressure is very evident. At all times the pressure of the air is resisted by the elastic forces within the condenser. If the pressure is reduced, the condenser expands, thus increasing the distance between the plates and at the same time increasing the area of the plates. In most cases the increase in the distance between the plates produces the more marked effect, so that the capacity decreases with a decrease of pressure. However, the one condenser examined which is an exception to this rule, is very firmly clamped between heavy brass plates, so that changes in atmospheric pressure produce very little effect on the thickness of the condenser; but it can expand with decreasing pressure so as to increase the area of the plates and thus increase the capacity.

The magnitude of these effects is such that the daily fluctuations of the barometric pressure produces a change in the capacity which can be measured, provided all other influences are eliminated. To accomplish this, a bridge was set up in which both the auxiliary condenser and the standard capacity were kept in an exhausted desiccator. These and the condensers under test were kept in a chamber whose temperature was controlled by a thermostat. After one or two trials reasonably satisfactory results were obtained and are given in Table VIII. While the agreement between the observed values and those computed from the previously determined pressure coefficient is not all that could be desired, yet it is all that can reasonably be expected considering the difficulties involved.

TABLE VIII

The Change in Capacity of Certain Condensers from 10.30 a. m., February 11, to 9.20 a. m., February 12, 1910

The Barometric Pressure Changed from 766 mm to 744 mm. The Temperature was Maintained Constant by a Thermostat

Condenser	Decrease in capacity in parts per million	Computed decrease
7765L	10	18
7766L	44	24
No. 6	2	12
" 7	6	6
" 8	4	6

All measurements were made at 100 cycles A. C.

The above results show that, while the ordinary fluctuations of barometric pressure produce only very small effects, yet in case there is any considerable change in pressure, careful account must be taken of it in work of the highest precision.

The effect of changes of the atmospheric humidity upon the capacity was also investigated with negative results. A condenser was even placed in water and left there for forty-eight hours without producing a measurable effect upon the capacity.

9. THE EFFECT OF VOLTAGE

When compared with an air condenser, the capacity of some mica condensers depends upon the voltage used in making the comparison. In the case of mica condensers built up in the usual way with tin-foil sheets interleaved with sheets of mica, the effect is always small; and, in the case of the better condensers, can not be detected. However, with silvered mica condensers the effect is very pronounced in every case which has been examined. Moreover there is a certain instability, in that with the same voltage different capacities will be obtained at different times. This effect is illustrated in the curve of Fig. 13, where the total increase in capacity on increasing the voltage from 20 to 100 volts is approximately 0.1 per cent.

While a condenser of this type can not be considered in selecting a standard, yet it seemed desirable to determine the cause of such a marked effect. On examining some sheets of silvered mica it was observed that there were flakes on the surface underneath which the plating solution only partially penetrated. On removing these and examining the surface under a microscope it was found that there were places underneath these flakes where there seemed to be spots of silver which were imperfectly connected with the main coating of silver. A chemical test showed that these spots contained silver.

In view of these facts the following explanation seems plausible. As the spots of silver are imperfectly connected to the plates of the condenser, at low voltage they will play no part in the charging of the condenser. As the voltage is increased there will be more and more cases where the charge will cross the intervening space, thus charging the spot of silver. In this way the virtual thickness of the plate is decreased, hence the capacity is increased.

If this explanation is correct, it would seem probable that at very high frequencies there would be little tendency to charge the small spots of silver at any voltage, so that there would be little change with voltage at high frequency. However, if the charging and discharging takes place slowly, the potential of the

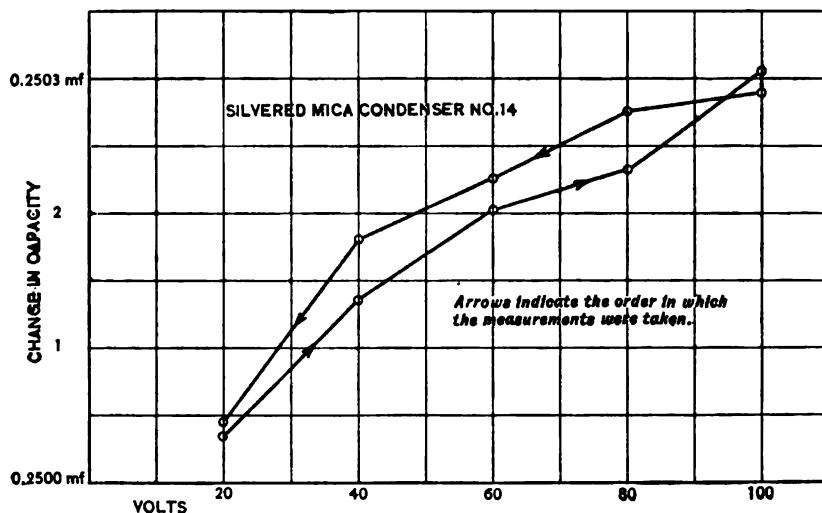


Fig. 13.—Change in capacity with voltage.

spots will be but little different from that of the plates at any time during the cycle. In this case also there would be little change over a moderate range of voltage. Thus we would expect a maximum effect at some intermediate frequency. That this is the case is indicated by Table IX.

TABLE IX

Effect of Frequency on the Increase in Capacity of Silvered Mica Condensers with Increasing Voltage

Time of discharge in D. C.	Frequency with A. C.	Increase in capacity on increasing the voltage from 30 to 70 volts, in parts per hundred thousand	
		Condenser No. 13	Condenser No. 14
0.1	3000	16	40
	1000	26	76
	100	50	50
	50	36	36
		18	10
0.5		12	

One would also expect that the change in capacity with changing frequency would be more pronounced with such a condenser than with ordinary condensers. However, this is so small in comparison with the change in capacity due to absorption, that it can not be determined whether or not any of the observed change is due to this phenomenon.

IV. THE STABILITY OF MICA CONDENSERS

It has been shown by Rosa and Grover that unclamped condensers are very unstable. It seemed desirable to see to what extent clamping a condenser improves it in this respect. Accordingly a number of condensers were twice carried through the temperature cycle from 15° to 30° C. The curves of Fig. 14 are fairly representative of the results. It will be seen that there is an uncertainty which often amounts to several parts in ten thousand. In some cases there is an increase in capacity on returning to 15° C, in others a decrease. While this is a great improvement over unclamped condensers, yet it was evident that mica condensers must be maintained at constant temperature if they are to be used in work of precision. It seemed quite possible also that condensers which are shielded from changes in the barometric pressure will be more constant than those which are not. To test this point four condensers were sealed in a desiccator and the air exhausted. In the first attempts the ground joint began to leak in three or four weeks, and three months elapsed before a satisfactory method of sealing the desiccator was devised. However, with the aid of Dr. Nutting, the desiccator was finally sealed in such a manner that for more than a year there has been so little change in pressure that it can not be detected by the Plücker tube which is attached to it. The total pressure is approximately a millimeter. These condensers with others have been constantly maintained at a temperature of 25° C.

One of these condensers (No. 12) is very sensitive at low pressures to any change in pressure, though at atmospheric pressure the pressure coefficient is not unusually large (1.5 parts per hundred thousand per centimeter change in pressure). In the first attempts at keeping the condensers in a vacuum, I was always able to detect the appearance of a leak by the change in capacity of this condenser. This sensitiveness to changes in pressure is fur-

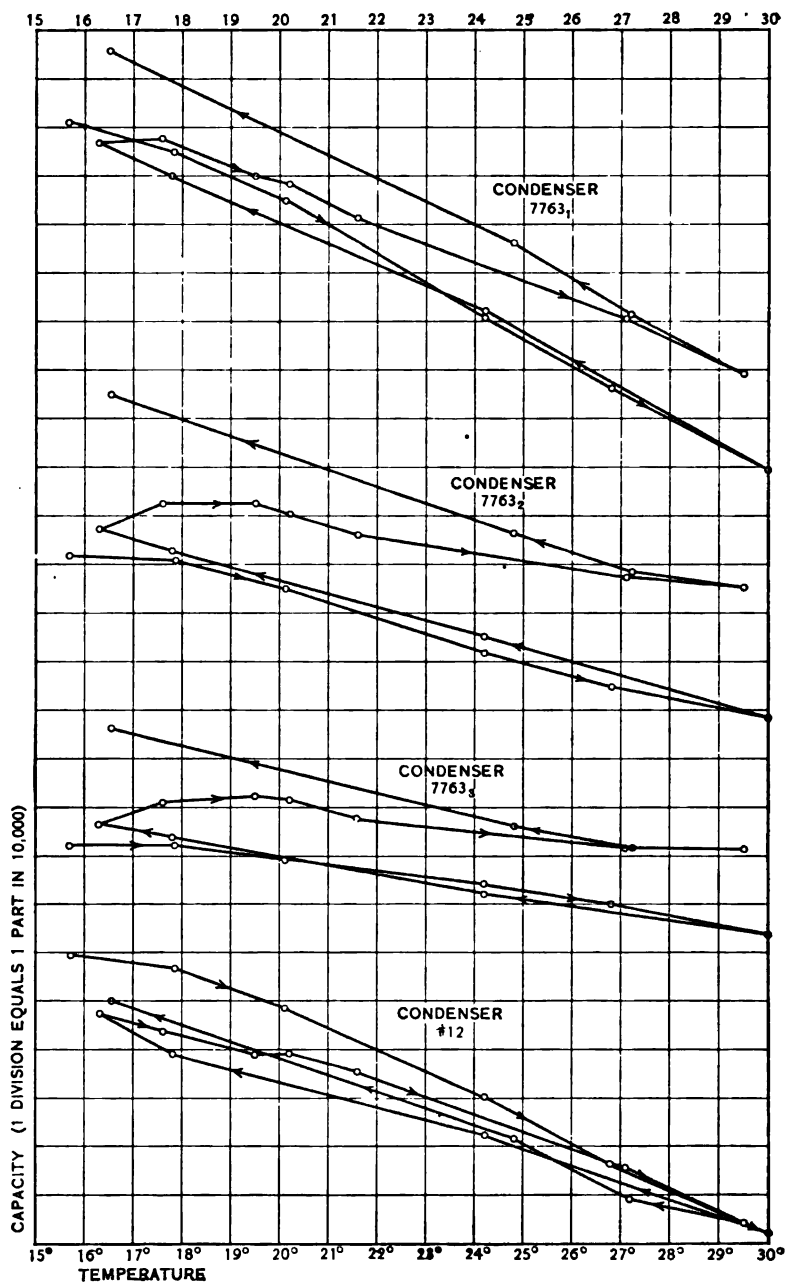


Fig. 14.—Curves showing the instability of clamped condensers

ther indicated by the fact that the decrease in capacity of No. 12 on going from atmospheric pressure to a vacuum is nearly 3 per cent, while for the other condensers it is less than 0.1 per cent. The plates between which the condenser is clamped are much thinner than are those of the other condensers, though it is not known that this is the cause for the exceptional behavior. For this reason the values of No. 12 are given much less weight than those of the other condensers in the vacuum desiccator.

In order to determine the behavior of condensers under different methods of treatment, a series of measurements extending over nearly five months was made upon the following condensers: Three condensers in a vacuum, maintained at approximately 25° C; five condensers, known by previous tests to be very stable, also maintained at 25° C under atmospheric pressure; four good condensers subjected to the ordinary variations of room temperature. After a few measurements it was evident that the condensers in the vacuum were relatively more constant than any of the others. Hence the mean of these three is taken as remaining constant, and all other values are referred to them. These condensers were put in a vacuum on March 5, 1909. The first measurements were made on March 9, but they did not seem to have adjusted themselves to the change in pressure until March 13. The temperature coefficients of the condensers which were left at room temperature had been previously determined, so that the temperature and capacity were measured, and the capacity at 20° C was computed from these measurements.

The results of these measurements are shown in the curves of Fig. 15. Those condensers which are in a vacuum are somewhat more stable than any of the others, though the improvement over those which are kept at constant temperature is not marked. However, both of these show a decided improvement over those which are kept at room temperature.

It should be stated that an accuracy of measurement of one part in a hundred thousand was in all cases attained, and often somewhat exceeded. The changes which appear are greater than the experimental errors. These changes may be due to errors in the temperature, for a condenser is a poor conductor of heat and the temperature is measured outside the condenser proper. Even with those controlled by a thermostat there is always a possi-

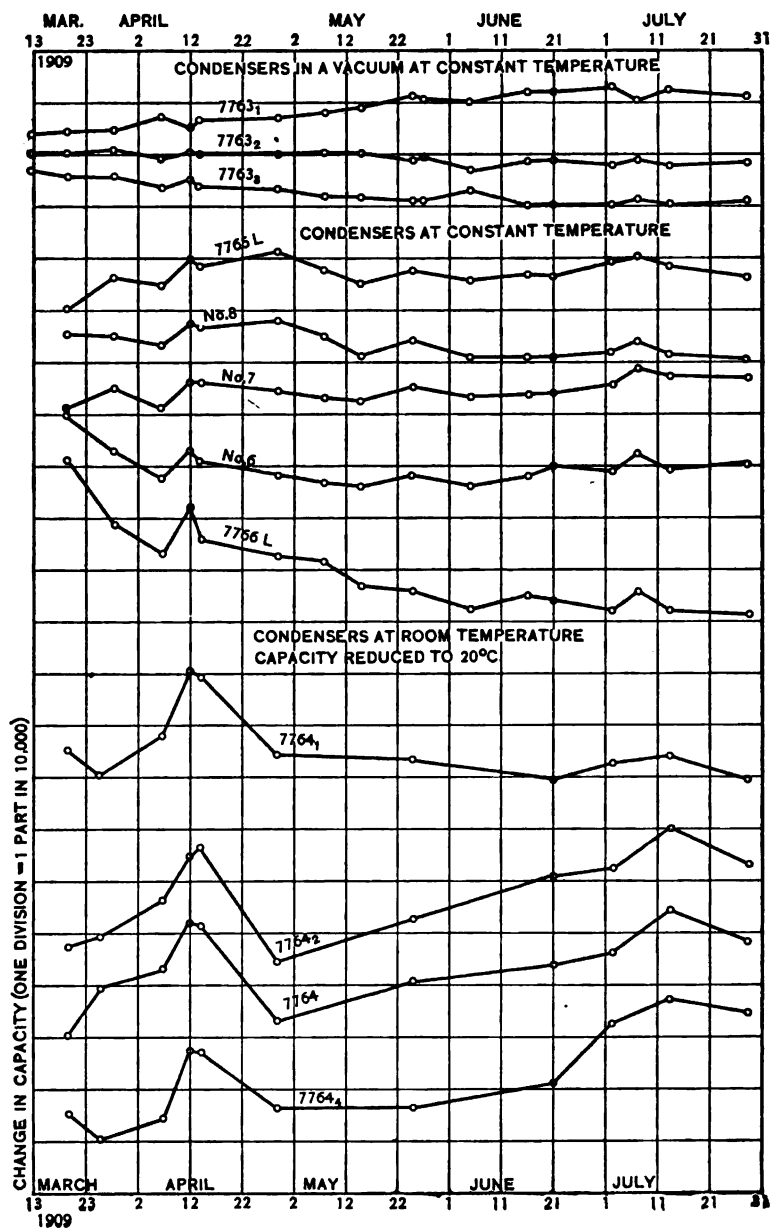


Fig. 15.—Curves showing the change of capacity with time

bility that slight temperature variations have occurred. On the other hand it should not be overlooked that these changes may be due to some cause of which we have no knowledge.

With the best condensers maintained at a constant temperature a constancy of one part in ten thousand over several months was attained. With those in a vacuum this was somewhat improved upon, so that the fluctuations are only a few parts in a hundred thousand, and these have taken place slowly. It should be noticed that one of the condensers, 7766L, which was maintained at constant temperature, showed a slow but continuous decrease in capacity. The condensers which were kept at room temperature showed fluctuations of two or three parts in ten thousand.

It should be noticed that the condensers in a vacuum are the same condensers as those whose behavior is shown in the previous curves (Fig. 14), where considerable variations in a comparatively short time are observed. It would seem to be established by these measurements, and it has been corroborated by several other series, that a condenser at room temperature shows fluctuations in its capacity amounting to several parts in ten thousand. However, the best condensers, when kept at constant temperature, will ordinarily remain constant to a part in ten thousand for a considerable time, and they will show even greater constancy if they are sealed in a vacuum.

V. THE RELATIONSHIP BETWEEN THE DIFFERENT APPARENT CAPACITIES

The principal cause for the difference in the apparent capacities of a condenser is, without doubt, its absorption. At the very high frequencies used in wireless telegraphy the question of the distribution of the charge on the condenser plates becomes a matter for consideration; but, as will be shown later, below 3000 cycles it need not be considered. Also in the case of silvered-mica condensers, and doubtless some others, the voltage may affect the apparent capacity for reasons already explained. But for the present no condenser in which this defect is known to exist will be considered. Hence the problem is to see whether the known facts in regard to absorption can explain the observed differences in capacity.

10. QUALITATIVE RELATIONS

Before attempting to reconcile all the differences in a rigid manner, some qualitative deductions from the curves of Figs. 7-12 may be of use. One of the most important conclusions that will be evident from an examination of these curves is that it can not be readily predicted from A. C. measurements at ordinary frequencies what a condenser will do when measured by D. C. with relatively long times of charge and discharge. There are condensers whose capacity changes but little with change of frequency on A. C., yet which do show a considerable change with length of discharge; though in the majority of cases if a condenser shows little change on A. C. it shows little change on D. C., and if it shows a relatively large change on A. C. it also shows a large change on D. C. Hence there seems to be no simple way of connecting the changes on A. C. with those on D. C.

However, the phase difference always gives an indication of the magnitude of the change in the A. C. capacity with change of frequency. One can not express this relationship accurately, but in all cases where the phase difference is large, there is a considerable change of capacity with change of frequency. Also the behavior in D. C. measurements may be roughly predicted from the value of the absorption. When a condenser has a large measured absorption, then its increase of capacity with increasing time of discharge is also large. Hence a knowledge of both the phase difference and the absorption of a condenser at any temperature, will give a fair idea of the behavior of the condenser at that temperature. The question then arises whether two or more constants can be found by means of which all the facts in regard to a condenser can be expressed.

11. APPLICATION OF MAXWELL'S THEORY OF ABSORPTION TO DIRECT-CURRENT MEASUREMENTS

Since the change in capacity is due to absorption, it is desirable to ascertain whether the necessary constants can not be deduced from one of the theories which have been proposed to explain this phenomenon. Several theories have been advanced, but the only undisputed one was proposed by Maxwell.¹² He showed that with

¹² Maxwell, *Electricity and Magnetism*, § 328.

a nonhomogeneous dielectric in which the product of the dielectric constant and the specific resistance are different for the different parts of the dielectric, there *will be* absorption. The question is whether this is of a magnitude sufficient to account for the observed effect. In order to discuss this more satisfactorily, a brief discussion of Maxwell's reasoning will be given.

NOMENCLATURE.

a_1 and a_2 are thicknesses.
 K_1 and K_2 are the dielectric constants.
 ρ_1 and ρ_2 are the specific resistances.
 E_1 and E_2 are the instantaneous differences of potential.

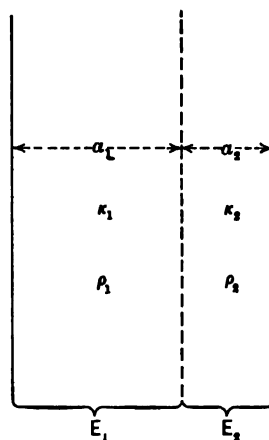


Fig. 16.

Consider for simplicity a condenser composed of two dielectrics in parallel strata, each of which has 1 sq. cm surface. It is easily shown that it is immaterial whether there are only two strata, one of each dielectric, or a number of strata of each material, whose combined thickness is the same as the single stratum. Therefore the case considered will cover the case of two dielectrics, provided that they lie in strata. The equations are easily modified in case there are more than two dielectrics, and the final conclusions apply to any number of dielectrics.

Let an emf, E , be applied to the terminals. Then a quantity Q will be instantaneously displaced through the dielectric, producing potential differences of E_1 and E_2 between the faces of the respective strata. If we consider only the left-hand stratum and designate the capacity due to it by C_1 , then

$$Q = C_1 E_1 = \frac{E_1 K_1}{4\pi a_1} \quad (9)$$

$$\text{Also } Q = C_2 E_2 = \frac{E_2 K_2}{4\pi a_2} \quad (10)$$

$$\text{Hence } \frac{Q}{C} = E = E_1 + E_2 = Q \left\{ \frac{4\pi a_1}{K_1} + \frac{4\pi a_2}{K_2} \right\} \quad (11)$$

where C is the capacity of the condenser due to an instantaneous charge.

$$\therefore \frac{1}{C} = 4\pi \left(\frac{a_1}{K_1} + \frac{a_2}{K_2} \right) \quad (12)$$

If the emf is applied for a finite time, it is necessary to consider the effect of conduction through the dielectric. In this case E_1 and E_2 are not constants, but vary with the time, though at all times $E_1 + E_2 = E$. If i represents the total current through the dielectric at any instant, i'_1 the conduction current through the left-hand dielectric, i''_1 the displacement current, and ρ_1 the specific resistance,

$$i'_1 = \frac{E_1}{a_1 \rho_1} \quad (13)$$

$$\text{and } i''_1 = \frac{dQ}{dt} = \frac{K_1}{4\pi a_1} \frac{dE_1}{dt} \quad (14)$$

$$\therefore i = i'_1 + i''_1 = \frac{E_1}{a_1 \rho_1} + \frac{K_1}{4\pi a_1} \frac{dE_1}{dt} \quad (15)$$

In a like manner

$$i = i'_2 + i''_2 = \frac{E_2}{a_2 \rho_2} + \frac{K_2}{4\pi a_2} \frac{dE_2}{dt} \quad (16)$$

and

$$\frac{E_1}{a_1 \rho_1} + \frac{K_1}{4\pi a_1} \frac{dE_1}{dt} = \frac{E_2}{a_2 \rho_2} + \frac{K_2}{4\pi a_2} \frac{dE_2}{dt} \quad (17)$$

since the total current in each dielectric is the same. If, now, the emf, E , is maintained until an equilibrium is reached,

$$\frac{dE_1}{dt} = 0 \text{ and } \frac{dE_2}{dt} = 0.$$

Let ${}_\infty E_1$ and ${}_\infty E_2$ represent the values of E_1 and E_2 when the equilibrium condition is reached. Then

$$i = \frac{{}_\infty E_1}{a_1 \rho_1} = \frac{{}_\infty E_2}{a_2 \rho_2} = \frac{{}_\infty E_1 + {}_\infty E_2}{a_1 \rho_1 + a_2 \rho_2} = \frac{E}{R} \quad (18)$$

where R is the total resistance between the plates of the condenser.

Let us suppose that after this condition has been reached, the outside emf, E , is removed, the terminals of the condenser connected together by a conductor of negligible resistance for a time just sufficient to allow the free charge Q to neutralize, and then both terminals insulated. Let ${}_0E_1$ and ${}_0E_2$ be the values of E_1 and E_2 at the instant the terminals are insulated.

Then ${}_0E_1 + {}_0E_2 = 0$

And

$$\left. \begin{aligned} {}_0E_1 &= {}_\infty E_1 - \frac{4\pi a_1 Q}{K_1} \\ {}_0E_2 &= {}_\infty E_2 - \frac{4\pi a_2 Q}{K_2} \end{aligned} \right\} \quad (19)$$

$${}_\infty E_1 + {}_\infty E_2 = E = 4\pi Q \left(\frac{a_1}{K_1} + \frac{a_2}{K_2} \right) \quad (20)$$

A comparison of this with equation (11) shows that the instantaneous discharge is the same as the instantaneous charge. If C represents the geometric capacity,

$$C = \frac{E}{Q} = 4\pi \left(\frac{a_1}{K_1} + \frac{a_2}{K_2} \right) \quad (21)$$

It may be measured by either the instantaneous charge or the instantaneous discharge.

After the terminals are insulated the current i is zero, hence from equation (15),

$$-\frac{dE_1}{E_1} = \frac{4\pi}{\rho_1 K_1} dt \quad (22)$$

Integrating, $\log E_1 = -\frac{4\pi t}{\rho_1 K_1} + \text{a constant.}$

Since ${}_0E_1$ is the value of E_1 when $t = 0$

$$E_1 = {}_0E_1 e^{-\frac{4\pi t}{\rho_1 K_1}} \quad (23)$$

Likewise $E_2 = {}_0E_2 e^{-\frac{4\pi t}{\rho_2 K_2}}$ (23a)

The difference of potential E' between the terminals at any instant, t , is

$$E' = E_1 + E_2 = {}_0E_1 e^{-\frac{4\pi t}{\rho_1 K_1}} + {}_0E_2 e^{-\frac{4\pi t}{\rho_2 K_2}} \quad (24)$$

The values of ${}_0E_1$ and ${}_0E_2$ are given in equations (19). From (18)

$${}_0E_1 = \frac{E a_1 \rho_1}{R} \quad \text{and} \quad {}_0E_2 = \frac{E a_2 \rho_2}{R}$$

$$\therefore E' = E \left\{ \left(\frac{a_1 \rho_1}{R} - \frac{4\pi a_1 C}{K_1} \right) e^{-\frac{4\pi t}{\rho_1 K_1}} + \left(\frac{a_2 \rho_2}{R} - \frac{4\pi a_2 C}{K_2} \right) e^{-\frac{4\pi t}{\rho_2 K_2}} \right\} \quad (25)$$

This equation gives the value of the potential difference of the condenser plates at any time, t , after they are insulated in terms of the dielectric constants and specific resistances of the strata. For simplicity let

$$D_1 = \frac{a_1 \rho_1}{R} - \frac{4\pi a_1 C}{K_1}; \quad D_2 = \frac{a_2 \rho_2}{R} - \frac{4\pi a_2 C}{K_2}$$

$$m_1 = \frac{4\pi}{K_1 \rho_1}; \quad m_2 = \frac{4\pi}{K_2 \rho_2}$$

Then $E' = E \{ D_1 e^{-m_1 t} + D_2 e^{-m_2 t} \}$ (26)

It should be noticed that m_1 and m_2 are necessarily positive, and it follows from (24) and (25) that $D_1 + D_2 = 0$, since ${}_0E_1 + {}_0E_2 = 0$.

The residual charge A_t is defined as the ratio of the quantity which reappears when a condenser has been completely charged, then instantly discharged, and after this left insulated for a time, t ; to the total free charge.

Hence, from the equation (26),

$$A_t = \frac{E' C}{E C} = D_1 e^{-m_1 t} + D_2 e^{-m_2 t} \quad (27)$$

This can be put into a somewhat simpler form for purposes of computation by expanding in a series. Then

$$A_t = D_1 \left(1 - m_1 t + \frac{m_1^2 t^2}{2!} - \frac{m_1^3 t^3}{3!} + \dots \right) + D_2 \left(1 - m_2 t + \frac{m_2^2 t^2}{2!} - \frac{m_2^3 t^3}{3!} + \dots \right) \quad (28)$$

Now, $D_1 + D_2 = 0$

$$\text{And } -\log(1 + m_1 t) = -m_1 t + \frac{m_1^2 t^2}{2} - \frac{m_1^3 t^3}{3} + \dots \quad (29)$$

Hence, if $m_1 t$ and $m_2 t$ are sufficiently small so that terms above the square may be neglected,

$$A_t = -D_1 \log(1 + m_1 t) - D_2 \log(1 + m_2 t) \quad (30)$$

This should hold with considerable accuracy during the first part of the discharge, but can not be used when t is relatively large.

We shall now apply a numerical test to this formula, to see whether it explains the observed phenomena. In the adjustment of some mica condensers, the capacity of a single sheet of mica was determined in a number of cases. These measurements gave a mean value in electrostatic units of about 105 cm per sq. cm. Also from measurements of the insulation resistance, the mean resistance, R , per sq. cm was found to be 8.3×10^3 electrostatic units. The thickness of the mica sheets being about 0.005 cm, and the known dielectric constant of paraffin about 2, and of mica 7, the thickness of the paraffin is computed to be 0.0001 cm. An examination of the plates shows that this is a reasonable value. Since, now, the insulation resistance of paraffin is very high, being usually given as between 10^{18} and 10^{19} ohms per cm cube or 10^6 to 10^7 electrostatic units, it follows that a very large part of the observed insulation resistance is due to the paraffin, so that the resistance of the mica can not be computed from this data, since there may be a wide variation in the resistance of the mica without producing any appreciable difference in the total insulation resistance. The published values of the specific resistance of mica vary from 10^{-1} to 10^4 electrostatic units. The lower values, which are due to Rood¹³ should not be given equal weight with the others, as he did not continue the application of the emf until a steady state was reached. Values obtained by Grover in this laboratory are in accord with those of Curie¹⁴ who found 10^4 electrostatic units. The value of the residual charge $A_{0.1}$ at the end of a tenth of a second, computed from the data already given and using various values of the specific resistance of mica are given in Table X.

¹³ Amer. J. Sci. **164**, p. 161; 1902. ¹⁴ Annal. Chim. Phys. (6), **18**, p. 229; 1889.

TABLE X

Given a Condenser with Paraffin and Mica as Dielectrics, to Compute the Residual Charge at the end of a Tenth of a Second, Using the Formula of Equation (30)

CONSTANTS: R=10 ⁹ ELECTROSTATIC UNITS; C=105 CM			
Mica			
$a_1=0.005$ cm $K_1=7$.			
ρ_1	$m_1 t$	D_1	$A_{\infty 1}$
10 ⁴	1.8×10^{-8}	-0.90	0.000016
10 ³	1.8×10^{-4}	-0.94	0.00017
10 ²	1.8×10^{-3}	-0.95	0.0017
50	3.6×10^{-3}	-0.95	0.0034
15	1.2×10^{-2}	-0.95	0.005
10	1.8×10^{-2}	-0.95	0.017
Paraffin			
$a_2=0.0001$ cm $K_2=2$ $\rho_2=10^7$ electrostatic units $m_2 t=6.4 \times 10^{-8}$ $D_2 \log (1+m_2 t)$ is negligible			

From the curves in Figs. 7-11 an idea of the value of the absorbed charge which is actually given up during the first tenth of a second can be obtained in the following way. As will be shown later, the value of the capacity at 1200 cycles differs but little from the geometric capacity. The difference between the apparent capacity using 0.6 second charge and 0.1 second discharge and the geometric capacity is due to the absorbed charge which reappears in the first 0.1 second. The value of this varies from 0.3 to 1 per cent in the case of these condensers. Hence, if the computed values are to agree with the observed, the specific resistance of mica must lie between 10 and 50 electrostatic units. As the best experimental results give in the neighborhood of 10⁴ electrostatic units, it seems probable that only a small part of the observed absorption is due to this cause.

12. APPLICATION OF MAXWELL'S THEORY OF ABSORPTION TO ALTERNATING-CURRENT MEASUREMENTS

A confirmation of the conclusions drawn from the D. C. measurements can be obtained by applying the theory to A. C. measurements. The formulas were first derived by Rowland,¹⁸ though he did not apply them to actual measurements. In the following derivation the nomenclature of Fig. 16 will be followed as far as possible:

From equation (17)

$$i = \frac{E_1}{a_1 \rho_1} + \frac{K_1}{4\pi a_1} \frac{dE_1}{dt} = \frac{E_2}{a_2 \rho_2} + \frac{K_2}{4\pi a_2} \frac{dE_2}{dt} \quad (31)$$

Let $m_1 = \frac{4\pi}{K_1 \rho_1}$ and $B_1 = \frac{4\pi a_1}{K_1}$

with similar values for m_2 and B_2 ,

then $\frac{dE_1}{dt} + m_1 E_1 = B_1 i$ (32)

If the current follows the sine law,

$$i = I \sin pt \text{ where } p = 2\pi \times \text{the frequency}$$

$$\therefore \frac{dE_1}{dt} + m_1 E_1 = B_1 I \sin pt \quad (33)$$

Integrating $E_1 = \frac{IB_1}{m_1^2 + p^2} (m_1 \sin pt - p \cos pt) + Ae^{-m_1 t}$ (34)

where A is the constant of integration. The last term becomes zero when t is sufficiently large. In the same manner

$$E_2 = \frac{IB_2}{m_2^2 + p^2} (m_2 \sin pt - p \cos pt) \quad (35)$$

But the emf also follows the sine law, so that

$$E_1 + E_2 = E \sin (pt - \phi)$$

where ϕ is the difference in phase of the current and emf. Substituting the values of E_1 and E_2 from (34) and (35)

¹⁸Amer. J. Sci. (4), 4, p. 429; 1897.

$$E \sin (pt - \phi) = I \left\{ \frac{B_1}{m_1^2 + p^2} (m_1 \sin pt - p \cos pt) + \frac{B_2}{m_2^2 + p^2} (m_2 \sin pt - p \cos pt) \right\} \quad (36)$$

Since this is true for all values of pt , it is true when $pt = 0$ and when $pt = \frac{\pi}{2}$

In the first case

$$E \sin \phi = Ip \left\{ \frac{B_1}{m_1^2 + p^2} + \frac{B_2}{m_2^2 + p^2} \right\} \quad (37)$$

In the second case

$$E \cos \phi = I \left\{ \frac{m_1 B_1}{m_1^2 + p^2} + \frac{m_2 B_2}{m_2^2 + p^2} \right\} \quad (38)$$

From Fig. 3, page 437, it is seen that

$$\frac{1}{pC} = \frac{E \sin \phi}{I}$$

where C is the apparent capacity. Hence, from equation (37)

$$\frac{1}{C} = \frac{B_1}{1 + \frac{m_1^2}{p^2}} + \frac{B_2}{1 + \frac{m_2^2}{p^2}} \quad (39)$$

If the frequency is infinite

$$\frac{1}{C} = B_1 + B_2 = 4\pi \left(\frac{a_1}{K_1} + \frac{a_2}{K_2} \right) \quad (40)$$

which is the same value as found for instantaneous charge or discharge.

In equations (37) and (38), $p^2 + m_1^2$ and $p^2 + m_2^2$ enter in the denominator, where

$$m_1 = \frac{4\pi}{K_1 \rho_1} \text{ and } m_2 = \frac{4\pi}{K_2 \rho_2}$$

In all cases K is greater than unity, and in the cases we are considering p is still larger. Hence, if p is more than 100, in no case will an error greater than 1 per cent be introduced by

neglecting m^2 . Making this approximation and dividing (38) by (37)

$$\cot \phi = \tan \theta = \frac{m_1 B_1 + m_2 B_2}{p (B_1 + B_2)} \quad (41)$$

Hence, with the values of K and ρ which have been assumed, it follows from this theory that the phase difference should vary inversely as the frequency. This is equivalent to saying that the energy loss per second is independent of the frequency. The values of the phase difference given in the curves of Figs. 7-12 show very plainly that this is not the case.

Interesting results are also obtained by inserting the constants which make the computed value of the residual charge agree with the observed value in the formulas which have just been developed. These results are given in Table XI. It will be seen that the computed change in the capacity is very much less than the observed. On the other hand, the computed values of the phase difference at

TABLE XI

The Observed and Computed Values of the Phase Difference and the Change in Capacity at Different Frequencies

Formulas (39) and (41) are Used in the Computation

CONSTANTS				
Mica			Paraffin	
$a_1=0.005$ cm			$a_2=0.0001$	
$K_1=7$			$K_2=2$	
$\rho_1=15$ electrostatic units			$\rho_2=10^7$ electrostatic units	
RESULTS				
Change in capacity from geometric capacity in parts in a hundred thousand			Phase difference	
Frequency	Observed	Computed	Observed	Computed
1200	7	0.00003	50''	20''
100	60	0.004	2' 0''	4' 0''
50	85	0.016	2' 30''	8' 0''

The observed values given are those of 5471B£0.1.

low frequencies is larger than that observed. This would indicate that the value of the specific resistance of mica has been taken too low, which is what the direct measurements of the resistance indicated.

From the above considerations it is evident that the observed absorption in mica condensers can not be accounted for on the basis of a stratified dielectric. If the true values of the constants are those which at present seem to be the most probable values, the absorption caused by the nonhomogeneity of the dielectric is a negligible part of the total absorption.

There remains the possibility suggested by Rowland¹⁶ that this theory be modified by supposing that electrolysis and thermoelectric currents are present. Still another theory¹⁷ has been advanced, which accounts for the phenomena of absorption by supposing that there is a polarization in the dielectric. Neither of these theories is sufficiently developed from a mathematical view point to allow of a numerical test. Hence the data which have been collected on the dependence of the apparent capacity on the method of measurement will be discussed from an empirical standpoint alone.

13. DETERMINATION OF THE GEOMETRIC CAPACITY

If the various values of the capacity are to be connected by some relation, it is necessary first of all to know whether the geometric capacity as determined by A. C. is the same as that determined by D. C. The A. C. value is found by plotting values of the apparent capacity as ordinates and values of the period (reciprocal of the frequency) as abscissa. This curve is projected backward to intersect the axis of zero period, and the intercept on this axis is taken as the geometric capacity. In the case of the better condensers there is no difficulty in determining the geometric capacity with precision.

Examples are given in the curves of Fig. 17. Here the frequencies used are 1200, 100, and 50. Frequencies from 10 to 3000 were available, but neither the highest nor lowest were found to be useful, for reasons which follow. Above 900 cycles the only indicating instrument at hand was a telephone. Above 1000 vibrations per second the sensibility of the ear decreases rapidly, so that

¹⁶ Amer. J. of Math., 1, 53; 1878.

¹⁷ First suggested by Kohlrausch, Pogg. Am., 91, p. 56; 1854.

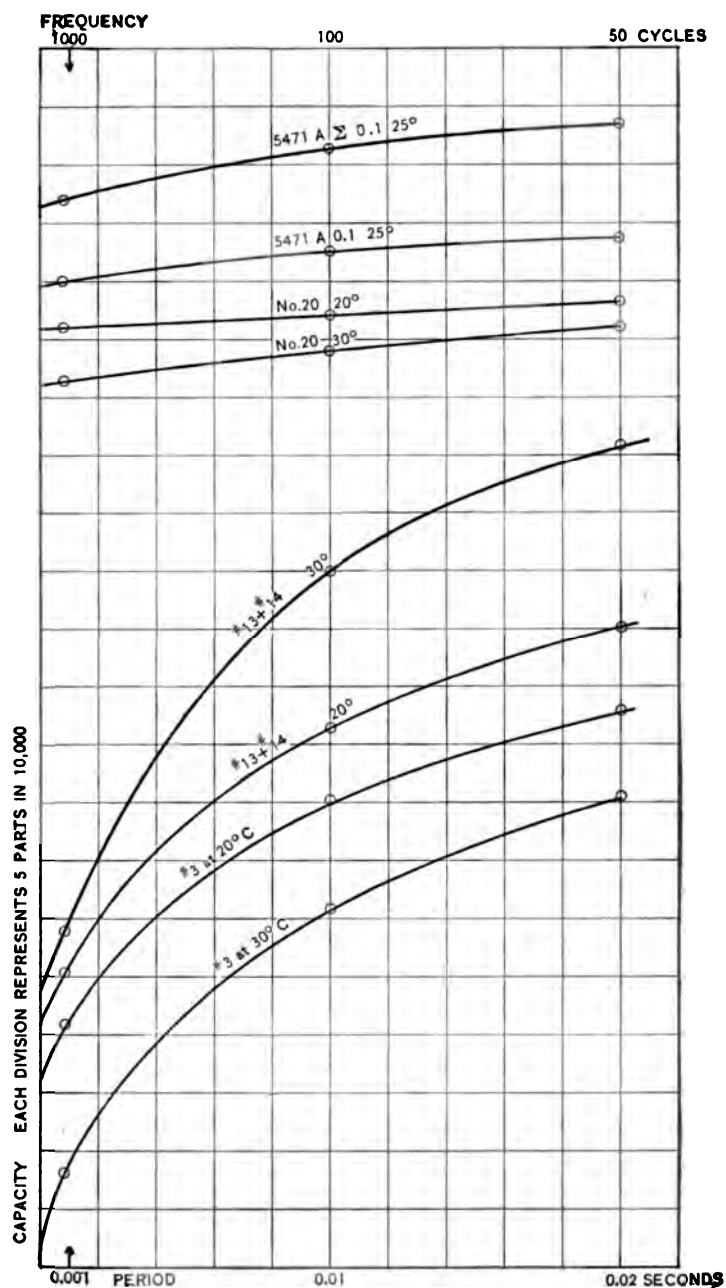


Fig. 17.—Curves showing method of determining the geometric capacity by A. C. measurements

it is impossible to make measurements of the highest accuracy at frequencies very much above this. Also at 3000 cycles the inductance of shortest possible leads often introduces an appreciable reactance, for which a correction must be introduced; and the resistance of the leads may prevent the condenser from becoming completely charged during each cycle, necessitating still another correction. The measurements made at 3000 cycles, using all possible precautions, showed a small decrease in the apparent capacity over that measured at 1200 cycles, and there is nothing to indicate that the capacity so determined does not lie on the curve as drawn. However, on account of the inferior accuracy, the use of these values is more of a hindrance than a help in determining the geometric capacity.

Also a number of condensers were measured at 12.5 cycles. The results in this case are interesting and are shown in the curves of Fig. 18. However, at this frequency the measurements are not very accurate and do not aid in the least in the determination of the geometric capacity, since the shape of the curve near the origin can not be inferred from points so far removed. If there is to be an improvement in the determination of the geometric capacity by this method, it will come from measurements in the interval between 1000 and 100 cycles. For the better condensers this is hardly necessary, but would be of some advantage with the poorer ones.

To determine the geometric capacity by means of direct current measurements¹⁸ it is necessary to measure the apparent capacity at definite and very short times of discharge. With these values a curve is plotted in which the apparent capacities are ordinates and the times of discharge are abscissa. Extending this curve to cut the axis of zero time, the capacity thus indicated is the geometric capacity. Such curves in the case of three condensers are shown in Figs. 19 and 20. In each case four values were obtained with short times of discharge, the longest being a tenth of a second. There is also drawn for comparison the curve representing the alternating current capacities, where the period of the applied emf. is plotted as abscissa.

¹⁸ This is similar to a method which was employed by Zeleny and Andrews, *Phys. Rev.* 27, p. 65; 1908.

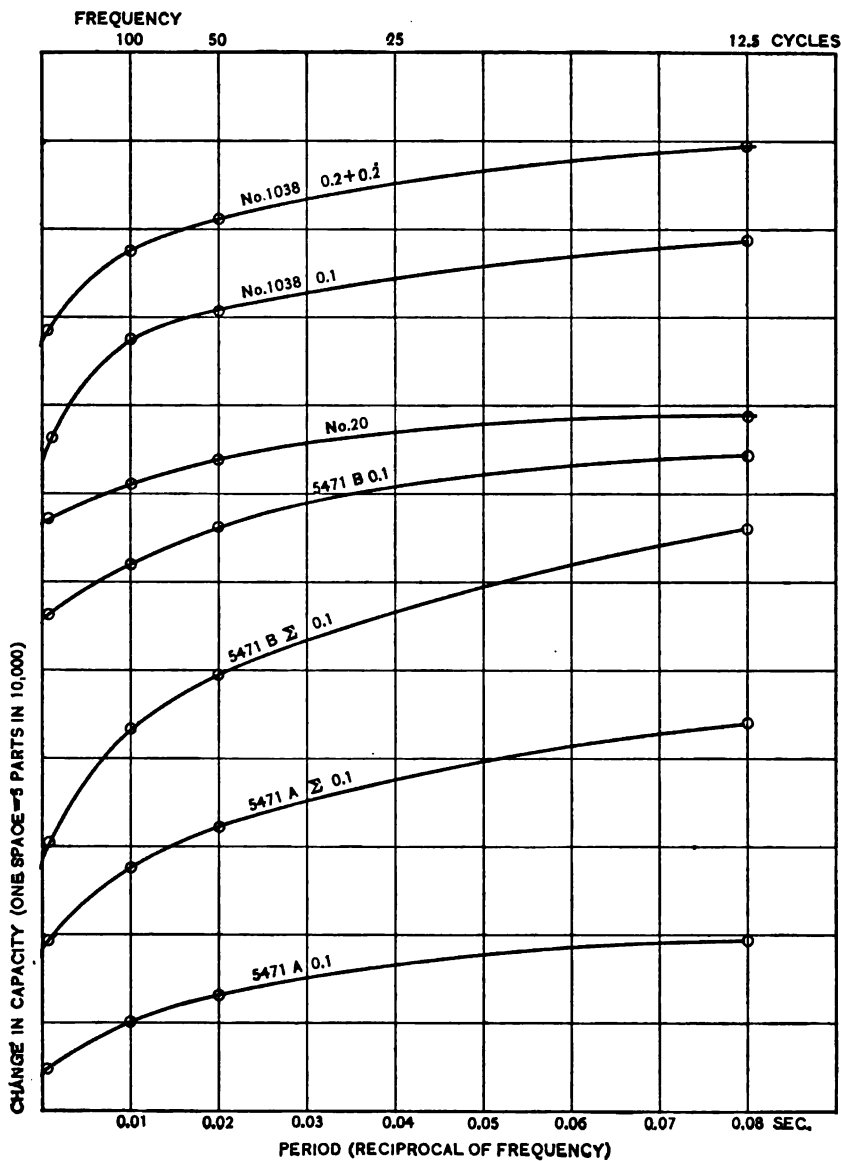


Fig. 18.—Curves showing the change of capacity at low frequency

The extrapolation in the case of the D. C. capacities is much less satisfactory than with the A. C. values, since the curve must be extended five times as far in the first case as in the second. However, it is sufficiently accurate to show that if there is any difference in the geometric capacity as determined by the two methods, it is very small; certainly less than a part in ten thousand, and in all probability less than a part in a hundred thousand.

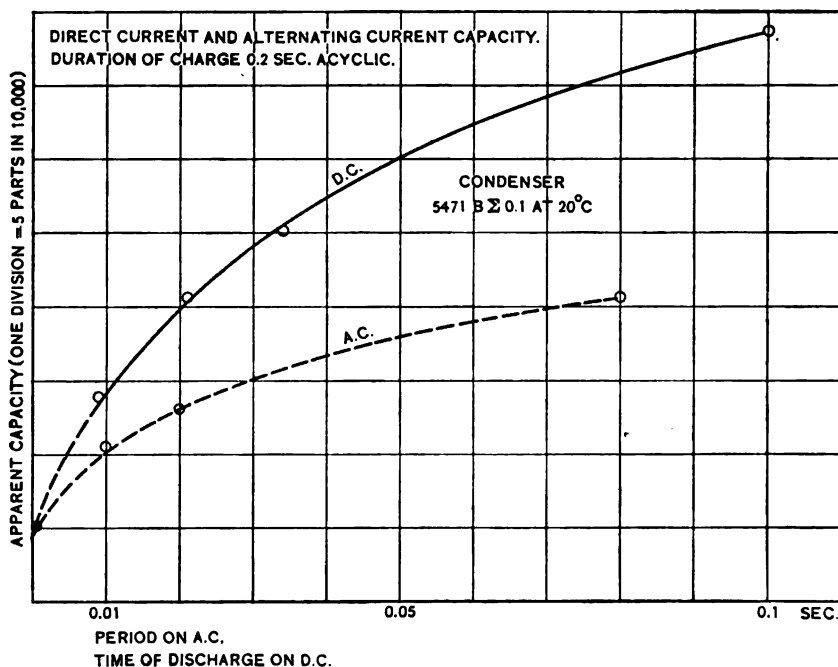


Fig. 19.—Curves showing agreement of the geometric capacity as determined by D. C. and A. C. measurements

14. EFFECT OF LEAKAGE

In the preceding work no account has been taken of the quantity of electricity conducted through the condenser. It is necessary to show that the resistance of the condensers is so high that this quantity is inappreciable. In the direct current work there was in each case an interval of a tenth of a second from the time the condensers were disconnected from the battery until they were discharged. In order that the leakage shall produce an effect of one part in a hundred thousand, a quantity equal to $E \times 10^{-12}$ coulombs must leak through a tenth-microfarad condenser in 0.1

second. This requires that the resistance shall be 10^{11} ohms. As each of the condensers used, including the air condenser, had resistances higher than this, the effect of leakage was negligible in all the D. C. measurements. With A. C. the effect on the measured capacity is much less than with D. C. It can even be shown that no appreciable part of the in-phase current in the case of the con-

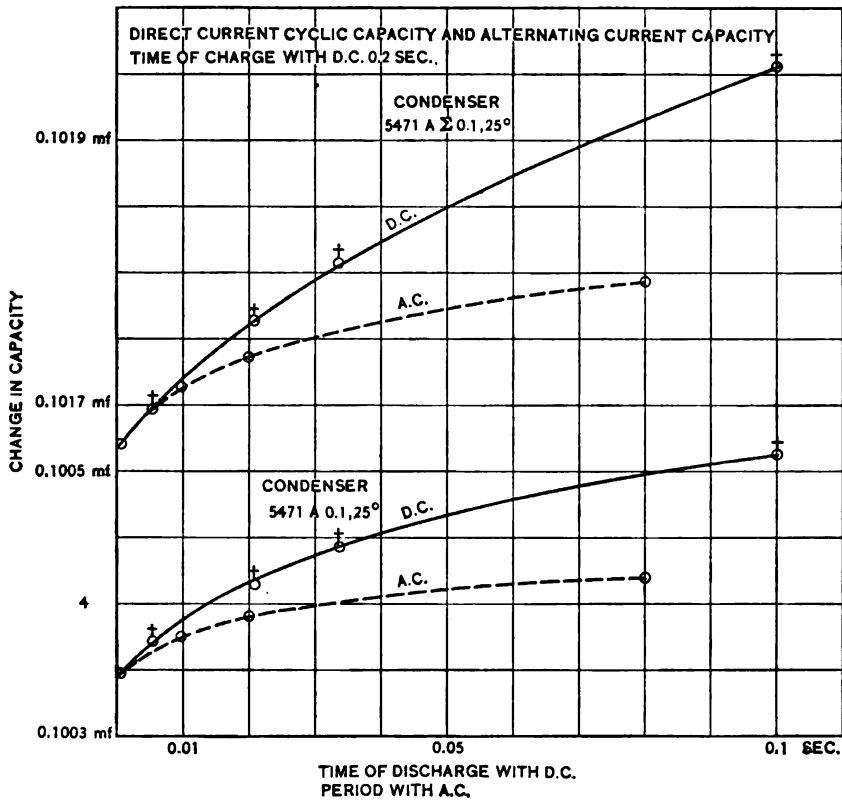


Fig. 20.—Curves showing the agreement of the geometric capacity as determined by D. C. and A. C. measurements

densers used in this investigation is due to leakage. Hence, the assumption that the insulation resistance is infinite is justifiable.

15. POSSIBLE SOURCES OF ERROR IN THE MEASUREMENT OF THE GEOMETRIC CAPACITY

Though the geometric capacity is the same whether determined by D. C. or A. C., yet there remains the possibility that this agreement is only accidental. In this case there must be two or more

causes whose effects exactly balance each other. There are three causes which may be operative to produce differences in the value of the geometric capacity by the two methods. The first lies in the possibility that, in the direct-current measurements, the total absorbed charge which is given up during the first instant of the discharge may be affected by the rate of dissipation of the free charge. The second is due to the fact that when alternating current of high frequency is used the distribution of the charge on the condenser plates is not uniform. The third lies in the possibility that the dielectric properties of mica are not the same with A. C. as with D. C., since with D. C. the charging or discharging may take place in a millionth of a second, while with A. C. these processes take place much more slowly.

The effect of the rate of discharge of the free charge on the reappearance of the absorbed charge has been investigated experimentally. In the method of mixtures, if the resistance, r (see Fig. 4), through which the condensers are discharged is varied over a wide range but always kept small enough so that no measurable part of the free charge will remain when the condensers are connected to the galvanometer, then if C_1 is an air condenser and C_2 a mica condenser, any change in the apparent capacity of C_2 on account of the variations of r will be due to a change in the amount of the absorbed charge which is given up during the dissipation of the free charge. This will now be subjected to a numerical test. The quantity of the free charge Q_t which is discharged in a time, t , is $Q_t = Q \left(1 - e^{-\frac{t}{rc}} \right)$ where Q is the total free charge.

If t is 0.1 second, $c = 10^{-7}$ farads, and $r = 10^5$ ohms, Q_t will differ from Q by only four parts in a hundred thousand, so that for lower values of r , Q_t is sensibly equal to Q . Experimentally it was found that when r was changed from 0.1 ohm to 1000 ohms no appreciable change in the apparent capacity occurred, but at $r = 10\,000$ ohms there was a decrease of four parts in a hundred thousand when a medium-grade mica condenser was used. This shows that if r is small this effect is negligible.

The effect of the frequency on the distribution of charge on the plates has been investigated mathematically by Coffin.¹⁹ An application of his formula to the air condensers used in these

¹⁹ Phys. Rev., 25, p. 123; 1907.

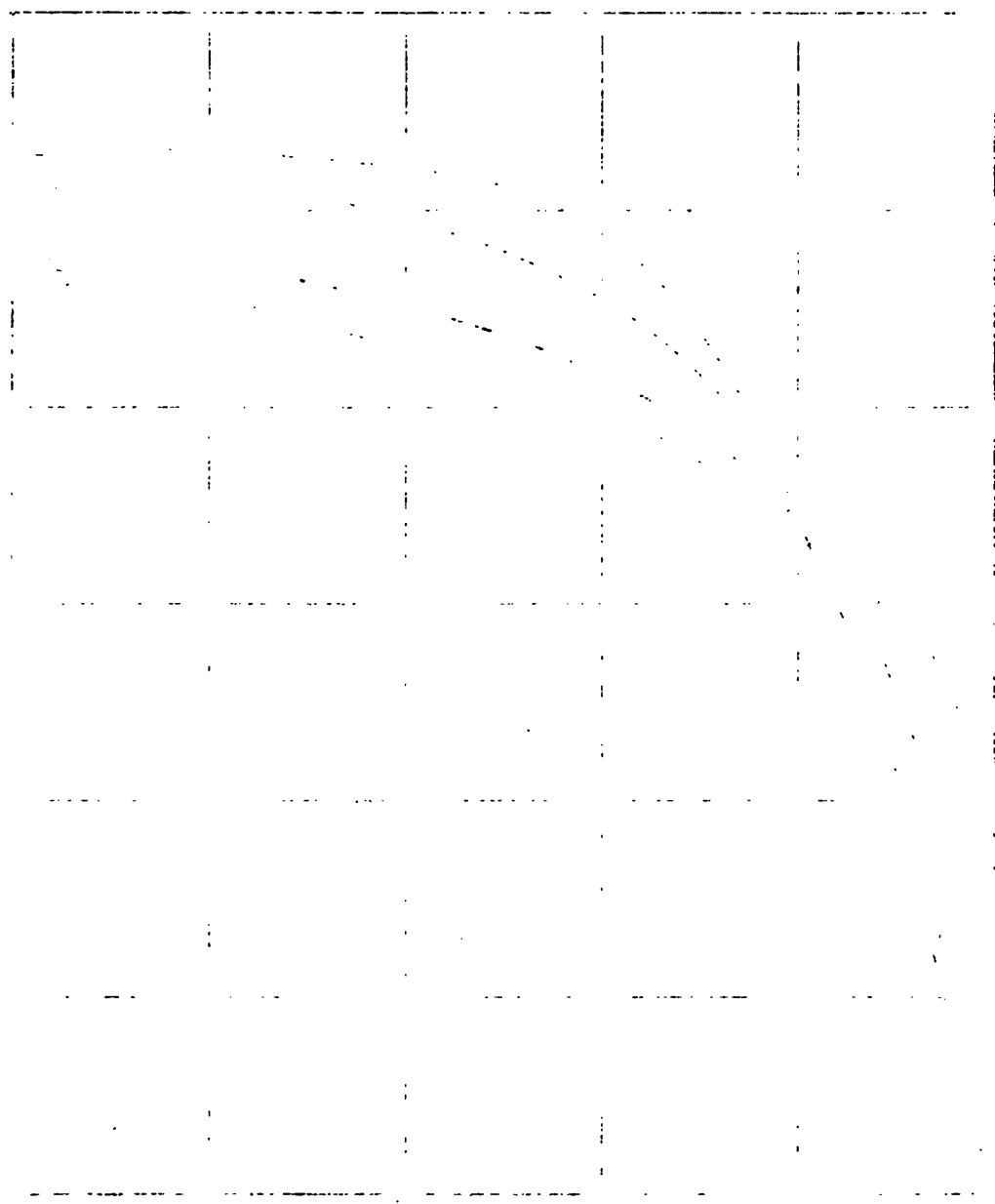
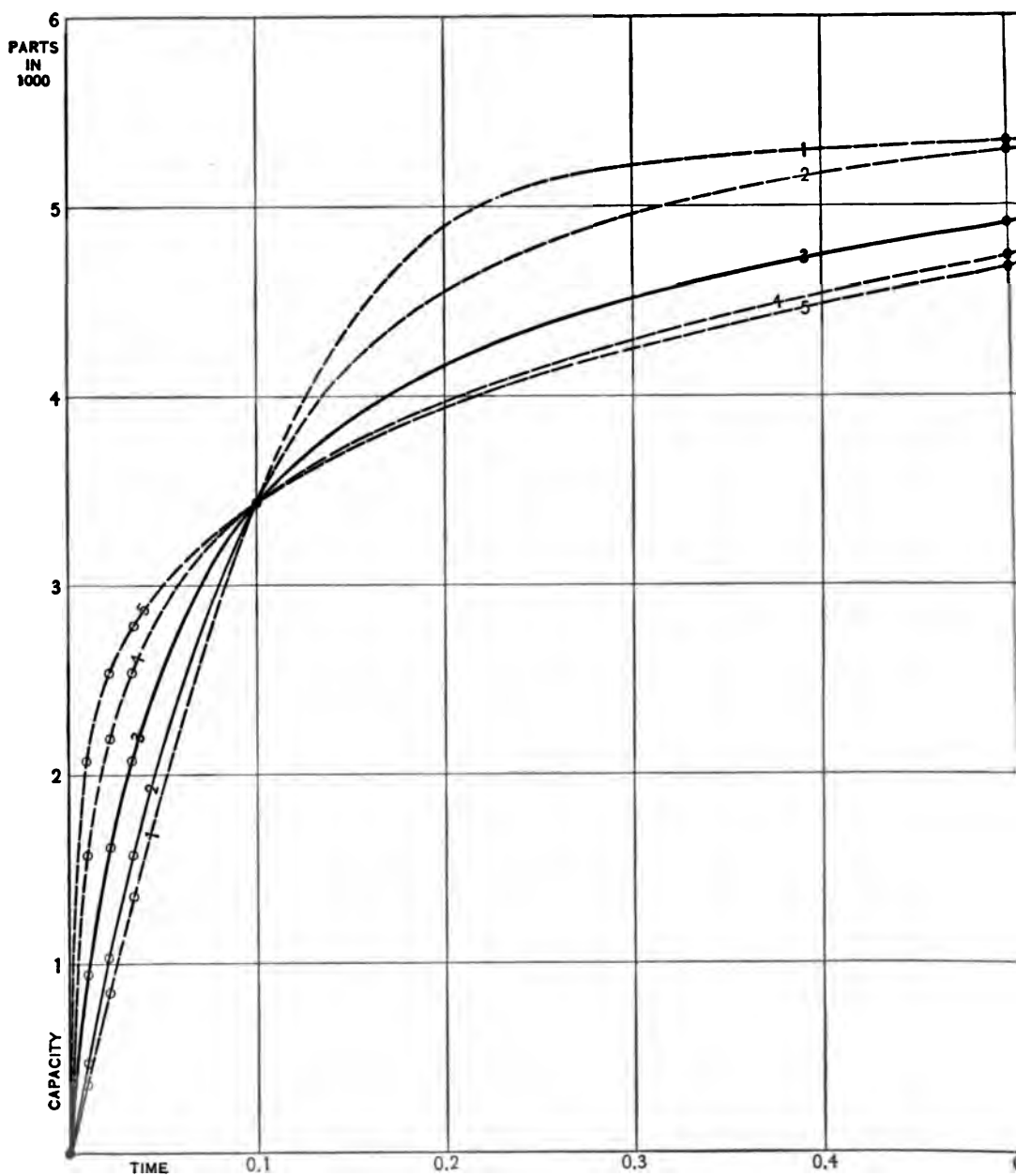
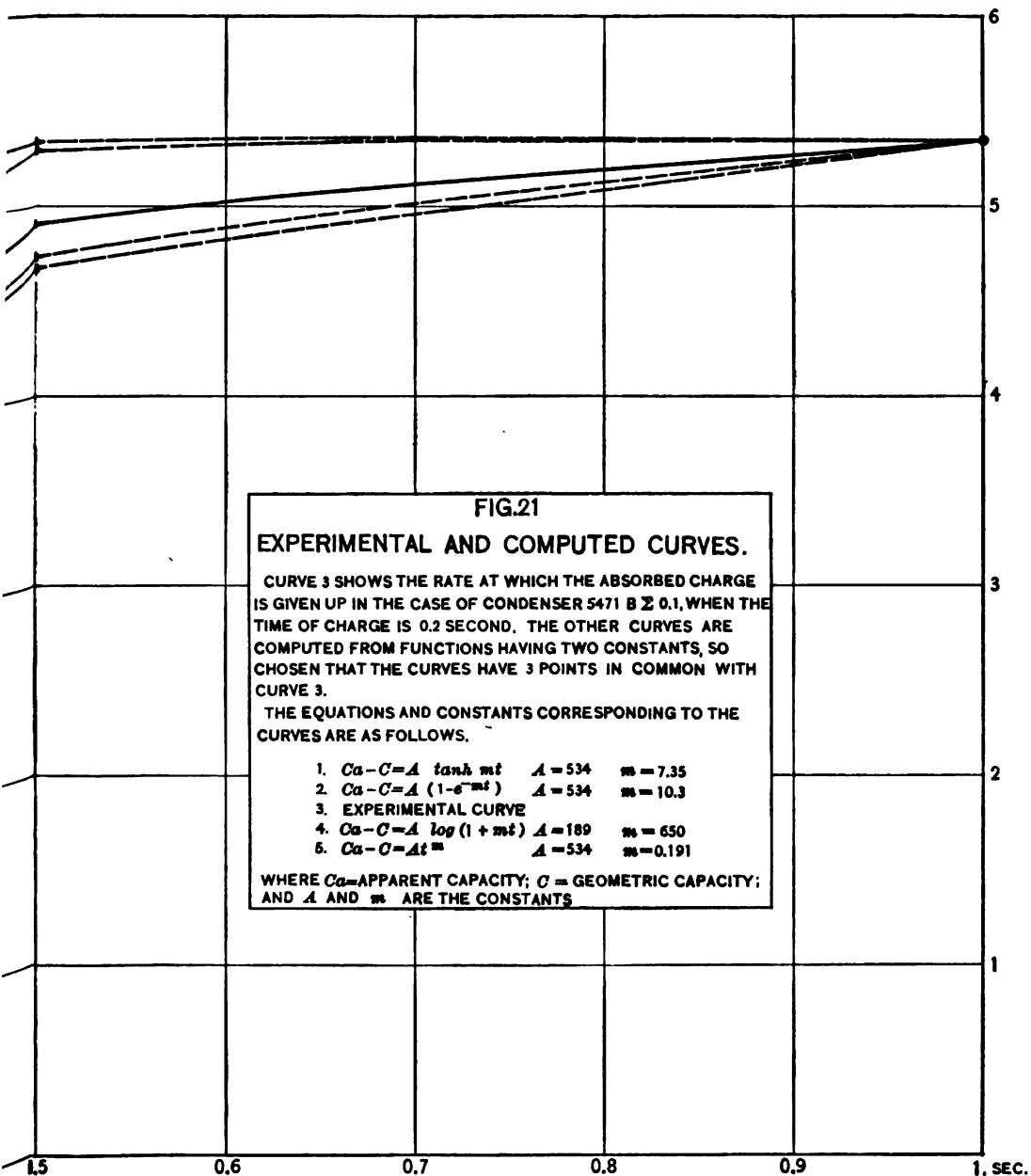


Figure 1. Age Structure, 1950-2050



48848—10. (To face page 485.)



experiments showed that the change in capacity at 3000 cycles due to the fact that the charge is not uniformly distributed on the plates is of the magnitude of one part in a hundred million. With mica condensers it is of the same order of magnitude.

Since two of the three possible effects have been shown to produce negligible errors, the conclusion must be drawn that none of them enter. Hence, the geometric capacity as determined is a function of the dielectric constants and dimensions of the dielectrics of which the condenser is constructed.

16. EMPIRICAL FORMULAS

While a relationship between all of the various capacities has not been established, yet a method of procedure for an empirical treatment can be indicated. It is first necessary to determine the equation of the discharge curve for several different times of charge and then to find the relationship between these curves. In other words, the absorbed quantity of electricity which reappears in a given time must be expressed as a function of the time of charge and the time of discharge. Then the apparent capacity with different A. C. frequencies must be expressed as a function of the frequency and the relationship between this function and the previous one must be established. It will be seen that these requirements are rather rigorous; and, unless the functions are comparatively simple, a knowledge of them will be of no practical use.

The discharge curve with 0.2 second charge was determined for one condenser (5471BΣ0.1) with great care. An attempt was then made to find a formula with two constants which would represent this curve.

The following formulas were tried:

Pellat's²⁰ $C_a = C + A (1 - e^{-mt})$

Wilson's²¹ $C_a = C + A \log (1 + mt)$

v. Schweidler's²² ... $C_a = C + At^m$

Curtis's $C_a = C + A \tanh mt$

In these equations C_a represents the apparent capacity, C the geometric capacity, and A and m are constants. The curves to

²⁰ Ann. chim phys. (7), 18, p. 150; 1899.

²² Ann. d. Phys., 24, p. 711; 1907.

²¹ Proc. Roy. Soc. A., 82, p. 409; 1909.

represent these equations, together with the experimental curve, are given in Fig. 21. The constants of each equation have been so chosen that the curve which it represents passes through the same points as the experimental curve when $t=0.1$ second and when $t=1$ second. It is seen that, for small values of t , the difference between the ordinates of the experimental and empirical curves is much larger than the experimental error.

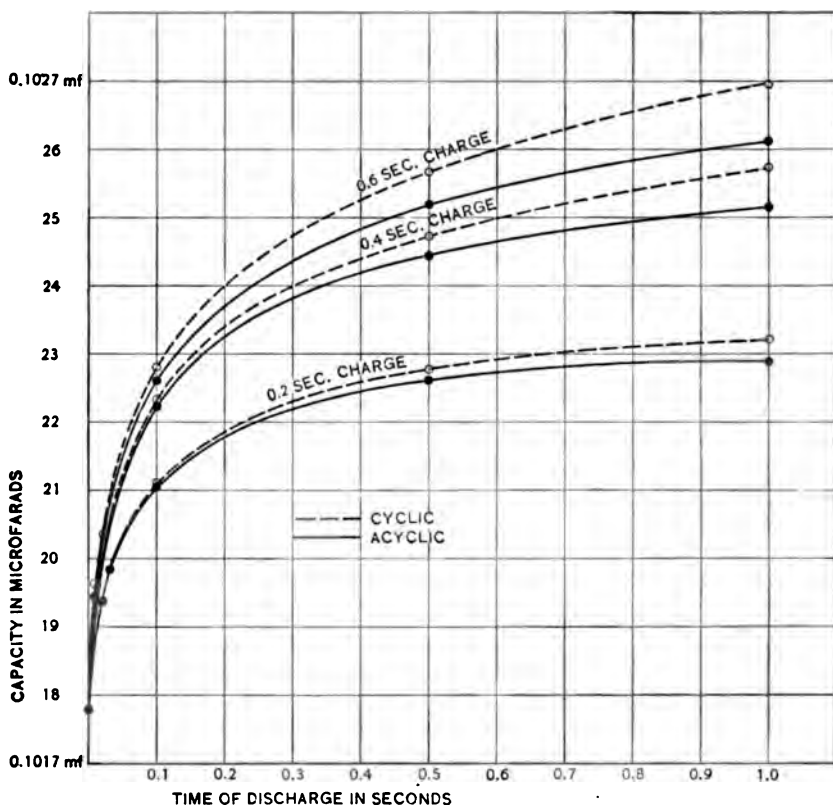


Fig. 22.—Cyclic and acyclic capacity with different times of charge and of discharge

By taking a series of terms any one of the above functions can be extended so as to represent the curve with any desired accuracy. However, such a function is so complicated that it does not seem feasible to attempt to express all the different D. C. and A. C. values of the capacity by means of a single function.²³

²³ Hill has found the same difficulty in the case of paper condensers, *Phys. Rev.*, 26, p. 400; 1908.

17. CASES OF PRACTICAL IMPORTANCE

There is one case of considerable practical importance where it is possible to pass from direct current to alternating current values with a fair degree of accuracy. If a good grade mica condenser is measured by means of a rotating commutator or other means for charging and discharging a given number of times per second, the capacity so determined will not differ greatly from the value determined by alternating current of the same frequency. It is evident that this agreement will be closer the higher the frequency.

A considerable number of comparisons at 100 cycles have been made, and the measured differences seldom exceed two or three parts in ten thousand. They may be either positive or negative, depending upon the length of contact.

One may be reasonably sure that with good mica condensers the difference of the A. C. and D. C. values of the capacity at any one frequency will be less than a tenth of 1 per cent.

Another case of some interest is the difference between the cyclic and acyclic values of the capacity at different times of charge and different times of discharge. This is illustrated in the curves of Fig. 22. It is evident that this difference is also a function of the length of the cycle, though this is not shown in the curves.

VI. SUMMARY.

(1) The various methods for measuring capacity with A. C. and D. C. are discussed, and the best methods to use when mica condensers are to be measured are indicated.

(2) A method is given by which the temperature coefficient of a mica condenser may be made very small, at most a few parts in a hundred thousand per degree.

(3) The effect of changes of the atmospheric pressure upon the capacity are shown to be small, though in some cases they are not negligible.

(4) An explanation of the change of capacity with voltage, which is observed in the case of silvered-mica condensers, is offered.

(5) Condensers which are kept in a vacuum and maintained at constant temperature can be relied upon to a few parts in a hundred thousand. Such condensers are somewhat more stable than condensers which are maintained at constant temperature,

but subjected to the ordinary fluctuations of atmospheric pressure. Condensers exposed to the ordinary changes of room temperature show unaccountable variations of the capacity of two or three parts in ten thousand.

(6) It is shown that, taking the measured values of the specific resistances of mica and paraffin, Maxwell's theory of a stratified dielectric does not account for the absorption actually observed in the case of a mica condenser. This theory is then applied to alternating current measurements, and it is shown that, according to this theory, the loss of energy per *second* in the dielectric is nearly independent of the frequency, whereas measurements show that the loss of energy per *cycle* is almost independent of the frequency. The conclusion is drawn that this theory alone will not explain the phenomena of absorption.

(7) The capacity of a condenser with A. C. of infinite frequency is shown to be the same as the capacity with D. C. when the time of discharge is infinitely short. This capacity is called the geometric capacity, since it is a function of the dimensions and dielectric constants of the dielectrics.

In conclusion, I wish to express my thanks to Dr. E. B. Rosa and to Dr. F. W. Grover for the interest they have shown in this research, and for the many valuable suggestions they have given.

WASHINGTON, D. C., April 22, 1910.

THE MUTUAL INDUCTANCE OF TWO PARALLEL COAXIAL CIRCLES IN TERMS OF HYPERGEOMETRICAL SERIES

By Frederick W. Grover

In the *Journal de Physique* for 1901 (vol. 10, p. 33) there appeared a paper by E. Mathy entitled "Application des signes de Weierstrass à l'étude de l'énergie potentielle de deux courants circulaires parallèles d'intensité un," in which the author, by the introduction of the Weierstrassian notation, obtains the mutual induction of two parallel, coaxial circles in terms of hypergeometrical series, instead of the usual forms involving elliptic integrals.

His formula is

$$\begin{aligned} \frac{M}{4\pi} = & \left[[(b^2 + r^2 + r_1^2)^2 + 12r^2r_1^2]^{\frac{1}{2}} \left\{ \frac{B}{3^{\frac{1}{2}}} F\left(-\frac{1}{12}, -\frac{1}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \right. \\ & \left. \left. - \frac{A}{6 \cdot 3^{\frac{1}{2}}} \sqrt{\frac{J-1}{J}} F\left(\frac{5}{12}, \frac{5}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} \right. \\ & \left. - \frac{b^2 + r^2 + r_1^2}{[(b^2 + r^2 + r_1^2)^2 + 12r^2r_1^2]^{\frac{1}{2}}} \left\{ \frac{A}{3^{\frac{1}{2}}} F\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \right. \\ & \left. \left. + \frac{B}{6 \cdot 3^{\frac{1}{2}}} \sqrt{\frac{J-1}{J}} F\left(\frac{7}{12}, \frac{7}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} \right] \end{aligned}$$

where

$$A = 1.311 \ 028 \ 777 \dots$$

$$B = 0.599 \ 070 \ 117 \dots$$

$$\sqrt{\frac{J-1}{J}} = \left[\frac{1 - 36 \left(\frac{rr_1}{b^2 + r^2 + r_1^2} \right)^2}{1 + 12 \left(\frac{rr_1}{b^2 + r^2 + r_1^2} \right)^2} \right]^{\frac{1}{2}}$$

and the Gaussian notation for the hypergeometrical series has been adopted, viz—

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 \\ + \frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$$

No numerical results were given by which one may judge of the degree of convergence of the series in a practical case, nor was, apparently, any comparison of the formula made with other expressions for the mutual inductance of two circles.

In order to obtain light on these questions, the numerical values of the mutual inductances of several pairs of circles were calculated by the above formula and compared with the results obtained by the use of the formulæ of Maxwell¹ and Nagaoka.² (See also This Bulletin, vol. 5, pp. 6, 8, 1908.) It was found, that only in the case where $\frac{J-1}{J}=0$ does Mathy's formula give correct results, and in that case M comes out with the negative sign. In the more general case where $\frac{J-1}{J}$ is not equal to zero, the formula of Mathy gives values in error by as much as 5 to 10 per cent.

I have therefore checked the derivation of the formula, with the result that a corrected expression was found which, if used within those limits in which it is rapidly convergent, agrees closely with the formulæ of Maxwell and Nagaoka; in other words, gives very accurate results. The derivation is given below, the notation being that of Mathy. Since the original derivation is very brief, and rather difficult to follow, the work will be given here somewhat in detail.

Let r and r_1 be the radii of the two parallel circles (Fig. 1), b the distance between their planes, and ds and ds' , respectively, elements of their circumferences. To get the mutual inductance of the two circles we have then to find

$$M = \int \int \frac{ds ds' \cos \epsilon}{R}$$

¹ Elect. and Mag. Vol. II, § 701.

² Phil. Mag., 6, p. 19; 1903.

where ϵ is the angle between the radii vectores to ds and ds' , and R is the distance between them, the integration being extended around both circumferences.

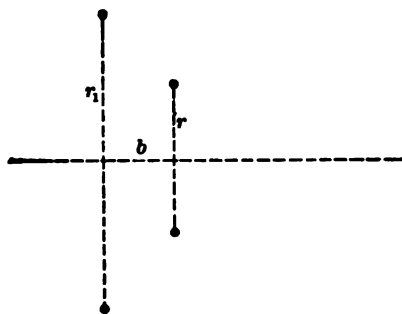


Fig. 1.

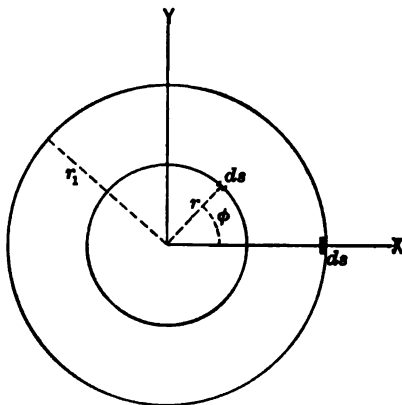


Fig. 2.

If we use polar coordinates (r, ϕ) and (r_1, ϕ_1) , and, taking $\phi_1 = 0$, integrate around the circle of radius r , we find

$$dM = r_1 d\phi_1 \int_0^{2\pi} \frac{r d\phi \cdot \cos \phi}{\sqrt{b^2 + r^2 + r_1^2 - 2rr_1 \cos \phi}}$$

Since the amount added to the mutual inductance by each element $ds' = r_1 d\phi_1$ is the same, we will have

$$M = 2\pi r r_1 \int_0^{2\pi} \frac{\cos \phi d\phi}{\sqrt{b^2 + r^2 + r_1^2 - 2rr_1 \cos \phi}}$$

Thus far, Maxwell's method has been used, but whereas he expressed the above elliptic integral in terms of the complete elliptic integrals F and E of the first and second kinds, Mathy next introduces the Weierstrassian function p in order to obtain a series development of the integral.

Taking rectangular coordinates as shown in Fig. 2

$$\cos \phi = \frac{x}{r}, \quad d\phi = -\frac{dx}{\sqrt{r^2 - x^2}}$$

and we find

$$M = 4\pi r_1 \int_{-r}^r \frac{x \, dx}{\sqrt{(r^2 - x^2)(b^2 + r^2 + r_1^2 - 2r_1 x)}}$$

The polynomial under the radical is

$$\begin{aligned} & 2r_1(x-r)(x+r)\left(x - \frac{b^2 + r^2 + r_1^2}{2r_1}\right) \\ &= 2r_1 \left[x^3 - x^2 \left(\frac{b^2 + r^2 + r_1^2}{2r_1} \right) - r^2 x + \frac{r^2(b^2 + r^2 + r_1^2)}{2r_1} \right] \end{aligned}$$

To reduce this to the canonical Weierstrassian form

$$4(y - e_1)(y - e_2)(y - e_3) = 4y^3 - g_2 y - g_3$$

where $e_1 + e_2 + e_3 = 0$, we must make the coefficient of y^3 equal to zero.

We put, therefore,

$$x = y + \frac{b^2 + r^2 + r_1^2}{6r_1}$$

and find

$$e_1 = - \left[\frac{b^2 + r^2 + r_1^2}{6r_1} - \frac{b^2 + r^2 + r_1^2}{2r_1} \right] = \frac{b^2 + r^2 + r_1^2}{3r_1}$$

$$e_2 = - \frac{b^2 + r^2 + r_1^2 - 6rr_1}{6r_1}$$

$$e_3 = - \frac{b^2 + r^2 + r_1^2 + 6rr_1}{6r_1} \quad \text{where } e_1 > e_2 > e_3$$

$$x = y + \frac{e_1}{2}$$

We may accordingly write

$$\begin{aligned} M &= 2\pi\sqrt{2r_1} \int_{e_1}^{e_3} \frac{\left(y + \frac{e_1}{2}\right) dy}{\sqrt{(y - e_1)(y - e_2)(y - e_3)}} \\ &= 4\pi\sqrt{2r_1} \left[\int_{e_1}^{e_3} \frac{y \, dy}{\sqrt{Y}} + \frac{e_1}{2} \int_{e_1}^{e_3} \frac{dy}{\sqrt{Y}} \right] \end{aligned}$$

where

$$Y = \sqrt{4y^2 - g_2y - g_3} \text{ and } y = pv$$

Now

$$v = \int_{e_1}^{e_2} \frac{dy}{\sqrt{Y}} = \int_{e_1}^{\infty} \frac{dy}{\sqrt{Y}} - \int_{e_1}^{\infty} \frac{dy}{\sqrt{Y}} = (2\omega + \omega') - (\omega + \omega') = \omega$$

ω and ω' being respectively the real and imaginary semi-periods of pv .

Consequently

$$\int_{e_1}^{e_2} \frac{y dy}{\sqrt{Y}} = -\zeta\omega = -\eta$$

and

$$M = 4\pi\sqrt{2r_1}\left(\frac{e_1}{2}\omega - \eta\right) \quad (1)$$

This differs from the expression found by Mathy only in the algebraic sign of M . A similar expression was found by Nagaoka, who developed it in terms of the rapidly convergent q series of Jacobi. Mathy, on the other hand, referring to Halphen's "Traité des fonctions elliptiques," part 1, p. 313 (Gauthier-Villars, Paris, 1886), expands ω and η in terms of hypergeometric series involving the absolute invariant J . The hypergeometrical differential equations for ω and η are deduced by Halphen (p. 313) and the final result for ω is given in equation (21), p. 343. The next two equations in Halphen in which the hypergeometrical series written in the $[q_1, q_2, q_3; x]$ notation are expressed in the equivalent $F(\alpha, \beta, \delta, x)$ notation are incorrect, and their use by Mathy is responsible for part of the error in his formula for M .

These equations should read

$$\left. \begin{aligned} \left[\frac{1}{2}, 0, \frac{1}{3}; \frac{J-1}{J} \right] &= F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \\ &= 1 + 5\frac{2}{12^2} \frac{J-1}{J} + \frac{2^2}{12^4} \frac{5 \cdot 13 \cdot 17}{3} \left(\frac{J-1}{J}\right)^2 + \dots \\ \left[-\frac{1}{2}, 0, \frac{1}{3}; \frac{J-1}{J} \right] &= F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \\ &= 1 + \frac{2}{12^2} 7 \cdot 11 \frac{J-1}{J} + \frac{2}{12^4} \frac{7 \cdot 11 \cdot 19 \cdot 23}{3 \cdot 5} \left(\frac{J-1}{J}\right)^2 + \dots \end{aligned} \right\} \quad (2)$$

where the $\frac{5}{12}$ replaces $\frac{1}{12}$, and the $\frac{11}{12}$ replaces $\frac{7}{12}$ in Halphen.

The relations connecting the absolute invariant J with the discriminant Δ and the invariants g_2 and g_3 of the Weierstrassian function $\wp$ are

$$\Delta = g_2^3 - 27g_3^2, \quad J = \frac{g_2^3}{\Delta}, \quad J - 1 = \frac{27g_3^2}{\Delta} \quad (3)$$

Halphen puts

$$x = \omega \Delta^{\frac{1}{12}}, \quad y' = \eta \Delta^{-\frac{1}{12}} \quad (4)$$

From page 307 he finds

$$\left. \begin{aligned} \frac{dx}{dJ} &= -\frac{1}{2\sqrt{3}}(J-1)^{-\frac{1}{2}} J^{-\frac{1}{2}} y' \\ \frac{dy'}{dJ} &= \frac{1}{24\sqrt{3}}(J-1)^{-\frac{1}{2}} J^{-\frac{1}{2}} x \end{aligned} \right\} \quad (5)$$

and, introducing the auxiliary quantity,

$$y = \frac{1}{2\sqrt{3}}(J-1)^{-\frac{1}{2}} J^{-\frac{1}{2}} y' \quad (6)$$

he derives immediately two differential equations for x and y as functions of J .

The first of these (equation 54) is

$$J(1-J)\frac{d^2x}{dJ^2} + \frac{1}{6}(4-7J)\frac{dx}{dJ} - \frac{x}{144} = 0 \quad (7)$$

The equation (55) for y is incorrectly given by Halphen. It should read

$$J(1-J)\frac{d^2y}{dJ^2} + \left(\frac{5}{3} - \frac{19}{6}J\right)\frac{dy}{dJ} - \frac{169}{144}y = 0 \quad (8)$$

the $\frac{5}{3}$ replacing $\frac{5}{6}$ and the $\frac{169}{144}$ replacing $\frac{53}{48}$.

Both these equations are in the Gaussian form

$$x(x-1)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta y = 0$$

and particular solutions may readily be written down from any treatise which gives the various forms assumed by the hyper-

geometrical series. Since g_3 and $\mathcal{A}$ are positive, we see from (3) that $(J-1)$ is positive, and therefore $\frac{J-1}{J} < 1$. The hypergeometrical series in $\frac{J-1}{J}$ will therefore converge.

Considering, first, equation (7) we find

$$\gamma = \frac{2}{3}, \quad \alpha = \frac{1}{12}, \quad \beta = \frac{1}{12}$$

Particular solutions are, therefore (see Weber, "Die Partiellen Differential-Gleichungen," II, p. 19, equations V)

$$x_1 = J^{-\frac{1}{2}} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$$

$$x_2 = J^{-\frac{1}{2}} \sqrt{\frac{J-1}{J}} F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$$

and the complete solution is

$$x = a J^{-\frac{1}{2}} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right) + ib J^{-\frac{1}{2}} \sqrt{\frac{J-1}{J}} F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$$

where a and b are arbitrary constants and $i = \sqrt{-1}$.

Accordingly, from (3) and (4)

$$g_3^{\frac{1}{2}} \omega = x J^{\frac{1}{2}} = a F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right) + ib \sqrt{\frac{J-1}{J}} F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \quad (9)$$

The constants are to be evaluated from the following considerations (Halphen, p. 342-343):

$$\lim_{J=1} (g_3^{\frac{1}{2}} \omega) = A \sqrt{2} = a$$

$$\lim_{J=1} \left(\sqrt{J-1} \cdot \frac{dx}{dJ} \right) = \frac{i}{2} b = -\frac{1}{2\sqrt{3}} \lim_{J=1} (\eta \mathcal{A}^{-\frac{1}{2}})$$

$$\lim_{J=1} (\eta \omega) = \frac{\pi}{4}$$

$$\therefore ib = -\frac{\pi}{4a\sqrt{3}}$$

The expression (9) becomes, therefore,

$$g_1^{\frac{1}{2}}\omega = A\sqrt{2} \cdot F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right) - \frac{B}{\sqrt{6}}\sqrt{\frac{J-1}{J}} \cdot F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \quad (10)$$

where A and B are Sterling's constants

$$A = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 1.311\ 028\ 777\ \dots$$

$$B = \frac{\pi}{4A} = 0.599\ 070\ 117\ \dots$$

Correcting Halphen's result for ω by means of equations (2) it agrees with (10).

The derivation of the value of η is similar.

In equation (8) $\alpha = \frac{13}{12}$, $\beta = \frac{13}{12}$, $\gamma = \frac{5}{3}$. Accordingly

$$y = cJ^{-\frac{1}{2}} \cdot F\left(\frac{13}{12}, \frac{5}{12}, \frac{3}{2}, \frac{J-1}{J}\right) + dJ^{-\frac{1}{2}}(1-J)^{-\frac{1}{2}} F\left(\frac{7}{12}, -\frac{1}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$$

and from (6)

$$\left. \begin{aligned} y' &= c2\sqrt{3}J^{\frac{1}{2}}\sqrt{\frac{J-1}{J}}F\left(\frac{13}{12}, \frac{5}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \\ &\quad - 2\sqrt{3}idJ^{\frac{1}{2}}F\left(\frac{7}{12}, -\frac{1}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \end{aligned} \right\} \quad (11)$$

or

$$\left. \begin{aligned} \eta g_1^{-\frac{1}{2}} &= y'J^{-\frac{1}{2}} = 2\sqrt{3}c\sqrt{\frac{J-1}{J}}F\left(\frac{13}{12}, \frac{5}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \\ &\quad - 2\sqrt{3}idF\left(\frac{7}{12}, -\frac{1}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \end{aligned} \right\} \quad (12)$$

To determine the constants we have

$$\lim_{J=1} (\omega\eta) = \lim_{J=1} (g_1^{\frac{1}{2}}\omega \cdot g_1^{-\frac{1}{2}}\eta) = \frac{\pi}{4}$$

$$\lim_{J=1} \left(\frac{dy'}{dJ} \sqrt{J-1} \right) = c\sqrt{3} = \frac{1}{24\sqrt{3}} \lim_{J=1} (x)$$

from which $c = \frac{a}{36\sqrt{2}}$, $-id = \frac{B}{2\sqrt{6}}$

Therefore, finally

$$\eta g_2^{-1} = \frac{B}{\sqrt{2}} F\left(\frac{7}{12}, -\frac{1}{12}, \frac{1}{2}, \frac{J-1}{J}\right) + \frac{A}{6\sqrt{6}} \sqrt{\frac{J-1}{J}} F\left(\frac{13}{12}, \frac{5}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \quad (13)$$

The value of η was found by Mathy from the equations (4) and (5) without the employment of the auxiliary quantity γ .

The differential equation for γ' is, as was also found by Mathy (in Mathy's notation γ' is γ)

$$J(1-J) \frac{d^2 \gamma'}{dJ^2} + \left(\frac{1}{3} - \frac{5}{6}J\right) \frac{d\gamma'}{dJ} - \frac{\gamma'}{144} = 0 \quad (14)$$

Here

$$\alpha = -\frac{1}{12}, \quad \beta = -\frac{1}{12}, \quad \gamma = \frac{1}{3}$$

$$\gamma_1' = J^{\frac{1}{2}} F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$$

$$\gamma_2' = J^{\frac{1}{2}} \sqrt{\frac{J-1}{J}} F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$$

Evaluating the constants as before we find from (4) and remembering that

$$F\left(\alpha, \beta, \gamma, \frac{J-1}{J}\right) = F\left(\beta, \alpha, \gamma, \frac{J-1}{J}\right)$$

exactly the same value for $\eta g_2^{\frac{1}{2}}$ as in (13). In Mathy's expression

$-\frac{1}{12}$ appears in place of $\frac{7}{12}$, and $\frac{5}{12}$ in place of $\frac{13}{12}$.

Substituting from (10) and (13) in (1) the corrected expression for the mutual inductance becomes

$$\begin{aligned} M = 4\pi\sqrt{2}r_1 \left[\frac{e_1}{2} \left\{ g_2^{-1} A \sqrt{2} \cdot F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \right. \\ \left. \left. - \frac{g_2^{-1} B}{\sqrt{6}} \sqrt{\frac{J-1}{J}} \cdot F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} - \left\{ \frac{g_2^{\frac{1}{2}} B}{\sqrt{2}} F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \right. \\ \left. \left. + \frac{g_2^{\frac{1}{2}} A}{6\sqrt{6}} \sqrt{\frac{J-1}{J}} \cdot F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} \right] \end{aligned}$$

Putting $x^2 = b^2 + r^2 + r_1^2$ we find

$$\begin{aligned} e_1 &= \frac{x^2}{3r_1}, \quad e_2 = -\frac{x^2 - 6rr_1}{6r_1}, \quad e_3 = -\frac{x^2 + 6rr_1}{6r_1} \\ g_2^2 &= 2(e_1^2 + e_2^2 + e_3^2) = x^4 + 12r^2r_1^2 \\ g_3 &= 4e_1e_2e_3 = \frac{x^2(x^4 - 36r^2r_1^2)}{27r_1^3} \\ \frac{J-1}{J} &= \frac{27g_3^2}{g_2^3} = \frac{\left[1 - 36\left(\frac{rr_1}{x^2}\right)^2\right]^3}{\left[1 + 12\left(\frac{rr_1}{x^2}\right)^2\right]^3} \end{aligned} \quad (15)$$

The final expression for M becomes, therefore,

$$\begin{aligned} \frac{M}{4\pi} &= \left\{ \begin{aligned} &\frac{x^2}{(x^4 + 12r^2r_1^2)^{1/4}} \left\{ \frac{A}{3^{1/4}} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \\ &\quad \left. - \frac{B}{6 \cdot 3^{1/4}} \sqrt{\frac{J-1}{J}} \cdot F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} \\ &- (x^4 + 12r^2r_1^2)^{1/4} \left\{ \frac{B}{3^{1/4}} F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \\ &\quad \left. + \frac{A}{6 \cdot 3^{1/4}} \sqrt{\frac{J-1}{J}} \cdot F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} \end{aligned} \right\} \quad (16) \end{aligned}$$

where

$$\begin{aligned} x^2 &= b^2 + r^2 + r_1^2 \\ A &= 1.311 \ 028 \ 777 \quad . \quad . \quad . \\ B &= 0.599 \ 070 \ 117 \quad . \quad . \quad . \\ \sqrt{\frac{J-1}{J}} &= \frac{1 - 36\left(\frac{rr_1}{x^2}\right)^2}{\left[1 + 12\left(\frac{rr_1}{x^2}\right)^2\right]^{1/4}} \end{aligned} \quad (17)$$

and

$$\begin{aligned} F(\alpha, \beta, \gamma, z) &= 1 + \frac{\alpha\beta}{1 \cdot \gamma} z + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} z^2 \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} z^3 \end{aligned}$$

The formula for M is by no means so formidable to use as might be expected from its appearance, since the constants which enter

and the coefficients in the hypergeometrical series may be calculated once for all. Using seven-place logarithms, the values come out

$$\log \frac{A}{3^{\frac{1}{2}}} = 9.759\ 7712 \qquad \log \frac{A}{6 \cdot 3^{\frac{1}{2}}} = 8.981\ 6199$$

$$\log \frac{B}{3^{\frac{1}{2}}} = 9.658\ 1974 \qquad \log \frac{B}{6 \cdot 3^{\frac{1}{2}}} = 8.880\ 0461$$

and the coefficients a_1, a_2, a_3 of z, z^2 , and z^3 in the series are given in the following table:

Series	a_1	a_2	a_3
$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	0.069 4444	0.035 5260	0.023 8485
$F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	-0.097 2222	-0.047 0358	-0.031 0523
$F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.300 9259	0.177 6300	0.126 0562
$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.356 4814	0.216 3645	0.155 2615

From the magnitudes of these coefficients it is evident that the practical use of the formula (16) will be restricted to those cases for which $z = \frac{J-1}{J}$ is not much greater than about 0.2, and for the most precise work it should be still smaller. It is, therefore, more special than the formulæ of Maxwell and Nagaoka.

For the case $z=0$, each of the four series becomes equal to unity, and we have

$$\frac{M}{4\pi} = \frac{A}{3^{\frac{1}{2}}} \frac{x^2}{(x^4 + 12r^2r_1^2)^{\frac{1}{2}}} - \frac{B}{3^{\frac{1}{2}}} (x^4 + 12r^2r_1^2)^{\frac{1}{2}}$$

which, remembering that in this case $x^4 = 36r^2r_1^2$, reduces to the remarkably simple form

$$M = 4\pi\sqrt{rr_1}(A - 2B) = 4\pi(0.112\ 888\ 543)\sqrt{rr_1} = 1.418\ 599\sqrt{rr_1} \quad (18)$$

If we introduce the distances

$$R_1 = \sqrt{(r+r_1)^2 + b^2}$$

$$R_2 = \sqrt{(r-r_1)^2 + b^2}.$$

into equation (15) we find that the condition $z=0$ is equivalent to

$$R_1^2 = 2R_2^2 \quad (19)$$

That is, in all cases where the greatest distance between the circumferences of the two circles is $\sqrt{2}$ times the shortest distance between them, the mutual inductance is given by (18).

The following pairs of circles satisfying equation (19) are tabulated to aid in interpreting this equation:

$\frac{r_1}{r}$	$\frac{b}{r}$	
$\begin{cases} 0.1716 \\ = 3 - 2\sqrt{2} \end{cases}$	0	Circles in the same plane.
0.2	0.4000	
0.3	0.8426	
0.4	1.1135	
0.5	1.3229	
0.6	1.4967	
0.7	1.6462	
0.8	1.7776	
0.9	1.8947	
1.0	2.0000	Equal circles.

Numerical results

Example 1. Formula (18): $r_1=25$, $r=25$, $b=50$

Therefore $z = \frac{J-1}{J} = 0$ (see table) $\sqrt{rr_1} = 25$

$$\begin{aligned} \frac{M}{4\pi} &= 25 \times 0.112 \ 888 \ 543 \dots \\ &= 2.822 \ 2136 \dots \end{aligned}$$

By Nagaoka's formula $\frac{M}{4\pi} = 2.822 \ 213$

By Maxwell's formula $\frac{M}{4\pi} = 2.822 \ 200$

The values by the latter two formulæ are those found using seven place logarithms. The value from (18) may be carried out without difficulty to as many places of decimals as desired.

Example 2. Formula (16).

$$r_1 = 25 \quad r = 20 \quad b = 40$$

$$x^2 = 2625 \quad \frac{rr_1}{x^2} = \frac{4}{21} \quad x^4 + 12r^2r_1^2 = 9\ 890\ 625$$

$$\frac{1}{4} \log (x^4 + 12r^2r_1^2) = 1.748\ 8059$$

$$\log \frac{x^2}{(x^4 + 12r^2r_1^2)^{\frac{1}{4}}} = 1.670\ 3234$$

$$1 - 36 \left(\frac{rr_1}{x^2} \right)^2 = -0.306\ 1225$$

$$1 + 12 \left(\frac{rr_1}{x^2} \right)^2 = 1.435\ 3742$$

$$\log \sqrt{\frac{J-1}{J}} = 9.250\ 4476n$$

$$\log z = 8.500\ 8952$$

	$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, s\right)$	$F\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, s\right)$	$F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, s\right)$	$F\left(\frac{5}{12}, \frac{13}{12}, \frac{1}{2}, s\right)$
	1.000 0000	1.000 0000	1.000 0000	1.000 0000
	0.002 2006	0.011 2962	-0.003 0808	0.009 5357
	0.000 0357	0.000 2173	-0.000 0473	0.000 1784
	0.000 0008	0.000 0049	-0.000 0010	0.000 0040
	1.002 2371	1.011 5184	0.996 8709	1.009 7181
$\log F$	= 0.000 9704	0.004 9738	9.998 6389	0.004 2002
$\log \text{const.}$	= 9.759 7712	8.880 0461	9.658 1974	8.981 6199
$\log \sqrt{\frac{J-1}{J}}$	=	9.250 4476n		9.250 4476n
$\log (x^4 + 12r^2r_1^2)^{\frac{1}{4}}$	=		1.748 8059	1.748 8059
$\log \frac{x^2}{(x^4 + 12r^2r_1^2)^{\frac{1}{4}}}$	= 1.670 3234	1.670 3234		
	1.431 0650	9.805 7909n	1.405 6422	9.985 0736n
	26.981 438	-0.639 427	25.447 327	-0.966 215
	-0.639 427		-0.966 215	
	27.620 865		24.481 112	

$$\therefore \frac{M}{4\pi} = 27.620\ 865 - 24.481\ 112 = 3.139\ 753\ cm$$

$$\text{By Nagaoka's formula } \frac{M}{4\pi} = 3.139\ 749$$

$$\text{By Maxwell's formula } \frac{M}{4\pi} = 3.139\ 766$$

For the more unfavorable case,

$$r_1 = 10, r = 5, b = 15, \therefore \log z = 8.562\ 0935$$

the values found were

$$\text{By formula (16)} \quad \frac{M}{4\pi} = 0.62398\ cm$$

$$\text{By Nagaoka's formula} \quad = 0.62399$$

$$\text{By Maxwell's formula} \quad = 0.62400$$

These examples will suffice to show that in those cases where the convergence is rapid the formula (16) gives the mutual inductance with precision. Before applying it in a given case, a preliminary rough calculation of z should be made to see if the convergence will be satisfactory. Because of the rather special applicability of the formula (16) it can not be regarded as in any way superseding the elliptic integral formulæ of Maxwell or the q series formulæ of Nagaoka.

The formula (18) gives, however, an exceedingly rapid and simple means for checking new formulæ, and should find extensive application in cases where the choice of the dimensions of the circles and their distance apart may be made to conform to equation (19).

WASHINGTON, February 1, 1910.

A NEW METHOD FOR THE ABSOLUTE MEASUREMENT OF ELECTRIC QUANTITY

By Burton McCollum

In the absolute measurement of current two classes of methods are available which are capable of considerable accuracy, viz, electro-dynamometer methods, in which the torque exerted between a fixed and a movable coil is measured and the current in the coils calculated from this measured torque and the known dimensions of the coil; and current balances, in which the force exerted between two coils carrying the current to be measured is balanced against a known weight and the current calculated from the weight and previously determined constants of the coils. By the latter form of apparatus measurements of current have been made in which a higher degree of accuracy has been attained than with the first-named type. There are, however, numerous possible sources of error in the use of the current balance, and, although most of these can be made small individually, they may on account of their number, introduce uncertainty in the result.

The ordinary electro-dynamometer methods as well as the current balance require in effect two separate experiments, the first for the determination of the value of the current in terms of a torque or force, and the second for the recording of this current in terms of the electro-chemical equivalent of silver or of a resistance and standard cell. This second experiment may introduce additional possibilities of error, owing to variations in the strength of the current or the voltage of the standard cell, errors in the resistances of the potentiometer circuits and thermo-electromotive forces in the galvanometer and potentiometer circuits, etc., and while these errors can be made small, still their possible presence increases the difficulties of the measurements.

Although the current balance is capable of yielding a high degree of accuracy, it is not desirable to depend on the value given by a single type of apparatus, even though the experiments may be repeated with good agreement by different experimenters and with different instruments. It is highly desirable that if possible the values be checked by some other method, using a totally different type of apparatus.

In electrodynometers of the ordinary types the dimensions of the windings must be known with extreme accuracy, thus requiring that the coils not only be wound with great regularity and measured with great care, but permanency of both size and form must be assured. In the Gray electrodyrometer a greater difficulty still is met with in measuring the torque. This is accomplished by balancing the torque due to the current against the torsion of a wire, the constant of which has been determined by a separate experiment. The great difficulty in this lies in the erratic behavior of all materials available for the torsion wire. They can not be depended upon to remain constant with time, they vary considerably even with slight changes in temperature, and they all have a greater or less tendency to take a permanent set when twisted, as they must be, in use. Again, both electrodynometers and current balances have a rather troublesome temperature coefficient, and, although artificial cooling may alleviate this condition somewhat, a certain amount of local heating is inevitable, and the consequent change in the dimensions may give rise to considerable variation in the constants of the instruments between the time of measuring the dimensions of the coils and the time of making a measurement of current.

In what follows, there is described a type of electrodyrometer which, when used to measure the electrochemical equivalent of silver, appears to be free from the objections referred to above, and at the same time it is believed that no other difficulties of a serious nature are introduced.

The instrument consists essentially of a relatively large fixed coil, C_1 Fig. 1, with its axis in a horizontal position, at the center of which is suspended the movable coil C_2 , with its axis parallel to the axis of the fixed coil when at rest. Attached to the movable coil is a cylinder K , of some homogeneous material placed with its axis vertical and coincident with the axis of suspension. This

cylinder should be of very regular dimensions, and should have a moment of inertia as large as, or larger than, that of the movable coil. If these two coils be connected in series in proper relation, and a current sent through them, there will exist a couple tending to hold the movable coil with its axis parallel to that of the fixed coil, and if the movable coil be given an angular displacement and released, it will oscillate as a torsion pendulum. If now, the form of the field due to the fixed coils be such that the restoring couple is directly proportional to the angular displacement, a condition

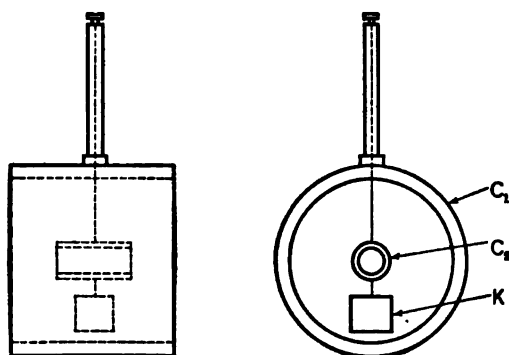


Fig. 1

that can readily be realized as explained later, the period of the oscillation will be independent of the amplitude, and we will have, in effect, a torsion pendulum subject to the laws of a damped oscillation.

The differential equation applying to this system is the well-known form

$$\frac{d^2\theta}{dt^2} + \frac{a}{K} \frac{d\theta}{dt} + \frac{R}{K} \theta = 0$$

where R is the restoring couple at unit angular displacement, a is the counter torque due to the damping forces at unit angular velocity and K is the moment of inertia of the moving system. From the well-known integral of this equation, it is readily derived that the time of vibration of the moving system is given by the equation

$$T = \frac{2\pi}{\sqrt{\frac{R}{K} - \frac{a^2}{4K^2}}} \quad (1)$$

Since, as assumed above, the coils are so proportioned that the torque due to the current i is proportional to the deflection θ , we will have

$$R = Ci^2 + b$$

where C is a constant of the coils, b the constant of torsion of the suspending wire, and R the total restoring couple. Putting this value of R in equation (1), we have,

$$T_1 = \frac{2\pi}{\sqrt{\frac{Ci^2 + b}{K} - \frac{a^2}{4K^2}}} \quad (2)$$

If, now, the current be switched off and the time of vibration again taken, we will have, since i in equation (2) becomes zero,

$$T_2 = \frac{2\pi}{\sqrt{\frac{b}{K} - \frac{a^2}{4K^2}}} \quad (3)$$

Eliminating $\left(\frac{b}{K} - \frac{a^2}{4K^2}\right)$ between (2) and (3), and solving for i , we get

$$i = \frac{2\pi}{T_1} \sqrt{\frac{K}{C} \left(1 - \frac{T_1^2}{T_2^2}\right)} \quad (4)$$

Equation (4) is an exact one, and it shows that damping introduces no error in the calculated value of the current.

If the apparatus be properly designed, the ratio $\frac{T_1^2}{T_2^2}$ can be made small compared to unity (not exceeding 1 per cent), so that a very large error in determining this ratio would introduce but a very small error in the value of i ; and, further, since the ratio can be determined with a high degree of accuracy, the factor in parentheses need not be regarded as an appreciable source of error. The ultimate accuracy with which i can be measured depends, therefore, on the accuracy with which the three quantities T_1 , K , and C can be measured. These quantities can be determined with a high degree of precision by methods to be indicated later; and, further, since K and C appear under

the radical, any error appearing in these values is reduced one half in the value of i . This apparatus and equation (4) may be used for the absolute measurement of current, if such is desired. However, since it is impossible to record the current directly, its value is usually determined only in order that it may be reproduced and maintained constant for a sufficient length of time to enable the electro-chemical equivalent of silver to be determined. This additional experiment is objectionable mainly because it introduces several possible sources of error, as pointed out above; but it is necessary with all forms of apparatus that have heretofore been used for the absolute determination of the electro-chemical equivalent of silver. It is not necessary, however, with the type of electro-dynamometer here considered.

The time of vibration T_1 , which appears in the denominator of equation (4) may be written $\frac{t}{n}$, where t is the total elapsed time and n is the number of vibrations occurring during that time. Equation (4) then becomes

$$i = \frac{2\pi n}{t} \sqrt{\frac{K}{C} \left(1 - \frac{T_1^2}{T_2^2} \right)}$$

$$\therefore it = Q = 2\pi n \sqrt{\frac{K}{C} \left(1 - \frac{T_1^2}{T_2^2} \right)} \quad (5)$$

where Q is the total quantity of electricity that has passed through the instrument while the movable system is making n vibrations. If, therefore, the instrument be connected in series with a silver voltameter and a current sent through them for any suitable period of time, during which the number of vibrations n is counted, the total quantity of electricity that has passed through the voltameter can be calculated from equation (5). Thus the instrument may be used for the direct measurement of electric quantity. By using equation (5) as the working formula, a very high degree of accuracy should be possible. An inspection of the equation will show that it is not necessary that the current be maintained rigorously constant. The only factor in the equation for Q that varies with i is the factor T_1 under the radical, which varies

nearly inversely as the current i . Differentiating equation (5) with respect to the variable T_1 , and dividing by Q , we get

$$\frac{dQ}{Q} = -\frac{T_1^2}{T_2^2 - T_1^2} \cdot \frac{dT_1}{T_1} \quad (6)$$

As pointed out above, T_1^2 need not exceed one one-hundredth of T_2^2 , and, therefore, it is seen from equation (6) that any given percentage change in T_1 (and therefore in i) would produce in Q a percentage change only about one one-hundredth part as great. Hence, the current might fluctuate continuously within a range of one-fifth per cent, or it might remain continuously too small or too large by as much as one part in a thousand, and the resulting error in Q would not exceed one part in one hundred thousand. It is evident, therefore, that the quantity in parentheses will not be an appreciable source of error, and no great care is necessary in maintaining a definite value of current. The ultimate accuracy with which Q can be determined depends, therefore, on the accuracy with which n , K , and C can be determined. We shall now describe suitable methods for determining these factors.

Determination of n

A typical arrangement of the apparatus is shown in Fig. 2. B is a battery supplying current through the regulating resistance r_1 , to the coils C_1 and C_2 of the electro-dynamometer connected in series. S_1 is a single pole double throw switch which at starting is thrown over to the terminal d , so that the current is made to flow through the control coil C_3 which holds the movable coil C_2 at any suitable initial deflection (which need not be measured). The resistance of the circuit df should be previously adjusted nearly equal to that of the circuit ef , which contains the voltmeter VA , and the terminals d and e adjusted so that the switch S_1 , when thrown over, makes contact with e an instant before breaking the current at d . When ready to start the experiment, the switch S_1 is quickly thrown from d to e which starts the current in the voltmeter circuit ef , and at the same time breaks the circuit of the control coil C_3 , thus releasing the movable coil C_2 and the vibrations begin. The instrument should then be permitted to swing freely for a sufficient length of time to permit the deposition of enough silver for accurate weighing, during which time the number

of complete vibrations of the movable coil is counted. When it is desired to stop the experiment, a reading should be taken of the last maximum deflection θ_0 at the end of n_0 complete vibrations, and about a quarter period later the current should be switched off, and at the same instant the position of the movable coil in its cycle, and also its direction of motion should be noted. From these data, the value of n , the total number of vibrations, can be readily determined as indicated later. In order to determine the position of the coil at the instant of opening the circuit, a telescope and scale g , and reflecting mirror, should be provided as in ordinary galvanometer work. Since the scale is in motion, its position

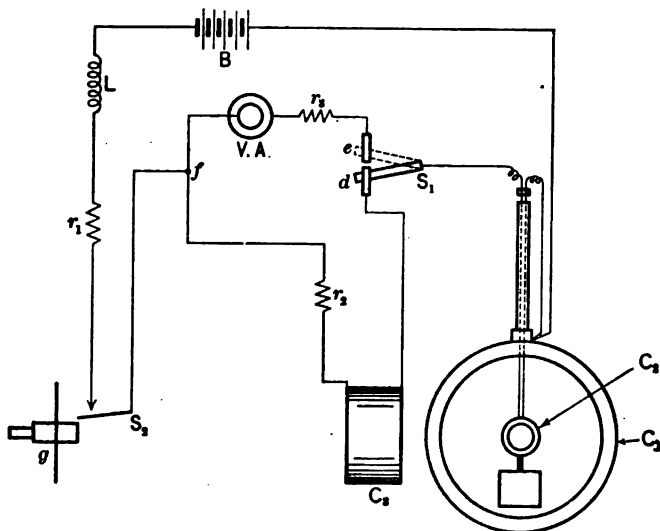


Fig. 2

must be determined by obtaining a momentary view of it at the instant of opening the circuit. Numerous ways suggest themselves for accomplishing this, one of the simplest, perhaps, being as follows: A portion of the scale, preferably the central portion, should be somewhat heavily shaded so that its reflection in the mirror will be very faint, if not invisible. A specially designed switch, S_2 , which is used to open the circuit, should be placed near the shaded portion, so that the flash caused by opening the switch will illuminate the scale for a brief instant. If, then, the switch be opened when the shaded portion of the scale is being reflected into

the telescope, the scale will appear stationary, and its deflection θ , can be easily read. The brightness of the flash can be made ample by including a self inductance L , in series with the circuit, if necessary.

Assuming, as above, that n_0 is the total number of complete vibrations, T_1 the time of a complete vibration, and t the time that elapses between the completion of n_0 complete periods and the opening of the circuit, we will have, obviously,

$$n = n_0 + \frac{t}{T_1} \quad (7)$$

where n is the factor to be used in equation (6). The ratio $\frac{t}{T_1}$ can be most accurately determined from the last maximum deflection θ_0 , and the final deflection θ , taken as described above. From the well-known laws of the torsion pendulum, we have,

$$\theta = \theta_0 \cos \sqrt{\frac{R}{K} - \frac{a^2}{4K^2}} \cdot t$$

$$\therefore \sqrt{\frac{R}{K} - \frac{a^2}{4K^2}} \cdot t = \cos^{-1} \frac{\theta}{\theta_0}$$

But,

$$\sqrt{\frac{R}{K} - \frac{a^2}{4K^2}} = \frac{2\pi}{T_1}$$

$$\therefore \frac{t}{T_1} = \frac{1}{2\pi} \cos^{-1} \frac{\theta}{\theta_0}$$

$$\therefore n = n_0 + \frac{1}{2\pi} \cos^{-1} \frac{\theta}{\theta_0} \quad (8)$$

It is important to determine the probable accuracy with which n can be determined, and the conditions under which the greatest precision may be attained. Differentiating equation (8), we have,

$$dn = - \frac{1}{2\pi \sqrt{1 - \left(\frac{\theta}{\theta_0}\right)^2}} \cdot d\left(\frac{\theta}{\theta_0}\right)$$

$$\therefore \frac{dn}{n} = - \frac{1}{2\pi \sqrt{1 - \left(\frac{\theta}{\theta_0}\right)^2} \left(n_0 + \frac{1}{2\pi} \cos^{-1} \frac{\theta}{\theta_0}\right)} \cdot d\left(\frac{\theta}{\theta_0}\right) \quad (9)$$

From this it appears that the greatest accuracy is attainable if the final reading is taken when θ is small, that is, when the coil is passing near the middle portion of its swing. Assuming that the experiment has continued for at least half an hour, and that the period of vibration is, say, fifteen seconds, thus giving a value of n_0 at least one hundred twenty, and assuming further that the final reading is taken when θ is between $+\frac{1}{10}\theta_0$ which would not be difficult, we may write equation (9) with close approximation:

$$\frac{dn}{n} = -\frac{1}{2\pi n_0} \cdot d\left(\frac{\theta}{\theta_0}\right)$$

From which we have the two relations:

$$(a), \frac{dn}{n} = -\frac{d\theta}{2\pi n_0 \theta_0} = \frac{dx}{4\pi n_0 \lambda \theta_0} \quad (10)$$

where x is the length of the arc θ and λ the distance from scale to mirror; and,

$$(b), \frac{dn}{n} = \frac{1}{2\pi n_0} \cdot \frac{\theta}{\theta_0} \cdot \frac{d\theta_0}{\theta_0} \quad (11)$$

For the purpose of making an approximate numerical calculation of the probable error in measuring n , we may assume that the scale is two meters from the mirror and that θ_0 equals $\frac{1}{25}$ radian. Putting these values in equation (10), we have

$$\frac{dn}{n} = \frac{\pm 25}{4\pi \cdot 120 \cdot 200} \cdot dx = \frac{\pm 1}{12057} \cdot dx$$

It follows, therefore, that an error as great as one millimeter in the final reading of the scale would introduce an error of less than one part in 120,000 in the value of n .

Substituting the assumed values in equation (11), we get

$$\frac{dn}{n} = \pm \frac{1}{7536} \cdot \frac{d\theta_0}{\theta_0}$$

Hence, any small error in reading θ_0 is reduced over 7500 times in the value of n . With reasonable care, therefore, the experimental error in determining n would be altogether negligible, and the

precision of the measurement of Q would depend only on the determination of K and C .

Determination of K

In order that K may be determined accurately, the movable coil should have attached to it a cylinder of some homogeneous material suspended with its axis vertical and coincident with the line of suspension. It should be made of some material that is known to be of uniform density and should preferably be of high electrical resistance so as to avoid excessive damping. Brass and other metal castings are objectionable because of the possibility of blow holes and other defects which can not be detected. Forged metal would be better, but there would still be uncertainty in regard to the uniformity of the material, and, further, the damping might become troublesome, for, although damping is not directly a source of error, it might, if excessive, bring the system to rest before enough time has elapsed to permit the deposition of a sufficient quantity of silver. Glass would seem to be a suitable material for this purpose. A carefully made cylinder of glass would be expensive, but it would have the great advantage that any inequalities could readily be detected by simple optical tests. It would also be free from damping effects due to induced currents. The moment of inertia of such a cylinder carefully ground to size could be determined with great precision by calculation from its mass and dimensions. The most obvious method of determining the total moment of inertia of the system would be to take readings of the time of vibration, first without the glass cylinder attached and again with this cylinder in place, and since the moments of inertia are proportional to the squares of the times of vibration, we would have at once, if K_1 and K_2 are the moments of inertia of the coil and cylinder respectively,

$$\frac{K_1}{K_2 + K_1} = \left(\frac{T_1}{T_2} \right)^2 \quad (12)$$

There are two objections to this method, however. The chief objection is that the torsion wire is subjected to considerably greater tension in the second part of the experiment than in the first, and this would undoubtedly affect appreciably the torsion of the wire. This disturbance could be greatly reduced by using a

multifilar suspension, but even then it would probably be troublesome. Further, it is practically impossible to find a torsion wire that can be depended on with certainty to remain constant even under constant stress, so that this method would give rise to considerable uncertainty, even though precautions were taken to keep the tension of the wire constant. These errors can be almost entirely eliminated, however, by making a constant current flow through the coils while taking the period. There would be no difficulty in making the electro-magnetic couple at least 99 per cent of the total couple, and, therefore, any change in the constant of torsion of the wire would be reduced one hundred times in its effect on the value of the ratio of the moments of inertia as calculated from equation (12). Equation (12) then holds, and the total moment of inertia $K = K_1 + K_2$, can be readily calculated if K_2 has been determined. It should be noted that the value of K thus determined is independent of any absolute errors in the instrument used for measuring time, provided that the instrument remains consistent with itself. Further, the moment of inertia of the air within and near the moving coil is taken into account in this way.

Determination of C

The only other factor that remains to be determined is C , the constant of the coils. In order to determine C , we shall make use of the formula given by Gray¹ for the mutual inductance between two coils, viz:

$$M = \pi^2 n_1 n_2 a_1^2 a_2^2 (K_1 k_1 Z_1 + K_2 k_2 Z_2 + K_3 k_3 Z_3 + \dots) \quad (13)$$

As pointed out by Gray, if the coils are concentric, the even terms all vanish and (13) becomes

$$M = \pi^2 n_1 n_2 a_1^2 a_2^2 (K_1 k_1 Z_1 + K_3 k_3 Z_3 + K_5 k_5 Z_5 + \dots) \quad (14)$$

¹ Gray, *Absolute Measurements in Electricity and Magnetism*, Vol. 2, part 1, pp 274, 275.

$$\left. \begin{aligned}
 \text{where } K_1 &= \frac{4\rho_1}{a_1 D} \\
 k_1 &= 2\rho_2 a_2 \\
 K_3 &= -\frac{\rho_1 a_1}{D^3} \\
 k_3 &= \rho_2 a_2^3 (4\rho_2^2 - 3) \\
 K_5 &= -\frac{\rho_1 a_1^3}{4D^5} (4\rho_1^2 - 3) \\
 k_5 &= 2\rho_2 a_2^5 \left(2\rho_2^2 - 5\rho_2^2 + \frac{5}{4} \right) \\
 K_7 &= -\frac{\rho_1 a_1^5}{4D^7} \left(4\rho_1^4 - 10\rho_1^2 + \frac{5}{2} \right) \\
 k_7 &= \rho_2 a_2^7 \left(4\rho_2^4 - 21\rho_2^2 + \frac{35}{2}\rho_2^2 - \frac{35}{16} \right)
 \end{aligned} \right\} \begin{array}{l} (a) \\ (b) \end{array} \quad (15)$$

where ρ_1 and ρ_2 are the ratios of length to diameter, n_1 and n_2 the number of turns per centimeter, and a_1 and a_2 the radii of the fixed and movable coils respectively, and D the half diagonal of the fixed coil. Z_1, Z_3, Z_5 , etc., are the zonal surface harmonics corresponding to the respective terms, with the angle between the axes of the coils as argument.

Differentiating equation (14) with respect to θ , we get, as the equation for torque,

$$\begin{aligned}
 T &= i^2 \frac{dM}{d\theta} \\
 &= \pi^2 n_1 n_2 a_1^2 a_2^2 i^2 \left(K_1 k_1 \frac{dZ_1}{d\theta} + K_3 k_3 \frac{dZ_3}{d\theta} + K_5 k_5 \frac{dZ_5}{d\theta} + \dots \right) \quad (16)
 \end{aligned}$$

Equating the expressions for K_3 and k_7 from equations (15) to zero and solving for ρ_1 and ρ_2 , we get,

$$\left. \begin{aligned}
 \rho_1 &= \frac{\sqrt{3}}{2} \\
 \rho_2 &= \begin{cases} 2.062 \\ .92 \\ .38 \end{cases}
 \end{aligned} \right\} \quad (17)$$

If then the coils are so wound that the ratio of length to diameter of the fixed coil is $\frac{\sqrt{3}}{2}$ and that of the movable coil either of the values of ρ_2 given in equation (17), the third and fourth terms in equation (16) disappear, and if the movable coil is small compared

to the fixed coil, all the terms beyond the fourth are altogether negligible. Equation (16) therefore becomes,

$$T = \pi^2 n_1 n_2 a^2 a'^2 i^2 \left(K_1 k_1 \frac{dZ_1}{d\theta} + K_2 k_2 \frac{dZ_2}{d\theta} \right) \quad (18)$$

$$= -\frac{2\pi^2 N_1 N_2 a^2 a'^2}{D} \left(\frac{dZ_1}{d\theta} + R_s \frac{dZ_2}{d\theta} \right) \quad (19)$$

where N_1 and N_2 are the total number of turns on the two coils, and R_s is, from equations (15), (a) and (b), equal to—

$$R_s = \frac{K_2 k_2}{K_1 k_1} = -\frac{a'^2 (4\rho^2 - 3)}{8a^2 (1 + \rho^2)^2} \quad (20)$$

In the beginning it was assumed that $T = -C\theta$, and if this condition is to be satisfied, we must have from equation (19),

$$-C\theta = \frac{2\pi^2 N_1 N_2 a^2 a'^2}{D} \left(\frac{dZ_1}{d\theta} + R_s \frac{dZ_2}{d\theta} \right) \quad (21)$$

It is not necessary that this equation shall hold for all values of θ , but only throughout the range of deflection used in actual experiment. An angle of oscillation of 6 or 8 degrees is entirely ample for present purposes, so that if equation (21) holds for all values of θ up to ± 4 degrees, it will be sufficient. We shall now determine the value of R_s that will make equation (21) hold within these, and even greater limits.

$$\text{Since } Z_1 = \cos \theta \text{ and } Z_2 = \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta,$$

$$\therefore \frac{dZ_1}{d\theta} = -\sin \theta$$

$$\text{and } \frac{dZ_2}{d\theta} = -\frac{15}{2} \cos^2 \theta \sin \theta + \frac{3}{2} \sin \theta.$$

Expressing these in the form of series, we have,

$$\frac{dZ_1}{d\theta} = -\sin \theta = -\theta \left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \frac{\theta^6}{5040} + \dots \right)$$

$$\frac{dZ_2}{d\theta} = -\theta \left(6 - \frac{17}{2} \theta^2 + \frac{19}{5} \theta^4 - \frac{1367}{1680} \theta^6 + \dots \right)$$

Substituting these values in equation (21) above, we have,

$$C\theta = \frac{2\pi^2 N_1 N_2 a^2 \theta}{D} \left[1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \frac{\theta^6}{5040} + \dots \right. \\ \left. + R_3 \left(6 - \frac{17}{2}\theta^2 + \frac{19}{5}\theta^4 - \frac{1367}{1680}\theta^6 + \dots \right) \right] \\ = \frac{2\pi^2 N_1 N_2 a^2 \theta}{D} \left[1 + 6R_3 - \theta^2 \left(\frac{1}{6} + R_3 \frac{17}{2} \right) + \theta^4 \left(\frac{1}{120} + R_3 \frac{19}{5} \right) - \dots \right] \quad (22)$$

If now we assign such a value to R_3 that the terms in θ^2 vanish, we will have,

$$R_3 \frac{17}{2} + \frac{1}{6} = 0. \\ \therefore R_3 = -\frac{1}{51} \quad (23)$$

Putting this value in (22) and reducing, we get,

$$C = \frac{2\pi^2 N_1 N_2 a^2}{D} \left(\frac{45}{51} - \frac{9}{136}\theta^4 + \frac{15}{952}\theta^6 - \dots \right) \quad (24)$$

For all values of θ between ± 5 degrees, the sum of all the terms involving θ in equation (24) amounts to less than .0000038, or about 4.3 parts in a million in comparison with the constant term. Even up to deflections of ± 8 degrees the time of vibration is independent of amplitude to within about one part in one hundred thousand. Hence, all these terms may be neglected even for the most accurate work, and the assumption of a constant value of C is therefore justified. We then have,

$$C = \frac{2\pi^2 N_1 N_2 a^2}{D} \cdot \frac{45}{51} \quad (25)$$

We are now in a position to determine which of the three values of ρ_3 given in equation (17) may be used. Whichever value is used must satisfy equation (17), and be consistent with equations (20) and (23). Substituting the value of R_3 from equation (23) in equation (20), and making $\rho_1 = \frac{1}{2}\sqrt{3}$ according to equation (17), and finally inserting in turn the three values of ρ_3 and solving for $\frac{a_2}{a_1}$, we get, approximately,

$$\begin{aligned} \frac{a_2}{a_1} &= \frac{1}{\sqrt{-5.04}} \dots (\rho_1 = .38) \\ \frac{a_2}{a_1} &= 1.1 \dots (\rho_2 = .92) \\ \frac{a_2}{a_1} &= \frac{1}{5.4} \dots (\rho_3 = 2.062). \end{aligned}$$

It is obvious at once that the first two of these values can not be used, since the first is imaginary, and the second gives a value for the radius of the movable coil 10 per cent greater than that of the fixed coil, which is obviously impracticable. The third value requires that the radius of the fixed coil be 5.4 times that of the movable coil, a value very desirable in practice, and one which not only insures that the neglect of the higher terms will not introduce any appreciable error, but it also gives such small values to the coefficients of the third and fourth terms of equation (16) that a comparatively large error in proportioning the coils would not appreciably affect the vanishing of these terms, as will presently be shown.

The value of C may be calculated from equation (25) if desired, but to do so requires a highly accurate knowledge of the dimensions of the coils. A better way of determining C is as follows:

Referring to equation (14), and noting that only the first two terms have any appreciable value under the conditions imposed, we have,

$$\begin{aligned} M &= \pi^2 n_1 n_2 a_1^2 a_2^2 (K_1 k_1 Z_1 + K_2 k_2 Z_2) \\ &= \frac{2\pi^2 N_1 N_2 a_2^2}{D} \left(Z_1 + \frac{K_2 k_2 Z_2}{K_1 k_1} \right) = \frac{2\pi^2 N_1 N_2 a_2^2}{D} (Z_1 + R_2 Z_2) \end{aligned}$$

If the coil be placed in the position of maximum mutual inductance M_o , we will have, since $\theta = 0$ and hence $Z_1 = Z_2 = 1$,

$$M_o = \frac{2\pi^2 N_1 N_2 a_2^2}{D} (1 + R_2) = \frac{2\pi^2 N_1 N_2 a_2^2}{D} \cdot \frac{50}{51} \quad (26)$$

Dividing (25) by (26), and solving for C , we get,

$$C = \frac{9}{10} M_o \quad (27)$$

It is necessary, therefore, to measure only the maximum mutual inductance of the coils for the determination of the constant of the instrument by equation (27). It is thus possible not only to determine the constant of the instrument by simply measuring the maximum mutual inductance, but there is the additional important advantage that the constant can be quickly redetermined electrically at any time without disturbing the coils. Substituting the value of C derived from equation (27) in equation (5), we have, as the final working formula,

$$Q = 2\pi n \sqrt{\frac{10K}{9M_0} \left(1 - \frac{T_1^2}{T_2^2} \right)} \quad (28)$$

DISCUSSION OF SOURCES OF ERROR

TEMPERATURE EFFECTS

It is evident that if we maintain the fixed coil at a constant temperature, as by water cooling or otherwise, the field at the center will be constant and the mutual inductance will vary nearly as the square of the linear dimensions of the movable coil. But the value of K is also proportional to the square of the linear dimensions of the moving system. The ratio of K to M_0 (28) would, therefore, be very nearly independent of temperature, and since these are the only quantities in the equation that are affected by temperature, the temperature coefficient of the instrument would be extremely small if not entirely negligible. Further, the mutual inductance can be readily measured immediately before and again immediately after the experiment, so that any slight drift in the constant of the instrument can be detected.

DIMENSIONS OF THE COILS

Since the above equation for C is derived on the assumption of certain relations between the dimensions of the coils, it is important to note what effect slight errors in these dimensions may have on the calculated value of electric current or quantity.

The neglected even terms need not cause any trouble, since each of these terms contains two factors, both of which approach zero as the centers of the coils are made to approach the intersection of the axes. This adjustment being entirely independent of the dimensions of the coils can, with a little care, be made as close

as desired, and there would be no difficulty in making these terms altogether negligible.

Following the method given by Gray,² we have for the general integrals for the ninth term of Gray's series for torque,

$$\frac{1}{n(n+1)} \int \frac{\psi Z'_n dx}{D^{n+2}} = -\frac{\rho_1 a^2_1}{11520 D^{17}} (576 \rho^6_1 - 3024 \rho^4_1 + 2520 \rho^2_1 - 315)$$

$$\int D^{n-1} \psi' Z'_n dx = \frac{15 \rho_2 a^2_2}{128} \left(\frac{128}{3} \rho^3_2 - 384 \rho^2_2 + 672 \rho_2 - 280 \rho^2_2 + 21 \right)$$

Substituting the limits, and substituting $\rho_1 = \frac{\sqrt{3}}{2}$ and $\rho_2 = 2.062$, and dividing by $K_1 k_1$, we have,

$$\frac{K_9 k_9}{K_1 k_1} = \frac{1}{5,313,000}. \quad (29)$$

By Roderigues Theorem, we have

$$Z_9 = \frac{1}{2^9 \cdot 9!} \cdot \frac{d^9}{dZ_1^9} (Z_1^9 - 1)^9$$

where, as above, $Z_1 = \cos \theta$. Hence, we have,

$$\frac{dZ_9}{dZ_1} = \frac{1}{2^9 \cdot 9!} \cdot \frac{d^{10}}{dZ_1^{10}} (Z_1^9 - 1)^9$$

Carrying out this differentiation, and substituting as close approximation, $\cos \theta = 1$, and putting the value of $\frac{dZ_9}{dZ_1}$ thus obtained in

(16) together with the value of $\frac{K_9 k_9}{K_1 k_1}$ from (29), we find that the ninth term amounts to a little less than 1 part in 120,000 in comparison with the first term.

We shall now investigate the error that may arise due to neglecting the third and fourth terms of equation (16). The ratio of the third term to the first term is

$$\mu_3 = \frac{K_3 k_3}{K_1 k_1} \cdot \frac{\frac{dZ_3}{d\theta}}{\frac{dZ_1}{d\theta}}$$

² Gray, *Absolute Measurements in Electricity and Magnetism*, Vol. 2, part 1, pp. 273, 274.

$$\text{Therefore } \mu_s = -\frac{a_2^4}{32a_1^4} \left(4\rho_2^4 - 10\rho_2^2 + \frac{5}{2} \right) \cdot \frac{dZ_s}{dZ_1} \cdot \frac{4\rho_1^2 - 3}{(1 + \rho_1^2)^4} \quad (30)$$

$$\text{and since } Z_s = \frac{63}{8} \cos^5 \theta - \frac{35}{4} \cos^3 \theta + \frac{15}{8} \cos \theta$$

$$\text{and } Z_1 = \cos \theta$$

$$\therefore \frac{dZ_s}{dZ_1} = \frac{315}{8} \cos^4 \theta - \frac{105}{4} \cos^2 \theta + \frac{15}{8}$$

Since θ is always very small, $\cos \theta$ will differ but slightly from unity, and we have, without appreciable error,

$$\frac{dZ_s}{dZ_1} = 15$$

Substituting this value of $\frac{dZ_s}{dZ_1}$ in equation (30) together with the numerical values of the other factors as given above, viz:

$$\rho_1 = \frac{\sqrt{3}}{2}, \rho_2 = 2.062, \frac{a_1}{a_2} = 5.4, \text{ we have, with close approximation,}$$

$$\mu_s = -\frac{1}{527} (4\rho_1^2 - 3) \quad (31)$$

In any case in which an accurate measurement is sought, the length of the fixed coil would be of the order of perhaps 50 centimeters, with the diameter greater in the ratio of $\frac{2}{\sqrt{3}}$. Choosing these as a basis for numerical calculation, we find that if the actual length of the coil differs from its assumed value by as much as half a millimeter, which would give an error of one part in one thousand in the ratio of ρ_1 , the value of μ_s from equation (31) would amount to only 1.1 parts in 100,000 in comparison with the first term. Since this is an error in the torque, it appears only half as great in the value of current or electric quantity, and hence is entirely negligible. Obviously, a somewhat smaller error would result if an error of one-half millimeter should occur in the diameter of the coil.

For the seventh term we have

$$\mu_7 = \frac{K_7 k_7}{K_1 k_1} \frac{\frac{dZ_7}{d\theta}}{\frac{dZ_1}{d\theta}}$$

$$= -\frac{a_2^3}{32a_1^3} \cdot \frac{4\rho_1^4 - 10\rho_1^2 + 5/2}{(1 + \rho_1^2)^3} \cdot \frac{dZ_7}{dZ_1} \cdot \left(4\rho_2^3 - 21\rho_2^2 + \frac{35}{2}\rho_2 - \frac{35}{16} \right) \quad (32)$$

and since $Z_7 = \frac{429}{16} \cos^7 \theta - \frac{693}{16} \cos^5 \theta + \frac{315}{16} \cos^3 \theta - \frac{35}{16} \cos \theta$

$$\therefore \frac{dZ_7}{dZ_1} = \frac{3003}{16} \cos^6 \theta - \frac{3465}{16} \cos^4 \theta + \frac{945}{16} \cos^2 \theta - \frac{35}{16}$$

Hence, for small values of θ , we have approximately

$$\frac{dZ_7}{dZ_1} = 28$$

Substituting this, and the proper numerical values of a_1 , a_2 , and ρ_1 in equation (32), we get

$$\mu_7 = +\frac{77 \cdot 10^{-5}}{216} \left(4\rho_2^3 - 21\rho_2^2 + \frac{35}{2}\rho_2 - \frac{35}{16} \right) \quad (33)$$

On the basis of the preceding assumption of 50 centimeters as the length of the fixed coil, the movable coil would have a length of a little less than 20 centimeters. An error of 1 millimeter here would be about one-half per cent, and assuming that ρ_2 is one-half per cent greater or less than its proper value of 2.062, we would have from equation (33), with substantial accuracy,

$$\mu_7 = 9 \times 10^{-6}$$

An error of 1 millimeter in the length of the movable coil would therefore give the seventh term a value equal to, roughly, 1 part in 110,000 in comparison with the first term, with a resulting error of 1 part in 220,000 in the quantity being measured. The same percentage error in the radius of the movable coil would give rise to an equal error.

It appears, therefore, that if the coils are wound and mounted with reasonable care, no correction will be necessary for the

neglected terms, even in work of the highest accuracy, and the simple form of equation (18) and those derived from it, may be regarded as substantially exact. It remains, therefore, to determine only the errors that may arise in the use of this equation.

From equation (22), we have, after omitting the higher terms,

$$C\theta = \frac{2\pi^2 N_1 N_2 a^2 \theta}{D} (1 + 6R_s)$$

$$\therefore C = \frac{2\pi^2 N_1 N_2 a^2}{D} (1 + 6R_s) \quad (34)$$

and from equation (14), we get

$$M_o = \frac{2\pi^2 N_1 N_2 a^2}{D} (1 + R_s) \quad (35)$$

Hence from (34), $C = \frac{M_o(1 + 6R_s)}{1 + R_s}$

and, therefore, from (5)

$$Q = 2\pi n \sqrt{\frac{K(1 + R_s)}{M_o(1 + 6R_s)} \left(1 - \frac{T_1^2}{T_2^2}\right)} \quad (36)$$

Differentiating this equation with respect to R_s , and dividing the resulting equation by (36), member for member, we have,

$$\frac{dQ}{Q} = -\frac{5}{2} \cdot \frac{dR_s}{1 + 7R_s + 6R_s^2} = -2.9dR_s \quad (37)$$

From equation (20), we have,

$$R_s = -\frac{a_2^2}{8a_1^2} \cdot \frac{4\rho_2^2 - 3}{(1 + \rho_1^2)^2}$$

Differentiating this equation with respect to the four quantities ρ_2 , ρ_1 , a_2 , and a_1 , separately, we have,

$$dR_s = -\frac{a_2^2 \rho_2^2}{a_1^2 (1 + \rho_1^2)^2} \cdot \frac{d\rho_2}{\rho_2} = -\frac{1}{20.7} \cdot \frac{d\rho_2}{\rho_2}$$

$$dR_s = +\frac{a_2^2 \rho_1^2 (4\rho_2^2 - 3)}{2a_1^2 (1 + \rho_1^2)^3} \cdot \frac{d\rho_1}{\rho_1} = +\frac{1}{29.6} \cdot \frac{d\rho_1}{\rho_1}$$

$$dR_1 = -\frac{a_2^2(4\rho_2^2 - 3)}{4a_1^2(1 + \rho_2^2)^2} \cdot \frac{da_2}{a_2} = -\frac{1}{25.3} \cdot \frac{da_2}{a_2}$$

$$dR_2 = \frac{a_2^2(4\rho_2^2 - 3)}{4a_1^2(1 + \rho_1^2)^2} \cdot \frac{da_1}{a_1} = -\frac{1}{25.3} \cdot \frac{da_1}{a_1}$$

Putting these values in equation (37) successively we have, very closely,

$$\begin{aligned} \frac{dQ}{Q} &= \frac{1}{7.2} \cdot \frac{d\rho_2}{\rho_2} \\ \frac{dQ}{Q} &= -\frac{1}{10} \cdot \frac{d\rho_1}{\rho_1} \\ \frac{dQ}{Q} &= \frac{1}{9} \cdot \frac{da_2}{a_2} \\ \frac{dQ}{Q} &= -\frac{1}{9} \cdot \frac{da_1}{a_1} \end{aligned} \quad (38)$$

These equations show that any small percentage errors in the value of ρ_1 will be reduced ten times in the calculated value of electric quantity. Errors in ρ_2 are reduced 7.2 times and those in a_2 and a_1 nine times. The same ratios hold in the equation for current. It appears, therefore, that if ordinary care be exercised in measuring the coils, any errors that may arise in these measurements will not affect appreciably the calculated value of electric current or quantity.

It may be well to state briefly the considerations that lead to the adoption of the particular proportions of the coils given above. In the Gray electro-dynamometer it is usual to give both fixed and movable coils a ratio of length to diameter equal to $\frac{\sqrt{3}}{2}$, in which case the third and fifth terms vanish, leaving only the seventh and higher terms, which are made negligibly small by making the movable coil small compared to the fixed coil. This is impracticable here, because, in that case, the torque would be proportional to the sine of the angle of deflection and the period would not be isochronous except for extremely small values of θ . The initial deflection could not exceed about one-half degree, and with the inevitable damping which would occur during a half-hour

run the final deflection would be too small to permit of a sufficiently accurate measurement of the value of n . To overcome this, one term after the first must be used in order to make the period independent of the deflection. By using the third term as described above, we can use deflections about sixteen times greater than would be possible with the Gray type of instrument before the period becomes appreciably affected by the amplitude of the vibration. If we desired to do so, we could choose such values of ρ_1 and ρ_2 that would cause the third and seventh terms to disappear and use the fifth term to produce the condition of isochronous vibration. This case has been worked out, but is found to be less suitable than the one given above for two reasons. In the first place, the limiting value of the deflection that can be used is only about two-thirds as great. The most serious objection, however, lies in the fact that on working out the approximation formulæ corresponding to equation (38), we find that the dimensions of the movable coil have to be determined with an accuracy about three times as great as in the present case in order to secure the same ultimate accuracy in the measurement of Q .

A possible method would be to make the third and fifth terms disappear as in the Gray electro-dynamometer and use the seventh term for making the time of vibration independent of the amplitude. To do this, however, would require that the radius of the movable coil be so large that the ninth and eleventh terms become troublesome. It seems, therefore, that the method described above of reserving the third term and making the others vanish possesses such important advantages as to make it the only one worthy of serious consideration in any practical case.

EFFECT OF EARTH'S MAGNETIC FIELD

To eliminate the effect of the earth's magnetic field, it is necessary only to permit the normal working current to flow through the movable coil alone, while the value of T_z is being determined. In this way the effect of the earth's field is eliminated along with the torsion of the suspending wire. It is true that in this case the time of vibration is not strictly isochronous, since the component of torque due to the earth's field varies as the sine of the angle of deflection. But since the angle of vibration is small, the total effect on the time of vibration is extremely small, and since

whatever effect there is enters only into the ratio $\frac{T_1}{T_2}$ in equation (5), the error would practically vanish. A simple calculation shows that it could not be greater than a few parts in a million. Likewise, any small variations in the value of the earth's field such as would ordinarily occur would disappear for the same reason. It will not be necessary, therefore, to use a compensating coil to neutralize the earth's field.

ELECTROSTATIC FORCES

Owing to the smallness of the movable coil in comparison with the fixed coil, and to the relatively low voltages used, it is evident that the total electrostatic forces that could act on the movable coil would be very small. Moreover, it is readily seen from the symmetry of the system that when the coils are coaxial any electrostatic force that might exist would be one of translation only, and could not give rise to any couple whatever; and since, in normal working, the angle between the axes of the coils would never be more than 4 or 5 degrees, no disturbance of appreciable magnitude is to be expected here. This point can be tested experimentally, however, by opening the movable coil at the center and applying between the two halves the normal potential drop across the terminals of the movable coil, at the same time permitting the normal working current to flow through the fixed coil. It is readily seen that the electrostatic couple thus set up would be slightly larger than that which would exist under normal working conditions; and if the effect of this couple is found to have an appreciable effect on the period of vibration, the reading thus obtained will permit of a fairly accurate calculation of the true effect. It should be noted that this reading is taken under the most favorable conditions, since, owing to the absence of the electromagnetic couple, the electrostatic couple would be relatively about a hundred times greater in comparison with the total couple than it would be when the current is flowing through both coils. It would thus be very easy to detect an effect that would be entirely imperceptible under normal working conditions.

The foregoing method for the absolute measurement of electric quantity has not yet been subjected to experimental test, so that the limits of the ultimate accuracy attainable by it can not be

stated with great precision. Inasmuch, however, as most of the important phases of the problem are amenable to exact mathematical calculation, the chief features of the performance of the instrument can be predicted with considerable confidence.

As compared with the Gray type of electrodymanometer and some types of the current balance, this instrument possesses, in common with the Rayleigh type of current balance, the advantage that errors in the dimensions of the coils are reduced manifold. The Rayleigh balance, however, possesses this advantage in greater degree. The method here described also possesses an advantage over the Gray electrodymanometer in that the torsion of the suspending wires is eliminated. Further, the fact that the indications of the instrument are in a large degree independent of the temperature of the movable coil and of moderate variations in the strength of the working current, contributes greatly to the convenience and to some extent to the accuracy of the method. As to the sensibility of the instrument, it seems that owing to the very great apparent travel of the scale, which, as a rule, would be from 100 to 200 meters, the sensibility of the instrument may safely be said to far exceed the limits of accuracy aimed at in absolute measurements at the present time. The constant of the instrument can be redetermined at any time without disturbing the coils by simply measuring their maximum mutual inductance, and this affords a convenient and accurate check on the stability of the instrument, which would in turn increase considerably the confidence that could be placed in individual measurements. The further fact that the quantity of electricity may be obtained directly without any knowledge of the value of the current being necessary, might often be of advantage where the electrochemical equivalent of silver or other metals is being determined, or, in other cases, where the time integral of current is required.

In conclusion, the writer wishes to express his obligation to Dr. N. E. Dorsey and Dr. F. W. Grover for valuable suggestions in connection with the preparation of this paper.

WASHINGTON, May 20, 1910.

THE COMPARATIVE SENSITIVENESS OF SOME COMMON DETECTORS OF ELECTRICAL OSCILLATIONS

By Louis W. Austin.

Although probably many comparisons of the sensitiveness of detectors have been made in the laboratories of the wireless companies, very little quantitative information on this subject has been made public. For this reason the experiments here described have been carried out. On account of the difficulties and the amount of time involved in the testing of detectors in wireless stations, methods have been developed during the last three years whereby these tests can be made in the laboratory more expeditiously and accurately than is possible in the regular stations. The difficulties usually encountered in laboratory tests of this kind are due mainly to the direct effect of the source of the oscillations on the receiving circuits and detector; but by using properly designed exciting circuits in connection with a buzzer as a source of oscillations these difficulties have been removed. As it is also possible in this way to produce oscillations of any required degree of damping as well as of any required wave length, all of the problems of wireless telegraphy can be successfully investigated in the laboratory except those which have to do with the properties of different types of antennæ and the passage of the waves from station to station. In the following paper some experimental comparisons of the sensitiveness of wireless detectors are described.

SYMBOLS

D = Galvanometer deflection	} In general proportional to
A = Telephone audibility	
I = Oscillatory current.	
c = Direct current.	

R = Direct current resistance.

R' = High frequency resistance.

$L_1 L_2$, etc. = Inductances.

$C_1 C_2$, etc. = Tuning condensers.

K = Large fixed condenser 0.04–0.1 mf.

M_{34} = Mutual inductance between coils 3 and 4.

d_{12} = Distance between coils 1 and 2.

δ_1 = Logarithmic decrement in circuit I.

λ = Wave length.

METHOD OF EXPERIMENT

The exciting circuit¹ in all the experiments is shown in Fig. 1. Here B is a buzzer run by two storage cells with considerable resistance in series. The oscillatory circuit consists of a variable air condenser C_1 having a maximum capacity of 0.005 microfarads, and for moderate wave lengths an inductance L_1 of 0.062 millihenries. These are connected directly across the contact of the buzzer.

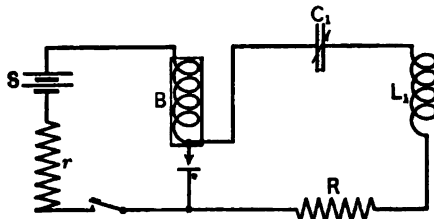


Fig. 1

The action of the circuit is as follows: When the buzzer contact is broken, the condenser C_1 is charged and this charge is retained until the contact is again closed, when the condenser discharges through the contact, producing oscillations in the circuit $L_1 C_1$. As these oscillations take place in a closed metallic circuit, they are very feebly damped unless extra resistance is introduced at R .

All buzzers are not equally suited to producing oscillations by this method. In some, considerable adjustment of the contact is required before sharp tuning is obtained. The buzzer used in

¹ Other forms of buzzer circuit such as are used in several well-known types of wave meters were also tried, but with the buzzer used they gave less powerful oscillations than the one here shown.

these experiments was of the 1905 United States Signal Corps type, and gave very satisfactory results with wave lengths varying between 350 and 2000 meters. With some care in adjustment the wave length could be pushed up beyond 3000. For telephonic reception of signals it was usually found necessary to cut down the intensity of oscillations from the buzzer, and this was done by placing an incandescent lamp across the buzzer contact.

The chalcopyrite-zincite rectifier known as the perikon detector² was used as a standard for the comparison of the various detectors. As its sensitiveness varies somewhat from time to time, it was standardized before each experiment in the following manner, the circuits being shown in Fig. 2. Here *I* is the buzzer circuit, *II* a

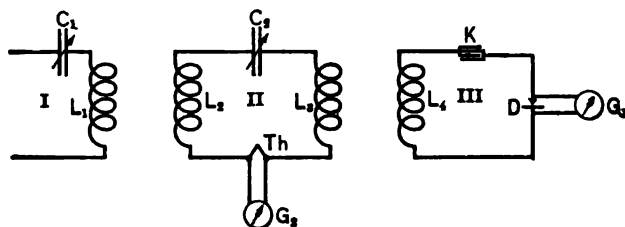


Fig. 2

circuit containing a thermoelement of known sensitiveness, and *III* the circuit consisting of the perikon *D*, its coupling inductance L_4 , and a suitable stopping condenser *K* of about 0.04 microfarad. With a given loose coupling between L_3 and L_4 , the relation was noted between the reading of the galvanometer attached to the thermoelement and that attached to the perikon,³ the condition of wave length and damping being kept constant. This relation gives at once the relative sensitiveness of the perikon. It was found better to take account of the variations in sensitiveness observed rather than to readjust each time to the same sensitiveness.

Table I shows the admirable proportionality between the deflections of the perikon detector and the square of the oscillatory current as measured by the thermoelement.

² The perikon was a new one which had been little used and was of normal sensitiveness.

³ The resistance in the perikon galvanometer circuit must be equal to that to be used in the experiment.

TABLE I
Comparison of Tellurium-Constantan Thermoelement and Perikon Rectifier

(See Fig. 2)

Thermoelement, Circuit II D	Perikon, Circuit III D	Ratio
2.3	25	10.9
5.3	54	10.2
9.7	100	10.3
19.8	198	10.0
31.2	330	10.6
41.3	480	10.9
51.7	540	10.4

$$L_2 = 0.03 \text{ mh} \quad \lambda = 900 \text{ m}$$

$$L_3 = 0.425 \quad \delta_1 = 0.16$$

$$L_4 = 0.420 \quad \delta_2 = 0.13$$

$$M_{34} = 0.025$$

Circuit III untuned.

Two general methods have been used for measuring the sensitiveness for telephonic reception of radiotelegraphic receivers: First, the shunted telephone method in which the relative loudness of response to the same signal in two different receivers can be compared, and, second, the variable coupling method by means of which the least audible signals for the different receivers can be compared.

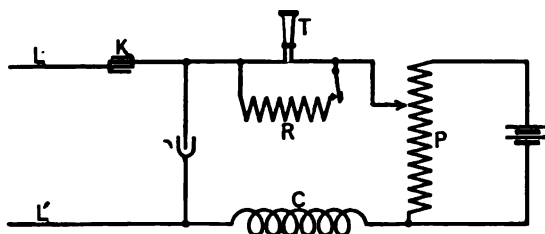


Fig. 3

A form of circuit for the shunted telephone method in the case of the electrolytic detector is shown in Fig. 3. Here L and L' are wires leading to the oscillatory circuit, K a stopping condenser of 0.04 microfarads capacity, T the telephones, R a vari-

able resistance in shunt across the telephones, and C a choke coil to prevent the oscillations from running around through R instead of passing through the detector when the shunt R is closed. Two 60-ohm telephones form a suitable choke. Whatever choke coil is used it is necessary to test it by placing it across $L L'$. If the choke is perfect, no oscillations will pass through it, and when it is placed across $L L'$ no change in the loudness of the signals will be observed.

The measurement of the intensity of signal is made as follows: After the receiving circuit and detector are adjusted to give maximum loudness in the telephone the shunt resistance R is closed and the resistance regulated until the signal just becomes audible. The value of the current pulses c in the telephone, which are proportional to the energy of the incoming waves in the detector, is expressed by the following well-known formula, where r is the value of the shunt, and t is the resistance of the telephones, and c' the least current audible in the telephones:

$$c = \frac{r+t}{r} c'$$

$$\text{The audibility, } A = \frac{c}{c'} = \frac{r+t}{r}.$$

The signal is often expressed in terms of A as being so many times audibility. With care a series of measurements of intensity may be made to agree among themselves to within about 10 per cent.

In the variable coupling method for comparing the least audible signals in two detectors, the deflections of the galvanometer connected to the perikon are observed for a number of different degrees of coupling between the inductances L_1 and L_2 . (See Fig. 2.) From these a curve is plotted which gives the variation of the received energy as determined by the distance between the coils. The coupling, for which the signals just become audible in the telephones of the detectors being compared, is then observed and the relative energy taken from the curve.

Fig. 2 shows the general plan of the circuits used in the comparison of the detectors. I is the exciting circuit adjusted to send

out wave trains having a wave length group frequency and damping similar to those occurring in wireless practice. *II* is a circuit whose constants were made to correspond to a receiving antenna of moderate size. *III* is an untuned circuit containing a detector connected to a galvanometer or to a pair of head telephones, as the circumstances of the work required. The electrolytic, perikon, and audion were used in circuit *III*, each being so coupled to circuit *II* as to give maximum effect. The magnetic detector was placed in circuit *II*, as it is generally used in the antenna. The Fleming vacuum valve detector was coupled to *II* by its own fixed coupling coils.

TABLE II

Electrolytic and Perikon Detectors, Shunted Telephone Method

		Relative sensitiveness	
		Strong signals	Weak signals
<u>Electrolytic</u>	⁴	2.5	2.4
<u>Perikon</u>			
<u>Electrolytic</u>	⁴	1.1	1.3
<u>Perikon (0.2 volt)</u>			
<u>Electrolytic</u>	⁵	2.3	2.5
<u>Perikon</u>			
<u>Electrolytic</u>	⁵	1.3	1.3
<u>Perikon (0.2 volt)</u>			
Telephone resistance 2100 ohms			
<u>Electrolytic</u>	⁵		2.7
<u>Perikon</u>			
<u>Electrolytic</u>	⁵		1.1
<u>Perikon (0.2 volt)</u>			
Telephone resistance 813 ohms			

Inductances, decrements, and wave length as in Table I. M_{24} adjusted for maximum sensitiveness.

⁴Wollaston wire 0.0012 mm diam.⁵Wollaston wire 0.003 mm diam.

THE ELECTROLYTIC DETECTOR

Comparison was made of the sensitiveness of the electrolytic detector and the perikon, using a wave length of approximately 900 meters and a decrement of the waves given out by the exciter circuit, amounting to 0.16. The values of the inductances and capacities of the circuits were so chosen that they simulated the conditions of commercial work.

In Table II are given the ratio of sensitiveness for both strong and weak signals. The perikon was used without external EMF and also with about 0.2 of a volt applied in the direction of the rectification. Comparison was made by the shunted telephone method, the same telephones being used for both detectors. In one set of experiments the telephones used were of 2100 ohms and in another of 813 ohms. The relative sensitiveness of the two detectors was little affected by this change, showing that their effective resistances must be about the same. It is seen that without external EMF the perikon is less than half as sensitive as the electrolytic, but that when the 0.2 volt is applied the difference is very small. The electrolytic used in the experiment was of the free-wire type and two positive Wollaston wires were used during the test, one 0.003 and one 0.0012 mm in diameter. This was done because it has been frequently stated that the finer wire was more sensitive to weak signals. No marked difference between the two, however, could be detected. This shows that it is only necessary for sensitiveness, that the capacity reactance for the given frequency should be large, and that beyond a certain limiting diameter there is no object in reducing the size of the positive wire.

Table III shows the relation between galvanometer deflections with the thermoelement in circuit *II* and the intensity of signals in the electrolytic in circuit *III* as measured by the shunted telephone method.⁶ This shows that the responses of the electrolytic as indicated by the telephone are proportional to the square of the oscillatory current.⁷

⁶ A comparison was also made of the sensitiveness of the perikon without external EMF and the electrolytic at the Washington Navy-Yard wireless station using incoming signals from a distant station. The results agreed within a few per cent with those found in the laboratory.

⁷ Very strong signals, i. e., more than about 300 times audibility, sometimes give irregular results.

TABLE III

Electrolytic

Received Energy and Audibility of Signals

Thermoelement in Circuit II D	Electrolytic in Circuit III A	Ratio
3.3	12	36
5.6	22.4	40
3.4	140	41
6.3	250	40
8.0	324	41
17.0	645	38

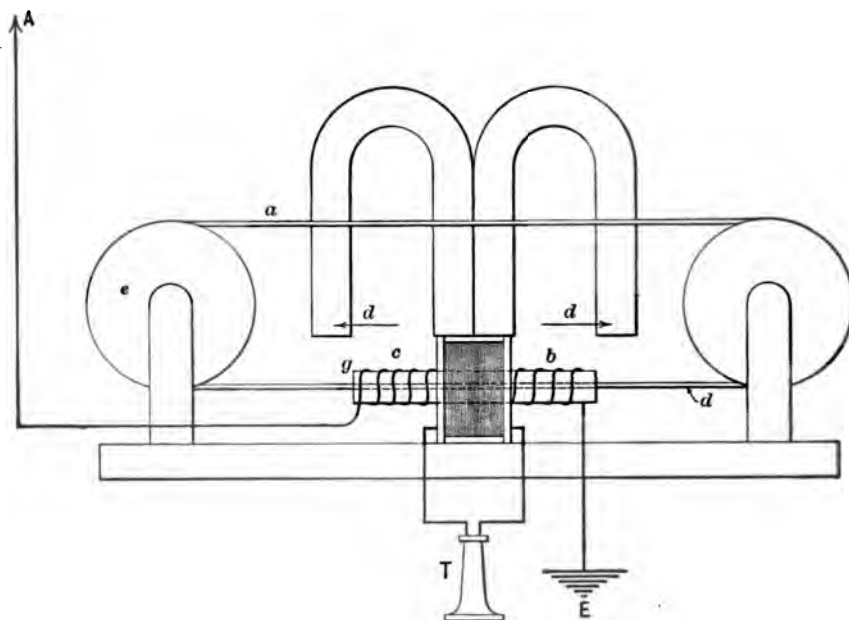
Inductances, decrements, and wave length as in Table I. $M_{34}=0.025$.

Fig. 4

THE MAGNETIC DETECTOR. (Fig. 4)

The circuits for the comparison of the magnetic and perikon detectors are shown in Fig. 2. The magnetic detector was connected directly in circuit II since it is usually used in the antenna. The magnetic detector used was of the ordinary

commercial type, the D. C. resistance of the oscillatory coil being 9.7 ohms and the telephone coil 120 ohms. The telephones furnished with the magnetic detector and used in the experiments had a resistance of 168 ohms. The telephones used with the perikon were of 1200 ohms resistance and medium sensitiveness. For the comparison of sensitiveness the variable coupling method was used, the coupling L_1 and L_2 being changed until the signals became inaudible. When the readings on the magnetic were being taken the circuit containing the perikon was removed, then for the perikon readings the magnetic was cut out and the coupling L_3 L_4 regulated for maximum strength of signals. The relative strength of signals for the different degrees of coupling between L_1 and L_2 was determined by means of the deflection of a galvanometer attached to the perikon. Such a series of readings is shown in Table IV.

TABLE IV

Relative Energy in Circuit II for Different Distances d_{12} Between L_1 and L_2

(See Fig. 2)

d_{12}	Perikon D
10 cm	400
12	245
14	172
16	115
18	75
20	54
24	30
28	14
32	8
38	3

$$\lambda = 900 \text{ m}$$

$$\delta_1 = 0.16$$

With waves from the buzzer circuit having a damping of 0.16 and a wave length of about 900 meters, it was found that the magnetic detector required 2.1 times as much energy to produce an audible sound as the perikon used without external EMF.

Further experiments showed that by putting a 0.05 microfarad condenser in series with the telephone of the magnetic, thus tuning it roughly to the rate of the buzzer, the loudness of signals could be increased so that there was very little difference between it and the perikon.

TABLE V
Magnetic and Perikon Detectors

Weak Signals

λ	Ratio of sensitiveness: $\frac{\text{Magnetic}}{\text{Perikon}}$
350 m	0.22
900	0.9
2000	1.0
3000	1.5

Comparisons were also made at a wave length of 350 meters and at 2000 and 3000 meters. At 3000 meters the magnetic with tuned telephone was approximately one and one-half times as sensitive as the perikon. At 2000 there was very little difference, while at 350 meters the perikon was nearly five times as sensitive as the magnetic.

In order to explain these differences in relative sensitiveness, the logarithmic decrement of the circuit containing the magnetic was determined for different frequencies. For this purpose the circuits were those shown in Fig. 2, the magnetic detector being in its place in circuit *II* and the perikon with its galvanometer in the untuned circuit *III*, very loosely coupled to *II*, so that no appreciable energy was taken up by it. The buzzer exciter circuit was adjusted so as to give feebly damped oscillations having a decrement of about 0.02. The decrement of circuit *II* was determined from the well-known formula

$$\delta_I + \delta_{II} = \pi \frac{C_m - C}{C_m} \sqrt{\frac{I^2}{I_m^2 - I^2}}$$

where C_m represents the reading of the condenser in circuit *II* for resonance and C any other reading slightly removed from reso-

nance. I_m then represents the value of the oscillatory current for resonance and I that for the setting of the condenser at C . This formula becomes much more convenient in practice if we choose I^2 so that it is just one-half I_m^2 , for then the expression under the radical reduces to unity. As the deflection of the galvanometer connected to the perikon is proportional to I^2 , our expression becomes

$$\delta_I + \delta_{II} = \pi \frac{C_m - C}{C_m}$$

where the change from C_m to C , either an increase or a decrease, is just sufficient to reduce the deflection of the galvanometer to one-half. The mean of several determinations of $C_m - C$ gives as accurate values for the decrement as can be obtained from a curve plotted in the usual manner and calculated according to the complete formula. In damping experiments it is necessary that all

TABLE VI

Effective Resistance of Magnetic Detector at Different Wave Lengths

λ	R'
350 m	110 ohms
900	32
3000	10

TABLE VII

Received Energy and Audibility of Signals in Magnetic Detector

Perikon D	Magnetic A
300 mm	120
90	53
55	40
20.5	16.5
7.5	10
2.0	2

the couplings be made so loose that any further loosening will not decrease the value of the damping.

δ_I , having been previously measured, the effective resistance R' of the circuit containing the magnetic detector was calculated from the formula

$$R' = 2nL\delta_{II}$$

where δ_{II} is the decrement, n the frequency, and L the inductance of the circuit. After subtracting the known ohmic high frequency resistance of the circuit, the other losses due to the detector for the different frequencies are shown in Table VI. This table would indicate that the amount of iron and the number of turns of the oscillatory circuit of the magnetic detector should be varied according to the frequency for which it is to be used.

In Table VII is shown the relation between loudness of signal as determined by the shunted telephone method in the magnetic and the oscillatory energy passing through it as determined by the deflections of the perikon galvanometer. It is seen that the loudness of signal in the telephone attached to the magnetic is not proportional to the square of the oscillatory current, but corresponds more nearly to the 1.4 power.

THE FLEMING VACUUM VALVE DETECTOR. (Fig. 5)

The vacuum valve detector furnished for the test was constructed with fixed coupling coils inclosed in its case, so that it was found impossible to test it fairly at any except very short wave lengths without taking it completely to pieces.

The circuits were arranged as shown in Fig. 2, the vacuum valve being coupled by its own coils to circuit II. Signals were taken alternately on the perikon, adjusted for maximum strength of signal, and on the vacuum detector, one detector circuit being removed when the other was in action. At a wave length of 350 meters and with a decrement of the incoming signals of 0.16, the ratio of sensitiveness of the perikon to that of the vacuum detector, as determined by the variable coupling method, was almost exactly two to one. For a wave length of 900 meters the perikon appeared to be at least ten times as sensitive as the vacuum

valve. As there is no reason to believe that the vacuum detector would change its sensitiveness with frequency, this difference is almost certainly due to insufficient coupling.*

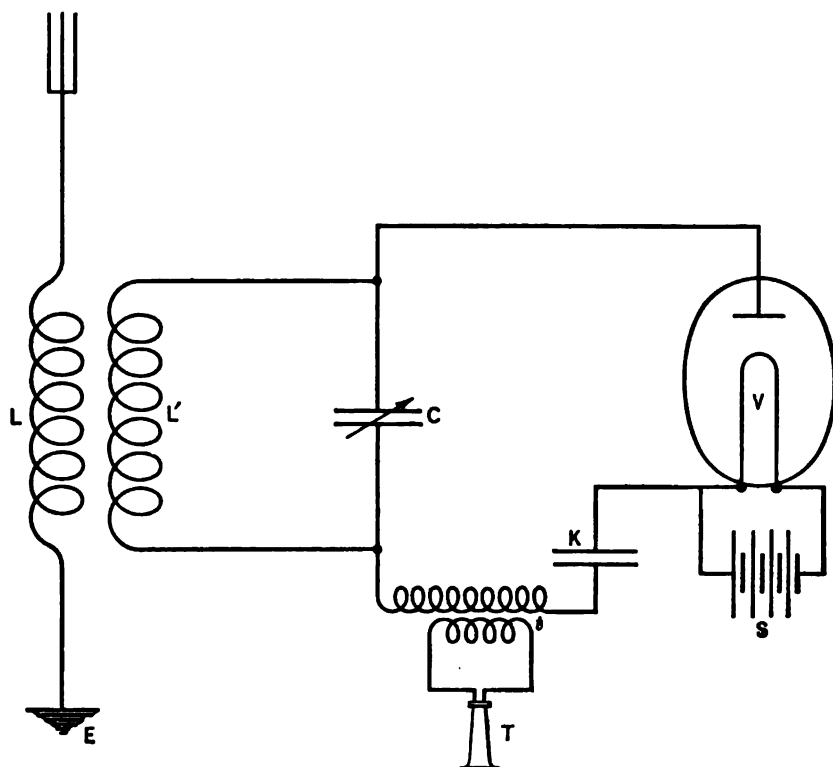


Fig. 5

In Table VIII is given the relation between oscillatory energy in the main circuit and the audibility of signals in the telephone of the vacuum detector, as determined by the shunted telephone method. The audibility is seen to be proportional to the energy. The telephones furnished with the vacuum detector had a resistance of 8300 ohms and were connected by means of a step-up transformer to the detector circuit.

*This shows the disadvantage of constructing receiving apparatus with fixed coupling.

TABLE VIII

Received Energy and Audibility of Signals in Fleming Vacuum Detector

Perfon D	Vacuum detector A
43	38
21	20
10	9
5	5

THE AUDION. (Fig. 6)

The De Forest form of vacuum detector, called by him the audion, is shown diagrammatically in Fig. 6. Here *V* is an exhausted glass bulb containing a tantalum filament *F*, heated by three storage cells *S*, a metal plate *P* and a metal grid *G* be-

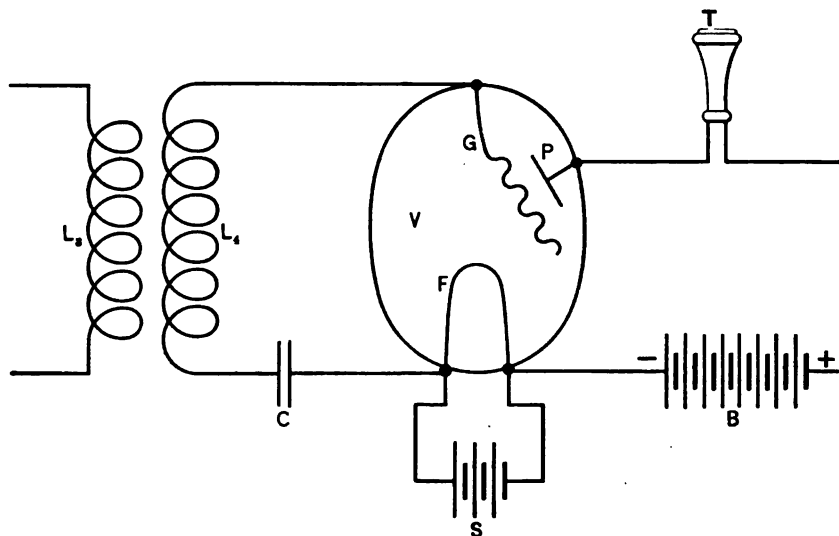


Fig. 6

tween the filament and the plate. The filament and plate are connected externally through a battery *B* of 36 small cells, any number of which can be thrown into the circuit, and a pair of head telephones of about 1200 ohms resistance. The filament

and grid are connected through a small fixed condenser C and an inductance L , which is coupled to the oscillatory circuit.

The action of the detector appears to be as follows: The heated filament F is charged negatively by the battery B and a steady stream of negative electrons flows from the hot filament to the cold plate producing a direct current through the telephone. When an oscillatory electromotive force is impressed on the circuit $F L_1 G$, the negative current can flow from F to G inside the bulb and not in the reverse direction, just as in the Fleming detector. The grid G thus becomes negatively charged and as it lies between F and P the direct current through the telephone is decreased as may be shown by placing a galvanometer in the telephone circuit.

The sensitiveness of the audion was compared with that of the electrolytic already described by the varying coupling method. (See Fig. 2.) Here L_1 and L_2 were the primary and secondary of a Wireless Specialty Apparatus Company tuner, the secondary condenser of the tuner being detached. The detectors were introduced alternately in circuit *III* and their most favorable conditions of coupling were determined separately. The audion, having a higher resistance, required a stronger coupling. The same telephones (1200 ohms) were used for both detectors, previous tests having shown that they were suited to each. High-resistance telephones up to 8000 ohms were tried with the audion, but were found to be less satisfactory than the ones used. This is remarkable, as with the Fleming vacuum detector 8000-ohm telephones are used and even then the current is stepped down through a transformer.

Below are given the data of the experiment:

$\lambda = 900 \text{ m.}$, $\delta_1 = 0.3$, $L_1 = 0.062$, $L_2 = 0.03$, $L_3 = 0.705$, $L_4 = 0.92 \text{ m. h.}$ (electrolytic). $L_4 = 2.3$ (audion).

Electrolytic coupling $L_3 L_4 = 7 \text{ cm}$ on coupling scale ^o of tuner; audion coupling = 2 cm. Distance between coils $L_1 L_2$ for least audible signal, 20.5 cm (electrolytic); 23 cm (audion).

^o The numbers on the scale represent the distance the two coils are drawn apart from the position of maximum mutual inductance.

Ratio of energy sensitiveness $\frac{\text{audion}}{\text{electrolytic}} = 1.5$. Some of the bulbs were slightly more sensitive than this when first tried, but the sensibility decreased with use.

The law governing the relations between the intensity of the oscillations and the loudness of response in the telephone, as determined by the shunted telephone method, seems to depend on the impressed direct electromotive force from the battery B and also on the temperature of the filament F , as would be expected from the known form of the current voltage curve in the case of the passage of electricity from a hot to a cold body through a gas. For most adjustments the response of the telephone is apparently nearly proportional to the square of the oscillatory current.

In addition to the above detectors, a very large number of other types have been tested in the laboratory, among which are the tellurium, silicon, and carbon already described,¹⁰ and also many crystal detectors. These detectors are all of the rectifying type and are remarkable in that when properly adjusted and used with telephones of proper resistance, and in some cases with small EMF in the direction of the rectification, they all give practically the same sensibility as the perikon. The usefulness of several of these is, however, limited by the difficulty of adjustment and a lack of constancy. In most of these detectors the loudness of the signal increases nearly in proportion to the square of the oscillating current, although in some of them the increase in loudness is less rapid.

UNITED STATES NAVAL WIRELESS TELEGRAPHIC LABORATORY,
March 1, 1910.

¹⁰ This Bulletin, Vol. 5, No. 1, p. 133; 1908.

PHOTOMETRIC UNITS AND NOMENCLATURE

By Edward B. Rosa

CONTENTS

	Page
I. Introduction	543
II. General Discussion and Derivation of Formulas	544
1. Point Source	544
2. Distinction between Luminous Flux and Energy	546
3. Definition of Intensity for Unsymmetrical Sources	547
4. Unit Disk	548
5. Extended Source—Circular Disk	550
6. Infinite Plane	553
7. Infinite Cylinder	554
8. Unit Length of Cylinder	555
9. Case of Large Sphere	556
10. Reciprocal Relations	558
11. Hollow Sphere	560
12. Luminous Flux within an Inclosure	561
III. Summary of Photometric Relations	562
IV. Problems for Illustrations	569
V. Collection of Formulas	571

I. INTRODUCTION

The subject of photometric units and nomenclature received a notable impulse by the paper of Blondel,¹ presented to the Geneva Congress of 1896. Since that time various modifications in the proposals there made have been put forward, and the units and nomenclature there proposed have come into use to a greater or less degree. There has, however, been a tendency to recognize as few separate photometric quantities as possible, and some of them have been employed rather loosely in more than one sense. This

¹ A. Blondel, Rapport sur les Unités Photométriques, L'Eclairage Electrique, Vol. 6, pp. 148-157; 1896.

is partly, at least, due to a lack of clearness in the perception of the physical relation of the various photometric quantities.

The Illuminating Engineering Society appointed a committee² on this subject something over a year ago, and it was due to my membership in this committee that I was led to give the matter careful attention.

The following paper is an attempt at a systematic discussion of the mathematical and physical relationships of the various photometric quantities. Some of the units that have been objected to are found to be most useful and to contribute materially to clear thinking and concise expression. Although many of the theorems derived below are not new, they are nevertheless useful in developing the desired relations between the various photometric quantities. Acknowledgment is made to Blondel, Palaz, Liebenthal, Hering, Kennelly, Sharp, Hyde, Jones, and others, whose writings and discussions have done much to develop the subject.

In what follows, some of the names are used in a different sense from what has been usual, and slight changes have been made in some of the symbols. These are in the interest of a more systematic arrangement and also of international uniformity.

II. GENERAL DISCUSSION AND DERIVATION OF FORMULAS

1. POINT SOURCE

We start with the idea of light as a *luminous flux*, radiating or flowing away from the source and illuminating bodies as it falls upon them. In the simple case of a point source the flux is equal in all directions, and since the entire flux falls uniformly upon the interior surface of any concentric sphere the quantity of the luminous flux per unit of area is inversely proportional to the square of the distance, a law which has been verified by experiment. The quantity of the luminous flux per unit of area or the flux density at the surface of the illuminated body is by definition the illumination E . If we represent the total flux by F , we have therefore

$$E = \frac{F}{4\pi r^2} \quad (1)$$

² The committee consists of Dr. C. H. Sharp, chairman, Prof. A. E. Kennelly, Prof. E. L. Nichols, Prof. A. Blondel, and the writer.

where r is the distance from the point source to the body illuminated.

Representing $\frac{F}{4\pi}$ by a single letter I , we have

$$E = \frac{I}{r^2} \quad (2)$$

and

$$F = 4\pi I \quad (3)$$

I is called the *intensity of the source*, and is equal to the flux per unit of solid angle.

The illumination is equal to the intensity of the source divided by the square of the distance (2), and the total flux is 4π times the intensity (3).

The intensity I is measured in *candles*,³ the flux F in *lumens*, and the distance r in centimeters. In practice r is often measured in meters or in feet. Thus from a point source of intensity I candles there is a luminous flux of $4\pi I$ lumens.

The *flux density* is the luminous flux per unit of area normal to the flux in the case of a point source, or the total flux F over an area divided by the area S ; thus the flux density is $\frac{F}{S}$, or $\frac{dF}{dS}$ when it is variable.

If the source is not a point but a small sphere of radius a , the flux $4\pi I$ passes out from a radiant surface $4\pi a^2$. Thus, the flux density of radiation or the *specific radiation*, is

$$\frac{F_s}{S} = \frac{4\pi I}{4\pi a^2} = \frac{I}{a^2} = E' \quad (4)$$

Thus, we may speak generally of the luminous radiation at any point in space, and of the flux density of such radiation. If it falls on a material surface the incident flux density is the *illumination* E ; as it comes from a luminous or other radiating or diffusing surface, the flux density is the *radiation*, E' . Although E and E' are quantities of the same nature, it is convenient thus to distinguish them.

³ It is proposed to call the new value of the American candle, which is the same as the English candle and the French bougie decimale, and which is also used by several other countries, the *international candle*.

The luminous flux density in space is analogous to electric displacement in electrostatics. We think of an electric displacement as occurring in space between two or more electric charges, but a surface density occurs only where there is a material conducting body on which the lines of electric force terminate. In the same way the terms luminous flux and flux density apply generally both at the surface of the luminous and the illuminated bodies and in the space between. The *radiation* is the flux density at the source of the flux, and the *illumination* is the flux density or flux per unit of area on the surface where the luminous flux is received.

2. DISTINCTION BETWEEN LUMINOUS FLUX AND ENERGY

The total luminous flux F is not to be confused with the total energy flowing from a luminous body. Luminous flux, or *light* as we ordinarily say, is the physical stimulus which applied to the retina produces the *sensation* of light. It is equal to the radiant power multiplied by the stimulus coefficient. This stimulus coefficient is different for every different wave frequency or wave length, and is of course zero for all frequencies outside the visible spectrum. Hence, if W_λ is the power (expressed in watts) for unit of wave length of the spectrum, and K_λ is the stimulus coefficient or *luminous efficiency* whose value varies with the wave length λ , we have for the total power radiated from a body

$$W = \int_0^\infty W_\lambda d\lambda$$

the integration being carried through the whole range of wave lengths, including of course the non-luminous radiation.

For the luminous flux

$$F = \int K_\lambda W_\lambda d\lambda$$

the integration being throughout the visible spectrum, K being zero elsewhere.

As the values of K_λ throughout the spectrum are not accurately known, it is not possible to calculate F in general. But by measuring W in watts and F in lumens, we can determine the ratio of the luminous flux to the radiant power, in any particular

case. One may properly say that luminous flux is due to and is always associated with radiant power; but luminous flux and radiant power can not in general be converted into one another like feet and inches; for, as stated above, the conversion factor, the stimulus coefficient, or luminous efficiency, is not a constant like the ratio of feet to inches, but is variable, having a different value for every different wave length, in the visible spectrum and falling to zero outside the visible spectrum. "Luminous energy" should not therefore be used as synonymous with luminous flux.

3. DEFINITION OF INTENSITY FOR UNSYMMETRICAL SOURCES

For a point source the intensity I has been defined as the total flux F divided by 4π . If the source is not symmetrical, but sends out a total luminous flux F unequally in different directions, then the mean value of the intensity is called the *mean spherical intensity*, and its value is

$$I_s = \frac{F}{4\pi} \quad (5)$$

We thus define the mean spherical intensity with respect to the total flux; and similarly, the intensity I in any particular direction is the ratio of the flux through a small solid angle in that direction to the angle. Thus

$$I = \frac{F}{\omega}, \quad \omega \text{ being a solid angle,} \quad (6)$$

$$\text{or, } I = \frac{dF}{d\omega}, \quad d\omega \text{ being an infinitesimal solid angle.}$$

Thus the *intensity* I is the *angular density* of the flux as the *illumination* E is *surface density*. Therefore both E and I are flux ratios, lumens per unit area and lumens per unit solid angle respectively. One lumen per square meter is the *lux*, and one lumen per unit solid angle is a candle.

In the case of a point source or unit sphere radiating equally in all directions, the intensity I is defined as the flux through a unit of solid angle, or steradian. That is, $I = F$ when $\omega = 1$. This is an angle subtended by $\frac{1}{4\pi}$ of a spherical surface, and in the case

where the solid angle is a circular cone, its section through the apex is a plane angle of $65^{\circ} 32' 28''$.

4. UNIT DISK

Concerning a body charged with electricity, we have the two ideas, (1) the electricity of density σ and total quantity $Q = \int \sigma dS$ on the surface of the charged body, and (2) the flux of force throughout the surrounding space, there being $4\pi Q$ lines of force for a quantity Q of electricity. We do not believe in the fluid theory of electricity in the same way that Franklin did, but we nevertheless find the idea of a surface density of electricity very useful. In the corresponding case with light we may have similarly two distinct ideas, (1) a surface distribution of light over a luminous area of *brightness* or *specific quantity* b , and total quantity $Q = \int b dS$, and (2) a luminous flux filling the surrounding space and producing an illumination E on any body equal to the flux per unit of area, or $E = \frac{F}{S}$.

We have so far defined illumination and intensity in terms of the flux. Let us now obtain their values in terms of the *quantity of light on the surface of the luminous source*.

The illumination from a very small source is inversely proportional to the square of the distance from the source and directly proportional to the brightness of the source. Hence for a luminous plane of area dS (Fig. 1) we may write

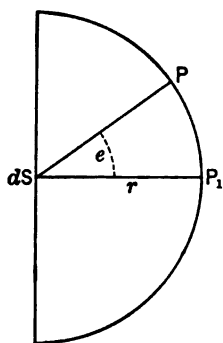


Fig. 1.

$$E_{P_1} = \frac{Q}{r^2} = \frac{bdS}{r^2} \quad (7)$$

where Q is the total quantity of light on the disk, and the radiation to P_1 at a distance r is normal. For a point P at an angle e from the normal the illumination would be

$$E_P = \frac{bdS \cos e}{r^2} = \frac{Q \cos e}{r^2} \quad (8)$$

The total flux over the hemisphere illuminated by the disk is

$$F = \int_0^\pi \frac{1}{2} E_p 2\pi r^2 \sin e \, de = Q \int_0^\pi \frac{2\pi r^2 \sin e \cos e \, de}{r^2}$$

$$F = \left[\pi Q \sin^2 e \right]_0^\pi = \pi Q \quad (9)$$

Thus the total luminous flux F from a small plane disk is π times the quantity of light Q on the disk.⁴

The average illumination over the hemisphere of radius r is $\frac{F}{2\pi r^2} = \frac{1}{2} \frac{Q}{r^2}$, whereas the maximum illumination E_n normal to the disk is $\frac{Q}{r^2}$. Thus the mean is half the maximum. The intensity I has been defined as the *angular rate of flux* in any particular direction. It is, therefore, proportional to the illumination produced in the given direction. Thus in the case of the luminous disk we have

$$I_n = \text{maximum intensity, (normal)} = Q$$

$$I_h = \text{mean hemispherical intensity} = \frac{Q}{2} \quad (10)$$

$$I_s = \text{mean spherical intensity} = \frac{Q}{4}$$

$$\text{Thus } F = \pi I_n = 4\pi I_s \quad (11)$$

That is, the intensity is numerically equal to the total quantity of light on the small disk *for all points on the normal*. It decreases to zero as we pass 90° away from the normal, having a mean value of half the maximum for the whole hemisphere, and is on the average only one-fourth the maximum for the whole sphere. We may, therefore, say that the *hemispherical reduction factor* for the disk is one-half, and the mean *spherical reduction factor* is one-fourth, the disk being supposed luminous on one side only.

⁴ In electrostatics the total flux is 4π times the quantity Q . The difference is due, first, to the fact that the luminous disk is supposed luminous on one side only and hence there is radiation only on one side, whereas the electric flux would be on both sides; secondly, the cosine law makes the average flux only half what it would be if the factor $\cos e$ were omitted.

Since the total flux F from an area is πQ , where Q is the quantity of light on the area, the flux from a unit of area is πb . This is the radiation E' . Hence, in general

$$E' = \pi b \quad (12)$$

For a small sphere of radius a , the total flux is

$$\begin{aligned} F &= E' \times \text{surface} \\ &= \pi b \times 4\pi a^2 = \pi Q \end{aligned}$$

$$\text{Also, } F = 4\pi I$$

$$\therefore I = \frac{Q}{4} \quad (13)$$

That is, for a unit sphere⁵ the intensity is one-fourth the quantity of light on the sphere. If the distribution of light over the sphere is not uniform, the *mean spherical intensity* is still one-fourth the total quantity of light on the sphere, as it is also for a disk. In other words, a sphere produces the same illumination at a given point as a disk of the same diameter and same brightness placed so that the radiation from the disk to the point is normal.

5. EXTENDED SOURCE—CIRCULAR DISK

Let dS be an element of a plane radiating surface of brightness b , defined by the equation

$$Q = bS$$

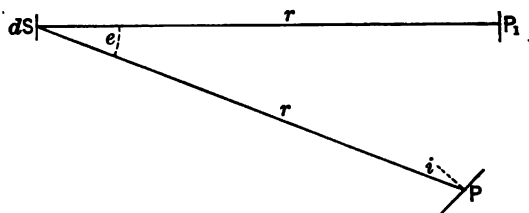


Fig. 2.

That is, the quantity of light Q is equal to the product of b into the surface S ; b , is the value of the quantity Q when the surface is unity, and is the quantity of light per unit of area measured in

⁵ By unit sphere or unit disk we mean a disk or sphere whose linear dimensions are negligible in comparison with the distance from source to receiver.

candles. Thus, the intensity I of such a source (or of any source) would be measured by comparing it experimentally with a standard light source, and it is equal to the intensity of a point source or unit sphere which produces the same illumination on a given test screen (of a photometer). Thus, while we *define* the intensity of a light source as the luminous flux per unit solid angle, we *determine* it by comparison with a concrete standard by means of the illumination produced on a test screen at a convenient distance, using a photometer and employing the law of inverse squares.

In Fig. 2 the illumination at P_1 in the normal to dS is

$$E_1 = \frac{bdS}{r^2}$$

while the illumination at P , the angles of emergence and incidence being e and i respectively is

$$E_2 = \frac{bdS \cos e \cos i}{r^2} \quad (14)$$

The cosine law is assumed to hold exactly for both surfaces.

To calculate the illumination due to a large circular disk of brightness b and any radius a on a small plane area P_1 , normal to the axis of the disk and situated on the axis at distance r from the disk (see Fig. 3) we integrate the effect of each elementary circular ring of the disk. Thus, in equation (14), putting $dS = 2\pi x dx$,

$$E = b \int_0^a \frac{2\pi x dx \cos e \cos i}{(r^2 + x^2)}$$

Since $\cos e = \cos i = \frac{r}{\sqrt{r^2 + x^2}}$,

$$E = \pi b \int_0^a \frac{2x dx \cdot r^2}{(r^2 + x^2)^2} \quad (15)$$

$$= \pi b \left[\frac{-r^2}{r^2 + x^2} \right]_0^a = \pi b \left[1 - \frac{r^2}{r^2 + a^2} \right]$$

$$\text{or, } E = \frac{\pi b a^2}{r^2 + a^2} = \frac{bS}{r^2 + a^2} = \frac{Q}{r^2 + a^2} \quad (15')$$

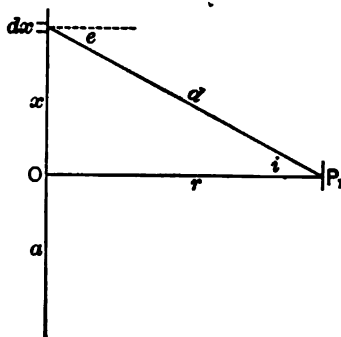


Fig. 3.

where Q is the product of the surface of the disk into the brightness b , and is the total quantity of light upon the disk measured in candles. If the disk were very small Q would be the same as the maximum intensity I_n of the source; but for an extended source we must distinguish between the equivalent intensity I_e and the surface integral of the brightness b , which is Q . The latter we have called the quantity of light upon the disk; it is proportional to the total luminous flux F coming from the extended source, and is equal to F/π , equation, (9). Q and F really measure the same thing, except that Q is located on the source and is measured in candles, while F is located in the surrounding space and is measured in lumens; their ratio is constant as $F = \pi Q$ always.⁶

In the case of the disk above mentioned, the illumination E on a small plane normal to the axis is proportional to the total quantity of light Q on the extended source (the circular disk) and inversely proportional to the square of the distance d from P_1 to the edge of disk. This holds true for all distances r from zero to infinity. Thus the law of inverse squares holds generally for the illumination along its axis due to a circular disk of any size, but the distance is measured, not to the center of the disk, but to the edge.

Thus we have

$$E = \frac{I}{r^2} \text{ for a point source or a unit disk,}$$

⁶ The total quantity of electricity on a disk of area S is equal to the integral of the surface density σ over the area. Thus

$$Q = \int \sigma dS$$

$$= \sigma S \text{ when } \sigma \text{ is uniform.}$$

The brightness b of a source corresponds to the surface density of electricity σ , and the total quantity of light over a surface is, in the same way, the surface integral of b . Thus

$$Q = \int b dS$$

$$= bS \text{ when } b \text{ is uniform over the area } S.$$

In the case of a sphere, the surface $S = 4\pi a^2$. Therefore, for a spherical source $Q = 4\pi a^2 b$, whereas the intensity $I = \pi a^2 b$. That is, the intensity I of a spherical source is one-fourth of Q , and is equal to the light on a disk of radius a and brightness b . That is, the intensity of the sphere is equivalent to that of a disk of the same diameter and the same brightness for points at a great distance.

and $E = \frac{Q}{d^2}$ for an extended disk. (16)

To illustrate the rate of variation of the illumination with the distance, let $a = 1$, $r_1 = 1$, $r_2 = 5$ (Fig. 4).

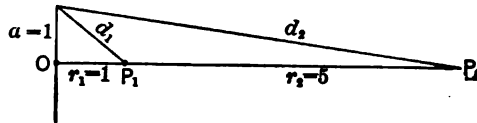


Fig. 4.

In the first case for the point P_1 , $E_1 = \frac{Q}{d_1^2} = \frac{\pi b}{2}$

In the second case for the point P_2 , $E_2 = \frac{Q}{d_2^2} = \frac{\pi b}{26}$

Thus in the first case the distance is 5 times less and the illumination is 13 times more instead of 25 times more, as it would be if the light Q were all concentrated at the center of the disk. If $r = 0$, the illumination is πb or twice as much as at P_1 , and not infinite as it would be at zero distance from a point source.

This theorem is useful in measuring the radiation from walls, as the radiating area may be quite large and the photometer relatively near.

6. INFINITE PLANE

The radiation from an infinite plane S (Fig. 5) upon a unit area of a parallel plane T is found by integrating equation (15) to infinity. Thus

$$E = \pi b \int_0^{\infty} \frac{2x dx \cdot r^2}{(r^2 + x^2)^2} = \pi b \left[\frac{-r^2}{r^2 + x^2} \right]_0^{\infty} = \pi b \quad (17)$$

Thus the flux density or *illumination* at any point P on the T plane is π times the brightness b on the radiating plane S and is independent of the distance r .

From each unit of area of S having a brightness b the total flux is πb , as shown above. The resultant flux at all points is the same as though the total flux πb from each unit of area of S was confined to a cylindrical

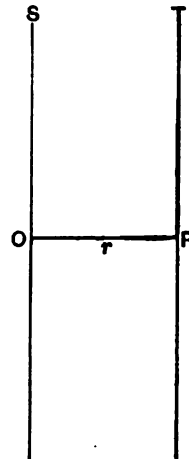
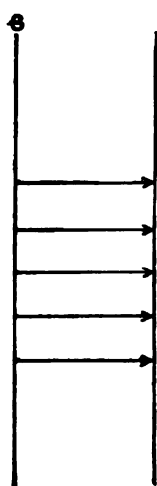


Fig. 5.

tube of unit area perpendicular to S , in which case the flux density would, of course, be constant at all sections—that is, at all distances (see Fig. 6).

7. INFINITE CYLINDER



In a similar manner we may consider the flux from an infinite circular cylinder of uniform brightness b and radius a .

The flux coming from unit length of the cylinder is πb times the area. Hence $F = 2\pi^2 ab$, whereas the flux falling on the inner surface of a concentric cylinder of radius r is E times the area, E being the illumination. Hence for a unit of length of the cylinder $F = 2\pi r E$. Therefore,

$$E = \frac{\pi ab}{r} = \frac{1}{2} \frac{Q}{r} \quad (18)$$

Thus the illumination due to an infinite cylinder varies *inversely as the distance*. This is interme-

Fig. 6. diate between the case of the point source, for which E varies inversely as r^2 , and the infinite plane, where E is independent of the distance—that is, proportional to r^0 .

The quantity Q for the luminous cylinder is b times the surface. Therefore the *quantity per unit of length* is

$$Q_1 = 2\pi ab \quad (19)$$

The total luminous flux F , as stated above, is $2\pi^2 ab$. Hence the total flux per unit of length F_1 is π times the quantity, or

$$F_1 = \pi Q_1$$

or, for any portion (or the whole) of an infinite cylinder of uniform brightness, the total flux is π times the quantity; that is,

$$F = \pi Q \quad (20)$$

as shown above for a circular disk.

8. UNIT LENGTH OF CYLINDER

Suppose a light source in the form of a very long cylinder of radius a and uniform brightness b . It is desired to determine experimentally its total luminous flux F . Suppose one has measured by means of a photometer the equivalent intensity I_1 of unit length of the cylinder (screening the photometer from all but a short section of the cylinder). We are to calculate the total flux F_1 from I_1 . The unit length of cylinder will produce the same illumination at a distance as a rectangular plane of breadth $2a$ and height unity of brightness b equal to that of the surface of the cylinder. Hence the equivalent intensity I_1 is equal to $2ab$ and the illumination produced on a photometer screen at distance r is

$$E = \frac{2ab}{r^2} = \frac{I_1}{r^2}$$

The quantity of light on the cylinder per unit of length is b times the surface, or $2\pi ab$, and the total flux F_1 is π times the quantity. Thus we have

$$\begin{aligned} F_1 &= 2\pi^2 ab \\ I_1 &= 2ab \\ \therefore F_1 &= \pi^2 I_1 \end{aligned} \quad (21)$$

Thus, to obtain the total luminous flux F_1 from the measured value of the equivalent intensity of a unit of length of the luminous cylinder we multiply this intensity I_1 by π^2 instead of multiplying by 4π , as in the case of a sphere.

The spherical reduction factor of a short cylinder (the convex surface only being luminous) is therefore $\pi^2/4\pi = \pi/4 = 0.785$. This would be nearly true for an incandescent lamp having one or more straight filaments. The value for a hairpin filament would be only slightly larger. Tantalum and tungsten lamps have reduction factors of about this value.

If the cylinder is long, we should then get the total flux F by multiplying F_1 by the length of the cylinder. This demonstration is of course based on the assumption that the cosine law holds for the cylinder. If the source is a long tube, like the Moore light, the result would be subject to any modification dependent on its departure from the cosine law.

Thus while the total flux F is always π times the quantity Q of the source, it is not always 4π times the intensity. It is 4π times the intensity I for a point source or sphere, $\pi^2 L$ times the equivalent intensity I_1 (measured at a relatively great distance) of unit length of a long cylinder, L being the length, and πS times the equivalent intensity I_1 of unit of area of a plane, S being the area of the plane.

It is, however, always 4π times the *mean spherical intensity* of the given source. The illumination produced by a short cylinder is approximately inversely proportional to the square of the distance. For all distances greater than five times the length, the departures are not greater than 0.2 per cent in a particular case worked out by Hyde; the diameter of the cylinder in this case was one-tenth the length. The exact expression for the illumination due to a finite cylinder is not simple, and the calculation is tedious.

9. CASE OF LARGE SPHERE

If a surface S_1 (supposed a portion of a spherical surface of radius r_1) has a brightness b and subtends a small solid angle ω ,

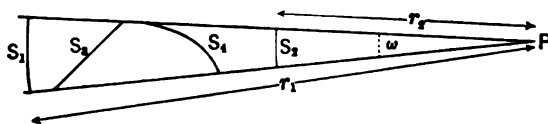


Fig. 7.

the illumination which it produces at P (Fig. 7) is

$$E = \frac{bS_1}{r_1^2} = \frac{b\omega r_1^2}{r_1^2} = b\omega \quad (22)$$

A second surface S_2 of the same brightness will produce the same illumination at P provided it subtends the same angle ω . A third surface, S_3 , at any angle will also produce the same illumination at P if it has the same brightness b and subtends the same solid angle ω .

For the radiation of each element dS_3 is $\frac{bdS_3}{r_3^2} \cos e = \frac{b\omega r_3^2}{r_3^2} = b\omega$ as before. So also with the curved surface S_1 . In every case the greater distance from P or the inclination of the angular position is compensated by the greater area included within the given (small) solid angle.

Let us calculate the illumination at P due to a large luminous sphere of radius a and brightness b , r being the distance from P (Fig. 8) to the center of the sphere. Let the solid angle APB subtended at P by the sphere be subdivided into a large number of elementary solid angles. Each of the latter incloses an area, as S_1 , on the surface of the sphere, and also a corresponding area S_1' , on the circular disk AB . As we have just seen, the illumination produced at P by each spherical area S_1, S_2 , etc., is exactly the same as that produced by the corresponding plane areas S_1', S_2' , etc., of the disk, if the brightness b is the same for the disk as for the sphere. Therefore the illumination at P due to the entire sphere is the same as that due to the disk AB , and we can calculate the latter by formula (15). That is,

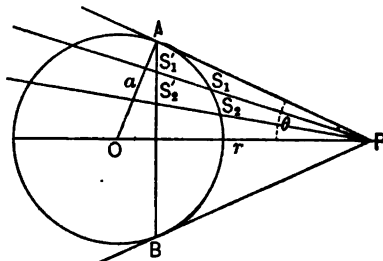


Fig. 8.

$$E = \frac{Q}{d^2}$$

where Q is the quantity of light on the disk and d is the distance AP from the point P to the edge of the disk. Q is equal to b times the area of the disk, or

$$Q = \pi(a \cos \theta)^2 b$$

$$d = r \cos \theta$$

$$\therefore E = \frac{Q}{d^2} = \frac{\pi a^2 b}{r^2} \quad (23)$$

$$= \frac{1}{4} \frac{Q_s}{r^2} = \frac{I_s}{r^2}$$

where Q_s is the quantity of light on the sphere $= 4\pi a^2 b$ and is constant for all distances, and I_s is the intensity of the equivalent point source. Therefore the illumination produced by a *sphere of any size* is inversely proportional to the square of the distance measured *from its center*, and is equal to the intensity of a point source (or unit sphere) having the same total amount of light

divided by the square of the distance. In other words, the inverse square law holds just as rigorously for large spheres as for points (always of course assuming the cosine law to hold for the spherical surfaces, and the brightness b to be uniform over the sphere).

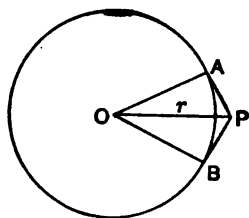


Fig. 9.

When P comes very near to the surface the area AB of the sphere (Fig. 9) available for illuminating P is very small, but the distance is just enough less to counterbalance. When P comes up to the surface, $r = a$, and

$$E = \pi b$$

the same as for an infinite plane, to which the sphere is equivalent when the distance from the surface is reduced to zero.

The same result is reached more simply as follows:

A luminous sphere of radius a and uniform brightness b gives off a total flux $F = 4\pi a^2 \times \pi b = 4\pi^2 a^2 b$. This produces an illumination on the inner surface of any concentric sphere, which by symmetry will be everywhere the same, and $F = 4\pi r^2 E$.

$$\therefore E = \frac{\pi a^2 b}{r^2} = \frac{I}{r^2}$$

Therefore the illumination produced by a sphere of uniform brightness is inversely proportional to the square of the distance from the center for all distances from the surface of the sphere to infinity.

10. RECIPROCAL RELATIONS

From what precedes we see that the illumination at any point P due to the hollow hemisphere ACB (Fig. 10) is the same as that due to the circular disk AOB . The latter is

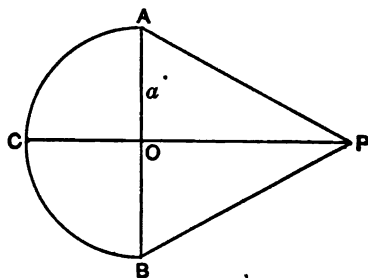


Fig. 10.

$$E = \frac{\pi a^2 b}{AP^2} \quad (24)$$

When OP is reduced to zero the illumination due to the disk is πb , and hence the illumination at O on an elementary plane area in

the diametral plane is π times the brightness b of the surface of the sphere. We have already seen that the total flux from a unit of surface of brightness b is πb . Hence the total flux through unit area S at O , due to the hemisphere, is equal to the total flux through the hemisphere due to the luminous unit area S , the brightness b being the same in each case.

This is a particular case of a more general proposition, namely, *the flux due to any surface S passing through an element dS is equal to the flux due to the latter passing through the former, the brightness being the same in each case.*

As shown above, the illumination E at P_1 , due to S_1 (Fig. 11) is equal to $b\omega$, where b is the brightness of S_1 and ω is the (small) solid angle subtended at P_1 by S_1 ; this is independent of the shape of S_1 or its distance from P_1 . The flux F passing through dS at P_1 is therefore

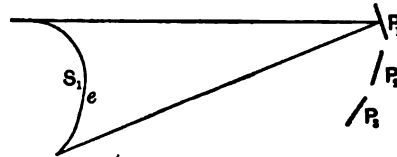


Fig. 11.

$$F = \int b d\omega dS \cos \theta, \text{ over the area of } S_1$$

Or,

$$F = b dS \int d\omega \cos \theta \quad (25)$$

Similarly, the flux due to dS at P_1 passing through S_1 is

$$\begin{aligned} F &= \int b dS \cos \theta d\omega \\ &= b dS \int \cos \theta d\omega \end{aligned}$$

In the integration every element $d\omega$ of the solid angle is to be multiplied by the cosine of the angle it makes with the normal to the area dS .

As the same theorem holds for the elementary areas P_2 and P_3 , etc., it holds for their sum, and hence for a finite surface S_2 (Fig. 12). Hence we see generally that *the luminous flux due to a surface*

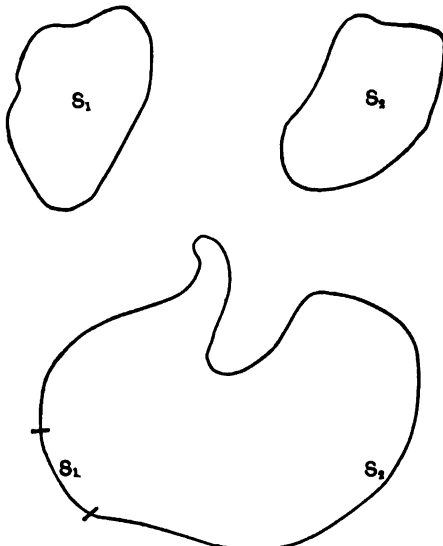


Fig. 12.

S_1 passing through S_2 is equal to the luminous flux due to S_2 passing through S_1 , the brightness being the same in each case. This is analogous to the theorem that the magnetic flux due to a magnetic shell S_1 , which passes through a second shell S_2 , is equal to that part of the magnetic flux of S_2 which passes through S_1 , the strength of the shells being supposed the same. Or, again, the number of lines of force due to unit current in an electric circuit S_1 passing through S_2 is equal to the number of lines of force due to unit current in S_2 passing through S_1 . It follows from the above that in any closed surface of uniform brightness the flux passing out from any portion S_1 is equal to that received from the remainder of the surface S_2 .

11. HOLLOW SPHERE⁷

Suppose a hollow sphere (Fig. 13) of uniform surface having a coefficient of diffuse reflection m ,

$$1 - m = \text{absorption.}$$

Let E = illumination at S .

$$E' = mE = \text{radiation from } S.$$

$$b = \frac{mE}{\pi} = \text{brightness of } S.$$

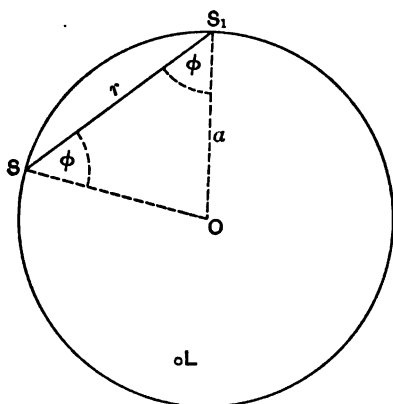


Fig. 13.

The flux falling on S_1 due to S is

$$S_1 dE_1 = \frac{eSS_1 \cos^2 \varphi}{r^2} = \frac{mE}{\pi} \cdot \frac{SS_1 \cos^2 \varphi}{r^2} \quad (26)$$

$$\left. \begin{aligned} \text{But } r &= 2a \cos \varphi \\ r^2 &= 4a^2 \cos^2 \varphi \\ \frac{\cos^2 \varphi}{r^2} &= \frac{1}{4a^2} \end{aligned} \right\} \therefore dE_1 = \frac{mE}{\pi} \cdot \frac{S}{4a^2}$$

and this is the same for every element of the sphere. Hence every element illuminates all other elements equally. Therefore the indirect illumination of the sphere must be the same everywhere, no matter how unequal the direct illumination may be. That is,

⁷ See Liebenthal, *Praktische Photometrie*, p. 301.

a light at L illuminates the sphere unequally, directly. But that part of the total illumination due to diffuse reflection is, notwithstanding, everywhere equal.

A light of mean spherical intensity I sends out $4\pi I$ lumens.

Of this there is reflected, 1st, $4\pi mI$ lumens.

“ “ “ “ “ 2d, $4\pi m^2I$ lumens.

“ “ “ “ “ 3d, $4\pi m^3I$ lumens, and so on.

Therefore total amount of flux reflected is

$$4\pi Im[1 + m + m^2 + m^3 + \dots] = 4\pi I \frac{m}{1 - m} = F_2$$

Hence the secondary illumination everywhere equal on the surface of the sphere is

$$E_2 = \frac{F_2}{4\pi a^2} = \frac{mI}{(1 - m)a^2} \quad (27)$$

Thus the indirect illumination is proportional to I , and the lamp of intensity I may be anywhere in the sphere. It is equal to $\frac{m}{1 - m}$ of what the direct illumination would be if the source were placed at the center of the sphere. For example, let a 16 candle-power lamp be placed within a sphere having a radius of 1 meter and a coefficient of diffuse reflection of 0.8.

Then $I = 16$

$a = 1$ meter

$m = 0.8$

$$E_2 = \frac{0.8}{0.2} \frac{16}{1} = 64 \text{ meter-candles}$$

$$E_1 = \frac{I}{a^2} = 16 \text{ meter-candles, if lamp is in the center}$$

$$E = E_1 + E_2 = 80$$

Thus the total illumination is five times what it would be if the walls were perfectly black. We can put this in another way: Of the total illumination of 80 meter-candles 20 per cent is absorbed by the walls. Therefore the lamp or source must supply only one-fifth of the total, just enough to make good the constant loss.

Thus the source is analogous to an exciter of electric waves that must supply just enough energy to make good the resistance losses in the circuit.

12. LUMINOUS FLUX WITHIN AN INCLOSURE

If the inner surface of the hollow sphere has a brightness b and a specific radiation $E' = \pi b$, a unit disk at the center of the sphere will receive an illumination $E = \pi b$, page 559. The same will be true wherever the unit disk is placed within the sphere and whatever the orientation of the disk; that is, the flux falling on the disk will be everywhere the same. The flux density within the hollow sphere is therefore everywhere uniform and equal to πb . The flux from a point source is thought of as in straight

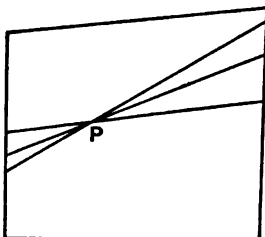


Fig. 14.

lines, and a disk can be placed normal to the direction of the flux. But within the sphere the flux has a uniform value, but no resultant direction.

Within a cube or enclosure of any shape, of which the walls have a uniform brightness b or uniform specific radiation E' the same condition obtains as in the sphere—namely, the luminous flux is everywhere the same, and a small area will have the same illumination no matter where it is placed or how it is oriented. This is seen by dividing up the space about any point P (Fig. 14) into elementary solid angles. The illumination due to the surface subtending an angle ω is independent of the distance from P , and hence it will be πb for the total angle 2π on either side of the surface at P , no matter where the surface is placed.

The same is true, therefore, for the space between two infinite planes of brightness b . The illumination is πb on a small plane at P_1 , P_2 , or P_3 (Fig. 15), anywhere between the two radiating planes S and T no matter how they may be placed. Evidently we can not think of the flux as normal to the planes, as with the lines of force due to electrostatic charges on the planes S and T . The luminous flux normal to P_3 is the same as normal to P_1 . On the other hand, the electric force normal to P_3 would be zero.

These theorems have a practical application in the lighting of rooms.

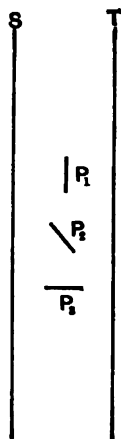


Fig. 15.

III. SUMMARY OF PHOTOMETRIC RELATIONS

The preceding discussion has shown the necessity for distinguishing several different photometric quantities which are sometimes confused. In order to fix our ideas more clearly, it will be advantageous now to state as concisely as possible the definitions of the several quantities and distinctions between them.

Luminous flux, or *light* as the term is used in photometry, is the usual physical stimulus which excites vision. It is propagated by means of the vibratory motion in the ether, and the frequency of the vibrations or the combination of frequencies present in any given case determines the color. The total quantity of flux F flowing away from a monochromatic luminous source is proportional to the total radiant energy (per second) and to a stimulus coefficient, the latter being the luminous efficiency K_λ for the particular frequency or wave length of the given radiation. Thus the equations

$$F = K_\lambda W$$

$$K_\lambda = \frac{F_\lambda}{W_\lambda}$$

express the luminous flux as the power W multiplied by the luminous efficiency K_λ , and if flux is expressed in *lumens* and the power in *watts*, the luminous efficiency is the number of lumens per watt of radiation of the wave length λ . For white or chromatic light K will have a value depending on the distribution of the energy in the spectrum. It is a maximum in the yellow-green region and falls off rapidly in either direction, reaching zero at the limits of the visible spectrum. The luminous efficiency of most light sources is greatly reduced by the amount of radiation outside the visible spectrum, chiefly of longer wave length than that of visible radiation, and the total efficiency of such a source

$$K = \frac{F}{W}$$

is the quotient of the total luminous flux divided by the total radiant power.

For the purposes of definition and of expressing the mathematical relations involved in photometry, it is permissible to confine ourselves to monochromatic light and to consider K a constant,

although it does in fact vary somewhat with the magnitude of the flux density. We also assume that all surfaces are perfectly diffusing and obey the cosine law and that there is no absorption in the atmosphere.

The intensity of a point source or uniform luminous sphere is measured by the luminous flux flowing through a unit solid angle whose apex is the given point or center of the given sphere. Thus from a source of intensity I light is flowing away at a rate of I lumens per unit solid angle or a total of $4\pi I$ lumens for the point source or uniform sphere. If the source is not uniform and light is flowing away at unequal rates in different directions, the intensity I in any direction is equal to the flux dF in an elementary solid angle $d\omega$ taken in the given direction. Thus

$$I = \frac{dF}{d\omega}$$

is a general expression applying to all point sources whether radiating equally or unequally in different directions. If the unsymmetrical source is extended—as, for example, an incandescent lamp or a diffusing globe—the same holds true if the distance at which the measurements are made are sufficiently great so that the distribution of light is practically the same as from an unsymmetrical point source. For less distances than this the intensity is not a constant in a given direction, but varies with r . In this case the equivalent intensity at any point is equal to that of a point source which gives the same flux density, or *lumens* per sq cm, at the point that the given source does. The mean spherical intensity I_s is the average value of the intensity and is equal to the total flux F divided by 4π .

The total flux from a given extended source is therefore a constant independent of distance, as is also the mean spherical intensity I_s . The intensity I in a particular direction, however, in the case of extended sources other than spheres varies with the distance, but at relatively great distances the variation is inappreciable. Thus the luminous flux is the fundamental quantity. But while we *define* I as the flux per unit solid angle, or *rate of flux with respect to solid angle*, we *determine* I by comparison with a concrete standard. Thus *photometric standards of intensity are standards of light flux, their values being expressed in candles.*

If f is the spherical reduction factor with respect to any particular direction, and I is the intensity of a source in that direction,

$$I_s = fI$$

For a unit disk—that is, a small circular disk of uniform brightness—the total flux is π times the normal intensity I_n , whereas the mean spherical reduction factor with respect to the normal is one-fourth. Hence, the total flux is

$$\begin{aligned} F &= \pi I_n \\ &= 4\pi I_s, \text{ as for a sphere.} \end{aligned}$$

In general, for any light source, $F_s = 4\pi I_s = 4\pi fI$, but for extended sources other than spheres, the value of f as well as I varies with the distance from the source for points relatively near the source.

The *specific flux* or *flux density* is the luminous flux per unit of area, or lumens per square centimeter. When the flux falls upon a material surface, we call the specific flux the *illumination*, E . When we speak of the flux coming *from* a surface, whether it be a self-luminous source at high temperature or a reflecting or radiating surface at low temperature, we call the specific flux the *specific radiation*, or simply the *radiation*, E' .

Thus the illumination E is

$$E = \frac{F_i}{S} = \frac{dF_i}{dS} = \frac{I}{r^2}$$

The radiation E' is

$$E' = \frac{F_e}{S} = \frac{dF_e}{dS}$$

F_i is the incident flux, F_e is the emitted or radiated flux. If m is the coefficient of diffuse reflection or transmission, $(1-m)$ being the absorption,

$$\begin{aligned} F_e &= mF_i \\ E' &= mE \end{aligned}$$

where the radiation consists in the diffuse reflection or transmission of a portion of the incident flux or illumination.

The radiation or illumination when large may be expressed in lumens per sq cm; when small in milli-lumens per sq cm. The milli-lumen per sq cm is nearly equivalent to the foot candle.

$$\begin{aligned} 1 \text{ lumen per sq cm} &= 10\,000 \text{ lumens per sq meter.} \\ &= 10\,000 \text{ meter-candles.} \end{aligned}$$

$$\therefore 1 \text{ milli-lumen per sq cm} = 10 \text{ meter-candles} = 10 \text{ lux.}$$

$$= \frac{1}{1.0765} \text{ foot-candles.}$$

The *brightness* b of a source is the intensity in candles per sq cm of area, taken normally. Thus

$$b = \frac{I}{S} = \frac{dI}{dS} = \frac{dQ}{dS}$$

Brightness, or *specific quantity*, refers to the quantity of light per unit of area of a source, and is measured in candles per sq cm. Brightness can refer equally to luminous sources of relatively high specific intensity or to reflecting and radiating sources of low intensities. The latter may be conveniently expressed in milli-lumens per sq cm. Thus we may say a flame has a specific radiation of 10 lumens per sq cm or a brightness of 0.8 candles per sq cm; and a wall has a specific radiation of 10 milli-lumens per sq cm, or a brightness of 0.8 milli-candles per sq cm or of 8 candles per sq meter.

The quantity Q is proportional to the total amount of light emitted by the source, and is equal to the surface integral of the brightness b . Thus

$$Q = \int b dS.$$

The quantity for a small luminous circular disk of radius a and uniform *brightness* b is

$$Q = \pi a^2 b = I_n$$

That is, the quantity is equal to the maximum intensity. In this case the whole surface is equally effective in producing the illumination on the test screen by which the intensity I_n is measured. But for an extended disk, the quantity and the normal intensity, as we have seen above, are not the same. Thus, the quantity is b times the surface, or

$$Q = \pi a^2 b$$

$$E_n = \frac{Q}{a^2 + r^2} = \frac{I_n}{r^2}$$

$$\therefore \frac{I_n}{Q_1} = \frac{r^2}{a^2 + r^2} = \frac{r^2}{d^2} = \cos^2 \theta$$

That is, the *normal equivalent intensity* I_n of the disk (Fig. 16) with respect to the point P on the axis of the disk is Q times $\cos^2 \theta$. When the distance is equal to the radius of the disk, the quantity Q is twice the normal intensity I_n .

The total luminous flux is $\pi b S$ or π times the quantity, and the

mean hemispherical intensity is $\frac{Q}{2}$ or half the quantity.

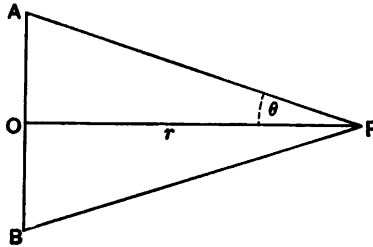


Fig. 16.

In the case of a sphere of uniform brightness b the quantity is $\int b dS = 4\pi a^2 b$. The intensity $I = \pi a^2 b$. Hence the intensity is one-fourth the quantity. In other words, the total radiation from the sphere is four times as great as from a unit disk of the same normal intensity. The relations between quantity and intensity for a few simple cases are as follows:

For a unit disk $I_n = Q$.

For an extended circular disk $I_n = Q \cos^2 \theta = Q \frac{r^2}{d^2}$.

For a sphere $I = \frac{1}{4} Q$.

For a unit cylinder $I_1 = \frac{1}{\pi} Q$.

The total luminous flux delivered in a given time—that is, the time integral of the luminous flux—may be expressed in lumen-seconds or lumen-hours, according to circumstances. Thus, putting L for the *total lighting* in the time T

$$\begin{aligned} L &= FT \\ &= \int F dT, \text{ if } F \text{ is variable} \end{aligned}$$

where F is in lumens and the time is expressed in the most convenient unit. The flash of a firefly may be expressed in lumen-

seconds; the total *lighting* per gram of an illuminant, or the total lighting given during the life of an incandescent lamp, is better expressed in lumen-hours.

Since flux of light may also be expressed in spherical candles ($\frac{1}{4\pi}$ times the lumens), we may also express the time integral or total lighting in terms of spherical candles and hours. Thus $L = I_s T = \int I_s dT$, if the spherical candlepower is a variable with respect to T , the value of L being here given in candle-hours.

The photometric quantities employed in the preceding discussion are shown in Table I, together with the units in which they are expressed and the equations of definition.

TABLE I

Photometric Magnitude	Symbol	Unit	Equation of Definition
1. Intensity of light	I	Candle	$I = \frac{F}{\omega}$
2. Luminous flux	F	Lumen	$F = I\omega = \frac{IS}{r^2} = ES = \pi Q$
3. Illumination	E	$\left\{ \begin{array}{l} \text{Lumens} \\ \text{cm}^2 \end{array} \right.$ or $\left\{ \begin{array}{l} \text{milli-lumens} \\ \text{cm}^2 \end{array} \right.$	$E = \frac{F_t}{S} = \frac{I}{r^2}$
4. Radiation	E'	$\left\{ \begin{array}{l} \text{Lux} = \text{meter-candle} \\ \text{Candles} \\ \text{cm}^2 \end{array} \right.$	$E' = \frac{F_e}{S} = \pi b = mE$
5. Brightness	b	$\frac{\text{Candles}}{\text{cm}^2}$	$b = \frac{I}{S \cos e}$
6. Quantity	Q	Candles	$Q = bS$
7. Lighting	L	Lumen-hours	$L = FT$

I, b, Q are expressed in candles.

F, E, E' are expressed in lumens.

L is in lumens or spherical candles.

$E' = \pi b = mE$.

$F = \pi Q$.

F_t = incident flux.

F_e = emergent flux.

m = coefficient of diffuse reflection or transmission.

$(1 - m)$ = coefficient of absorption.

The symbol F has been employed for the flux (as originally proposed by Hospitalier) instead of Φ , for the following reasons:

1. Φ is the only Greek letter in the series, and it is more consistent to use a Latin letter; F is the initial letter of the word flux.

2. The letter Φ is more or less unfamiliar to many illuminating engineers and also to many printing offices, and it is often confused with the small letter ϕ , which is used for an angle.

The symbol E' is used for radiation instead of R (as proposed by Hospitalier), because it is so closely related to the illumination, and because the letter R is employed for the distance from the source. Blondel and others proposed to employ the same letter, E , for illumination and radiation, but that gives rise to confusion. On the other hand, E' gives sufficient distinction and at the same time recalls their close connection. The letter b is used for brightness instead of i for specific intensity because i is used for the angle of emission, and specific intensity is a less desirable name. The quantity is a *specific intensity* strictly only for small plane areas, not for small spheres or large sources of any form. Quantity of light, Q , is here used as the surface integral of b instead of the time integral of F . It is analogous to quantity of electricity in electrostatics and is more properly employed in the sense here used than with the other meaning. The term *lighting for flux times time* is use in harmony with the usage in France and Germany.

IV. PROBLEMS FOR ILLUSTRATIONS

Problem 1.—A lamp of 200 candlepower (supposed uniform in all directions) is placed in the center of a spherical diffusing globe of 40 cm diameter, the absorption of which is 30 per cent. Required, the intensity of the globe, its brightness, its specific radiation, the illumination on its inner surface, and the illumination it produces at a distance of 3 meters from the center of the globe.

The illumination on its inner surface is $E = \frac{I}{a^2} = \frac{200}{400} = 0.5$ lumens per sq cm (formula 1). The radiation E' is mE , where m is one minus the absorption; it is here 0.7. Therefore the radiation is 0.35 lumens per sq cm. The brightness b is $\frac{E'}{\pi}$ or 0.112 candles per sq cm. The intensity I of the globe is $200 \times 0.7 = 140$ candles. The illumination E at a distance of 3 meters is

$$\begin{aligned} E &= \frac{140}{300^2} = .00156 \text{ lumens per sq cm} \\ &= 1.56 \text{ milli-lumens per sq cm} \\ &= 1.45 \text{ foot-candles} \\ \text{or } E &= \frac{140}{3^2} = 15.6 \text{ meter-candles} \\ &= 15.6 \text{ lux.} \end{aligned}$$

Problem 2.—A circular area S , 2 meters in diameter, on the side of a wall is uniformly illuminated, E being 4 meter-candles. A photometer placed 1 meter from the wall, perpendicular to the center of the illuminated area, measures the equivalent intensity I_0 of the area S , and finds it to be 1 candle. What is the absorption coefficient of the wall?

The illumination E being 4 meter-candles, and the area S being π square meters, the flux F falling on the area S is 4π lumens. The measured intensity I at a distance $r=1$ meter is 1 candle. Therefore, the quantity of light on the disk is

$$Q = I_0 \frac{d^2}{r^2} = 1 \times \frac{2}{1} = 2 \text{ candles}$$

The total flux from the disk is π times the quantity Q . Therefore, the total flux coming from the area S is 2π lumens, whereas the flux falling upon it is 4π lumens. Therefore, the coefficient of absorp-

tion is $\frac{1}{2}$ or 50 per cent.

Problem 3.—Suppose a room of 900 square meters total wall surface is to be so lighted that the walls shall have an average illumination of 10 lumens per square meter, the coefficient of absorption of the walls being 40 per cent on the average. How many lamps of 15 mean horizontal candlepower will be required?

Part of the illumination will be due to light reflected from the walls. The lamps must supply that which is absorbed. The flux to be supplied is therefore $F = 0.40 \times 900 \times 10 = 3600$ lumens. If each lamp has a spherical reduction factor of 80 per cent, it will supply $4\pi \times 0.80 \times 15 = 150$ lumens, approximately. Hence, 24 lamps will be required.

(Examples 1 and 3 are borrowed from one of Blondel's papers.)

V. COLLECTION OF FORMULAS

1. $E = \frac{I}{r^2}$ for point source, unit sphere or sphere of any size.

$I = \pi a^2 b$, where a = radius of sphere and b = brightness of surface.

2. $E = \frac{\pi a^2 b}{r^2}$ for sphere of radius a .

= πb when $r = a$; that is, at the surface of the sphere, same as for an infinite plane.

3. $E = \frac{\pi a^2 b}{a^2 + r^2} = \frac{Q}{d^2}$ for disk of radius a , at distance r on axis.

d = distance of point on axis to edge of disk.

4. $E = \frac{\pi ab}{r}$ for infinite cylinder, b = brightness, a = radius.

= $\frac{Q_1}{2r}$, where Q_1 = quantity of light per unit of length

= πb at surface. $I_1 = \frac{2ab}{r^2}$ = intensity per unit of length

5. $E = \pi b$ for infinite plane, at all distances.

6. $E = \frac{bdS \cos e}{r^2} = bd\omega$ for any small surface dS subtending a small angle $d\omega$, at any distance.

7. $E = \frac{2b}{r} \cos e$, for infinitely long, very narrow strip of b units of light per unit of length
= $2 \frac{Q_1}{r}$

8. $Q = \int bdS$ over sphere, cylinder, disk or other surface where
 b = normal intensity = quantity of light per unit of area.

9. $F_s = \pi Q$, for sphere or other extended source

10. $E = \frac{Q}{d^2} = \frac{I_0}{r^2} \therefore \frac{I_0}{Q} = \frac{r^2}{d^2} = \cos^2 \theta$, for a disk.

I_0 = equivalent point source, Q = quantity of light over disk
 $I = Q/4$ for a sphere.

WASHINGTON, May 10, 1910.

The letter b is used for brightness instead of e as was done in an earlier paper in the Transactions of the Illuminating Engineering Society in accordance with the preference of the Committee on Nomenclature, following the suggestion of Prof. Blondel.

A MODIFIED METHOD FOR THE DETERMINATION OF RELATIVE WAVE-LENGTHS

By Irwin G. Priest

CONTENTS

	Page
I. INTRODUCTION	574
II. PURPOSE AND SCOPE OF THIS INVESTIGATION.....	577
III. FUNDAMENTAL PRINCIPLES OF RELATIVE WAVE-LENGTH DETERMINATION.....	578
IV. REVIEW OF PREVIOUS METHODS	579
1. Earliest Method.....	579
2. Grating Methods	580
Simple dispersion method. The method of coincidences.	
3. Interferometer Methods.....	582
Michelson's method. Fabry and Perot's method of coincidences. Fabry and Perot's method of diameters.	
V. GENERAL EXPOSITION OF THE PRESENT METHOD.....	586
1. Characteristic Features	586
Use of circular fringes in reflected light. Use of double increment in distance between mirrors as difference of path. Use of method of flexure to measure fractional part of order of interference.	
2. Abstract Outline of Procedure.....	588
Essential arrangements and conditions.....	588
Determination of order of interference for standard wave-length.....	589
Determination of order of interference for unknown wave-length.....	590
Reduction of required quantities to directly measured quantities.....	590
3. Summary of Requisite Data	591
VI. THE APPARATUS AND ITS ARRANGEMENT.....	592
VII. EXPERIMENTAL DETAILS AND PRECAUTIONS	594
1. Preliminary Adjustments.....	594
2. Course of a Single Determination.....	595
3. Special Precautions.....	596
VIII. COMPUTATION	597

	Page.
IX. TESTS OF PRECISION AND RELIABILITY.....	597
1. Definition of Terms.....	597
2. Precision.....	598
Tests and tabulated results. Conclusions as to precision.	
3. Reliability.....	600
General discussion. Test applied. Results of test. Sources of error not discovered by test applied.	
X. ADVANTAGES OF METHOD.....	603
XI. CONCLUSION.....	605

I. INTRODUCTION

For a proper understanding of the subject of this paper the reader should be familiar with the work that has been done in the past decade for the establishment of a new system of standard wave-lengths.¹

An excellent résumé of the work of Michelson and of Fabry and Perot, as well as an exposition of the circumstances of the loss of confidence in the extreme accuracy of Rowland's system of spectroscopic standards, will be found in Chapter IX of Baly's *Spectroscopy* (Longmans, 1905). The history of the organized effort that has been made since 1904 to establish an "International System of Wave-Length Normals" is recorded in the transactions of the International Union for Cooperation in Solar Research, Volumes I and II (Manchester, 1906, 1908).

In sections II-IV of this paper is presented such theoretical and historical matter as the author considers requisite to form a logical introduction to his own work. A comprehensive historical résumé has not been attempted, and it is not expected that these sections will be sufficient introduction for the reader entirely unacquainted with the previous researches noted above; on the

¹ The following are the important memoirs on this subject.

Michelson, *Trav. et Mem. du Bur. Int.*, **11**, p. 3; 1895.

Fabry and Perot, *Ann. de Chim. et de Phys.* (7), **16**, p. 289; 1899.

Fabry and Perot, *Ann. de Chim. et de Phys.* (7), **25**, p. 98; 1902.

(*Trans. in Astrop. J.*, **15**, pp. 73 and 261.)

Rayleigh, *Phil. Mag.* **11**, p. 685, 1906, and **15**, p. 548; 1908.

Buisson and Fabry, *J. de Phys.* (4), **7**, p. 169; 1908. (*Trans. in Astro. Jour.* **28**, p. 169.)

Evershein, *Zs. wiss. Photo.*, **5**, p. 152; 1907. (*Trans. in Astro. Jour.*, **26**, p. 172.)
Ann. der Physik, **30**, p. 815; 1909.

Benoit, Fabry and Perot, *C. R.*, **144**, p. 1082. *Trans. Inter. Union Solar Res.*, **2**, p. 109; 1908.

Pfund, *Astro. Jour.*, **28**, p. 197; 1908.

other hand, III and IV will be of no interest to readers already familiar with such investigations. Section V is intended to be sufficient in itself to give anyone already acquainted with this field of investigation a clear idea of the principles applied and the general procedure followed in the present method. Sections VI, VII, and VIII are taken up wholly in details of experiment and computation. They will only be of interest in case it is desired to examine the method critically or to duplicate the apparatus or procedure. Section IX presents the tests we have made of the precision and reliability of the method, and Section X summarizes the advantages that are claimed for it.

Schedule of Symbols—Symbols not defined in the immediate context where they appear should be understood as follows.

λ , wave-length.

λ_x , the unknown wave-length whose accurate value is sought.

λ_s , the standard reference wave-length, relative to which λ_x is to be determined.

λ_{xa} , the approximate value of λ_x , known before the present determination.

λ_r , the auxiliary reference wave-length, i. e., a wave-length known with an accuracy about the same as that of λ_s and used in the process of determining ΔP_{ee} .

t , perpendicular distance between interferometer mirrors.

d , difference of path.

d_o , difference of path at center of circular fringe system = (subject to a small correction) $2t$.

$n = \frac{d}{\lambda}$, order of interference.

$p = \frac{d_o}{\lambda}$, order of interference corresponding to center of circular fringe system.

N , integral part of n .

η , fractional part of n .

P , integral part of p .

ϕ , fractional part ² of p .

² ϕ as defined here is not rigorously equal to ϕ as found in the computing forms; but as ϕ only enters the computation in the form $(\phi_2 - \phi_1)$ and the errors of these two may be assumed equal, no error will be produced in the result. It is therefore not thought necessary to introduce another symbol.

t_1 and t_2 , particular values of t such that $t_1 < t_2$.

$$\Delta t = t_2 - t_1.$$

$$\Delta p = \frac{2\Delta t}{\lambda}.$$

ΔP , integral part of Δp .

$\Delta \phi$, fractional part of Δp .

Subscripts 1 and 2 indicate that the symbols to which they are appended represent particular values for the conditions $t = t_1$ and $t = t_2$, respectively. Subscripts s , r , and x indicate that the symbols to which they are appended represent the indicated quantities for the conditions $\lambda = \lambda_s$, $\lambda = \lambda_r$, and $\lambda = \lambda_x$, respectively.

t_{1a} indicates an approximate value of t_1 , never entering our computation and only given as incidental datum.

Δt_a indicates an approximate value of Δt , determined by $\Delta t_a = M_2 - M_1 + K$ where M_2 and M_1 are micrometer-microscope readings, and K is the scale correction.

Δt_e indicates the accurate value of Δt determined by $\Delta t_e = \Delta P_{se}\lambda_s$.

ΔP_{sa} indicates the approximate value of ΔP_s determined by
$$\Delta P_{sa} = \frac{2\Delta t_a}{\lambda_s}.$$

ΔP_{se} indicates the exact value of ΔP_s .

Δp_{xa} indicates the approximate value of Δp_x determined by
$$\Delta p_{xa} = \frac{2\Delta t_e}{\lambda_{xa}}.$$

Δp_{xe} indicates the accurate value of Δp_x .

$\Delta \phi_{rm}$ indicates experimental value of $\Delta \phi_r$, determined by $\Delta \phi_{rm} = \phi_{r2} \text{ (or } 1 + \phi_{r2}) - \phi_{r1}.$

$\Delta \phi_{rc}$ indicates value of $\Delta \phi_r$, determined by $\Delta \phi_{rc} = \text{fractional part of } \Delta P_{se} \cdot \frac{\lambda_s}{\lambda_r}.$

$\text{\AA}$, Ångström unit.

$$A_M, \frac{\lambda, \text{ red cadmium radiation}}{6438.4722}.$$

δ prefixed to the symbol for any quantity indicates a small error in that quantity.

E indicates probable error as defined in the theory of errors.

The quantity to which E refers is indicated by writing its symbol as a subscript to E .

This work has been done in the laboratory of Dr. S. W. Stratton, to whom is due the design of the interferometer mounting and the flexure apparatus as well as general supervision of the work. The author has also had the privilege of consulting with Dr. A. H. Pfund, of the Johns Hopkins University, who has kindly reviewed and criticised this paper while in manuscript. Our hearty thanks are due him for his interest and service in this connection. Mr. C. F. Snyder has assisted in observing and computing.

II. PURPOSE AND SCOPE OF THIS INVESTIGATION

The purpose of this research has been. (1) to develop, along the lines indicated under V, a method for the determination of relative wave-lengths, which would meet the present requirements for the establishment of secondary spectroscopic standards; and (2) to test the precision and reliability of this method.

Let us recall what the requirements are. The consensus of opinion among spectroscopists in regard to wave-length standards is: (1) that some one wave-length should be adopted as the primary standard; (2) that a large number of secondary standards distributed through the spectrum at intervals not greater than $50\text{\AA}$ should be established relative to this primary by methods as direct and simple as possible; (3) that the uncertainty in the established values of these standards should not be greater than the uncertainty incident to a wave-length determination by the method of interpolation between neighboring standards in a grating spectrum. The accuracy of this method has heretofore been given³ as about $\pm 0.005\text{\AA}$ or one part in 1 000 000. More recently it has been asserted⁴ that, under the most favorable conditions, this uncertainty is not greater than about $\pm 0.001\text{\AA}$ or, for the wave-lengths in question, one part in 4 000 000. We may then take as our ideal of accuracy one part in 4 000 000, mean-

³ Kayser, *Trans. Inter. Union Solar Research*, 1, p. 39; 1906. Fabry and Buisson, *Trans. Inter. Union Solar Res.*, 2, p. 138.

⁴ St. John. *Astro. Jour.*, 81, p. 156; March, 1910.

ing thereby that we desire the difference between the established and the true value to be certainly not greater than that amount. At the same time let us note that determinations having two or three times this uncertainty would still be of great value, although they should not be regarded as final so long as there is any possibility of improving them.

The method which I have developed, apparently satisfies the above requirements very closely and is believed to possess some peculiar advantages. The purpose of this paper is (1) to describe the method as it is now being used in this laboratory, and (2) to present the experimental evidence which has been obtained relative to its precision and reliability.

It will be convenient, first, to outline the principles upon which any determination of relative wave-length must be based.

III. FUNDAMENTAL PRINCIPLES OF RELATIVE WAVE-LENGTH DETERMINATION

Any comparison of two wave-lengths must, of necessity, be made by some interference method; and, on the other hand, the wave-length of any radiation relative to that of any other may be derived from suitable observations on some form of interference phenomenon. The fundamental equation of condition that must be used in evaluating one wave-length in terms of another is

$$n_x \lambda_x = n_s \lambda_s \quad (1)$$

The essential problem of a determination of relative wave-length is, then, the determination of two numbers, n_x and n_s . In particular, the problem is: (1) to select some interference phenomenon yielding a measurable quantity or quantities of which n is a function; (2) to set up some interferometer and auxiliary apparatus enabling these quantities to be measured when the interferometer is illuminated by lights of wave-lengths λ_x and λ_s , and the condition, $n_x \lambda_x = n_s \lambda_s$, is satisfied; (3) to measure the quantities from which n_x and n_s can be evaluated; and (4) to derive n_x and n_s from the experimental data and evaluate λ_x from

$$\lambda_x = \frac{n_s \lambda_s}{n_x} \quad (2)$$

In general, n will not be exactly integral, but will be the sum of an integer and a proper fraction. The exigencies of experiment and discussion frequently make it convenient to speak of these two parts separately. So we define

$$N \text{ is the integral part of } n \text{ and} \quad (3)$$

$$\eta \text{ is the fractional part of } n \quad (4)$$

$$\text{so that} \quad n = N + \eta. \quad (5)$$

Since the relative error, $\frac{\delta\lambda}{\lambda}$, in the measured wave-length is directly proportional to $\frac{\delta\eta}{n}$, the relative error in n , there are obviously two methods by which $\frac{\delta\lambda}{\lambda}$ can be made small, viz: (1) N may be chosen very large or (2) η may be measured with great precision. The degree of homogeneity of the light sets a natural maximum limit to N , which limit is of the order of hundreds of thousands (10^5) for even the most nearly homogeneous radiations. But long before this limit is reached the increase in N becomes antagonistic to the precise measurement of η . In all methods now available, including the present, this limitation comes to be markedly felt in the region where N is of the order 10^4 .

Variations in the procedure for obtaining n_x and n_s constitute the differences, more or less great, in methods of relative wave-length determination. Several of these methods that have been of great importance will now be mentioned.

IV. REVIEW OF PREVIOUS METHODS

1. EARLIEST METHOD

It is merely of historic interest to recall that the earliest discovered phenomenon affording the requisite data for the determination of relative wave-lengths was "Newton's Rings" (1676). The precision attainable in this classic method is limited both by the low values of N available and the comparatively great uncertainty in the measurement of η . A higher degree of precision than 0.1% would probably be unattainable adhering to this method in its simplicity.

2. GRATING METHODS

While Young actually did compute from Newton's data not only the relative, but also the absolute wave-lengths for the several spectrum "colors," we can hardly class these values as standards of wave-length; and we may say that all spectroscopic standards established prior to Michelson's evaluation of three wave-lengths in terms of the meter, were derived from observations on diffraction spectra produced by the grating. Let us outline the fundamental principles of the grating methods, considering the grating as a particular type of interferometer. If we throw the grating data for the determination of relative wave-lengths into the form of the fundamental equation (1) we have, in general, for any grating and any combination of wave-lengths and orders,

$$n_x \lambda_x = \left[\frac{n'_s (\sin i \pm \sin \theta_{n_x})}{(\sin i \pm \sin \theta_{n'_s})} \right] \lambda_s \quad (6)$$

where

i is angle of incidence,

θ_{n_x} is angle of diffraction for the point at which the difference of path equals $n_x \lambda_x$

$\theta_{n'_s}$ is angle of diffraction for the point at which difference of path equals $n'_s \lambda_s$, and

the expression in brackets equals n_s expressed as a function of the experimental data.

To obtain the requisite data for the evaluation of λ_x , n_x and n'_s , are always chosen arbitrarily as integers and then the corresponding angular measurements (or their equivalents) are made. To indicate that n_x and n'_s are integral we write N_x and N'_s hereafter. N_x and N'_s are never greater than a few units and are obtained by direct count.

There are two special cases of importance. The first is distinguished by the condition

$$N_x = N'_s \quad (7)$$

all the observations being on the same "order" of spectrum, so that n_s may have any value between 0 and 1. According to Kayser⁵ this method, using plane gratings, would have an accuracy

⁵ Spectroscopie, Vol. I, p. 715.

of about $0.1\text{\AA}$. In practice it is replaced by the method of linear interpolation between known standards.

The second is distinguished by the condition

$$N_x \lambda_x \text{ is very nearly equal to } N'_s \lambda_s \quad (8)$$

so that η_s is either very nearly 0 or very nearly 1. Unless λ_x is nearly equal to λ_s , this condition requires the condition

$$N_x \neq N'_s \quad (9)$$

so that observations are made on spectra of different orders. This is the well-known "method of coincidences." With plane gratings it is of small importance because it is embarrassed by the chromatism of the auxiliary lenses, but with the concave grating in Rowland's hands, it afforded the means of establishing his great system of spectroscopic standards. In Rowland's experimental method both θ_{N_x} and $\theta_{N'_s}$ were made either equal to zero or very nearly so. If

$$\theta_{N_x} = \theta_{N'_s} = 0 \quad (10)$$

we have simply $n_s = N'_s$ and

$$N_x \lambda_x = N'_s \lambda_s \quad (11)$$

which is the ordinary text-book form of presenting this method. However, the method would be of very limited application if only wave-lengths *exactly* satisfying (11) could be compared by it. Actually, Rowland obtained the small departure of n_s from N'_s (i. e., η_s or $1-\eta_s$) not from angular measurements as indicated by (6), indeed, but from linear measurements in the normal spectrum. We have only presented the method from this point of view in order to emphasize its fundamental relation to the methods which are commonly distinguished as "interference methods." ⁶ To summarize, we may say that Rowland's method, the first important method that we have to consider, is distinguished, regarding it from our present point of view, by: (1) the choice of an integral value for n_x ; (2) the choice of very small unequal values for N_x and N_s readily obtained by count without any other artifice; (3) the choice of a very small value for η_s ; (4) extreme accuracy of measurement of η_s (about 0.000 001).

⁶ Baly, Spectroscopy, Chap. IX.

Rowland's method has been most thoroughly examined and discussed in recent years. It is now universally admitted that the actual errors in his *relative* values, due to causes not recognized when it was published, may amount to several hundredths of an Ångström unit, although they were originally thought to be accurate to 0.01Å . The inaccuracies of Rowland's method of coincidences were due to errors in the ruling of the grating. The essential defect of the method is thus the great mechanical difficulty of making a sufficiently good grating. A great advance was made by adopting radically different and much simpler apparatus. This simpler instrument, having much less serious intrinsic defects, is known as the interferometer. The two important forms have been the Michelson-Morley and the Fabry-Perot.

3. INTERFEROMETER METHODS

Incident to his absolute determination of the wave-lengths of three cadmium lines, Michelson obtained the relative values of these wave-lengths, and it is noteworthy that while the accuracy claimed for the absolute values was only about one part in 2 000 000, the relative values are thought to be correct to about one part in 20 000 000.⁷ The method of Michelson thus presents a *possible* means of determining relative wave-lengths. But Michelson's interferometer is subject to a limitation which would make its use in the establishment of an extensive system of standards quite inconvenient. This limitation is that the accuracy of measurement of η is only about one part in 10. This means that to obtain an accuracy of one part in 4 000 000 in λ , N would have to be 400 000, a value quite impracticable. In his work on the meter Michelson overcame this difficulty by his method of "stepping-up" from one difference of path to another. It is the inconvenience of this process which would make the method unsuitable for an extensive research on a large number of lines. It has been found better to obtain the requisite accuracy in λ by making the measurement of η more accurate.

The more accurate measurement of η with simple apparatus was made possible by Fabry and Perot's invention of an interferometer consisting of two *parallel transparent* mirrors of *high reflecting power*.

⁷ Light Waves and their Uses, p. 104.

Fabry and Perot's first method, known as their method of coincidences, is similar in fundamental principle to Rowland's method of the same name in that

$$\eta_x = \eta_s = 0 \quad (12)$$

This first method did not make full use of the property of the new instrument in the accurate measurement of η . The error in η may be as large as 0.05.⁸ To obtain the required accuracy in n Fabry and Perot sought coincidences among spectra (fringes) of very high order ($N=10^5$). They found it necessary to devise a very elaborate procedure to determine the integers N_x and N_s ; and although they determined a few relative wave-lengths by this method, they decided it was not entirely satisfactory for general application and soon abandoned it in favor of their second method or method of diameters.

Fabry and Perot's second method makes use of the same interference phenomenon as their first method, and, in so far as the optical parts are concerned, the interference apparatus in the two methods is identical. The radical difference between the two methods is this: In the first method η_x and η_s are each made equal to 0 by varying t until this condition is satisfied. In the second method t is constant and η_x and η_s , which may have any values between 0 and 1, are evaluated from measurements on the interference fringes. The accidental uncertainty in η is probably about 0.01 and N is of the order 10^4 . The still outstanding discrepancies between different investigators using this method vary between less than one part in 4 000 000 to more than one part in 1 000 000 for different lines and average roughly about one part in 2 500 000. There is some reason for suspecting small systematic differences between different investigators, and there remain outstanding very serious discrepancies between Eversheim's recent determinations by this method and previously unquestioned determinations by other methods.⁹

⁸ Ann. de Chim. et de Phys. (7), **16**, p. 317; 1899.

⁹ [Note added, November, 1910.] See Priest, Certain Peculiarities of the Discrepancies among Recent Wave-Length Determinations. Communication to the American Physical Society, New York Meeting, October, 1910. *Phy. Rev.*, November, 1910. While this article has been going through the press, results have been published by Kayser showing that the accuracy of grating determinations has not yet been attained in the standards established by the method of diameters. See *Astroph. J.*, **82**, p. 217; October, 1910.

As this is the only method in the field to-day, and so is the one with which our own method will naturally be compared, it is desirable to present it in a little more detail than the others. We shall therefore give a brief exposition of the essential features of this method as now practiced, not, however, taking space to justify the procedure followed. Also it will not be convenient or necessary to follow in detail the actual computation forms used; but the equations we shall give will exhibit the fundamental relations between the measured and computed quantities, and we will then summarize the nature and number of the measured quantities.

The interferometer used in this method consists of two plane, transparent, metallic mirrors supported on glass plates and accurately parallel to each other at a constant distance (t) apart, this distance being maintained by the supporting action of small pieces of metal or alloy against which the glass plates are pressed by suitable springs. Light which has traversed the interferometer and suffered multiple reflections within it falls upon a convergent lens or mirror, in the principal focal plane of which is formed an interference figure consisting of alternately bright and dark concentric circles. The difference of path on which the computation is based is that giving rise to the interference at the center of this figure, and, to a first approximation, is equal to $2t$. The required condition, $n_x\lambda_x = n_s\lambda_s(1)$, is then nearly satisfied by the approximate equation

$$p_x\lambda_x = 2t = p_s\lambda_s \quad (13)$$

The fundamental relation used to evaluate p is the theoretically established equation

$$p = \frac{n}{\cos i_n} \quad (14)$$

or, to a sufficient approximation

$$p = n \left(1 + \frac{i_n^2}{2} \right) \quad (15)$$

where n is the order of interference of the bright ring which is the locus of image points for beams of parallel rays incident, on the interferometer mirrors, at the angle i_n . The exact integer n may

be determined by artifices which give its value *without experimental error*. Consequently the method of its determination need not be discussed here. The evaluation of i_n is based upon the theoretically derived relation

$$2i_n = \alpha_n \quad (16)$$

where α_n is the angle subtended at the second principal point of the lens (or the pole of the mirror) by D_n , the linear diameter of the ring of order n .

Three general methods have been used to determine α_n . Fabry and Perot measured this angle by means of a telescope previously calibrated in terms of angle. Lord Rayleigh obtained it by measuring the angle through which the interferometer was turned to pass the distance D_n past a fixed reference mark in the focal plane. Fabry and Buisson, Pfund, and Eversheim have evaluated α_n and α_{n_x} by means of the relations

$$\alpha_n = \frac{D_n}{F_{\lambda_s}} = \frac{h}{h'_{\lambda_s}} \frac{D'_{n_s}}{F_{\lambda_s}} = \frac{\theta_{\lambda_s} D'_{n_s}}{h'_{\lambda_s}} \quad (17)$$

$$\alpha_{n_x} = \frac{D_{n_x}}{F_{\lambda_x}} = \frac{h}{h'_{\lambda_x}} \frac{D'_{n_x}}{F_{\lambda_x}} = \frac{\theta_{\lambda_x} F_{\lambda_s} D'_{n_x}}{h'_{\lambda_x} F_{\lambda_s}} \quad (18)$$

where the newly introduced symbols have significance as follows.

h , distance between two marks on a line coincident in direction and position with D_n and D_{n_x} .

F_{λ_s} and F_{λ_x} , focal lengths of the objective for light of wave-lengths λ_s and λ_x , respectively.

θ_{λ_s} , angle subtended by h as viewed through the objective when illuminated by light of wave-length λ_s .

D'_{n_s} and D'_{n_x} , the negative photographic images of D_n and D_{n_x} formed by the action of an optical train containing convergent and dispersive elements.

h'_{λ_s} and h'_{λ_x} , the negative photographic images of h formed, respectively, by light of wave-lengths λ_s and λ_x through the action of the just-mentioned train. The quantities experimentally measured were: by Fabry and Buisson, θ_{λ_s} , h'_{λ_s} , h'_{λ_x} , D'_{n_s} , D'_{n_x} , F_{λ_s} and F_{λ_x} ; by Pfund and Eversheim, h , h'_{λ_s} , h'_{λ_x} , D'_{n_s} , D'_{n_x} , F_{λ_s}

and F_{λ} . However, with Pfund, $F_{\lambda} = F_{\lambda}$ rigorously, there being no chromatic aberration in his system; and Eversheim, in the examples he gives,¹⁰ puts $F_{\lambda} = F_{\lambda}$, thus implying that in the particular case he gives the aberration is negligible.

In addition to the foregoing, data must be obtained for evaluating the correction for the error due to "dispersion of phase" on reflection. The procedure for obtaining this data involves essentially another determination of the quantities D'_{n_1} , D'_{n_2} , h'_{λ_1} , and h'_{λ_2} for another value of t , or an operation about equivalent to this in number and character of observations.

Let us now examine the nature of these several quantities, reducing each to actual *scale readings* in so far as we can. D'_{n_1} , D'_{n_2} , h'_{λ_1} , h'_{λ_2} , and h are simple lengths, each obtained as the difference of two scale readings. θ_{λ} is an angle measured by a telescope with micrometer ocular. It is the difference of two scale readings. All of the foregoing quantities depend upon the previous accurate calibration of micrometer screws. None of the authors gives in any detail his method for the determination of F_{λ} , and F_{λ} , so that we are not in a position to reduce these to original scale readings.

Taking into account the observations necessary to make correction for "dispersion of phase," each wave-length determination is dependent upon 14 or 18 scale readings plus whatever readings are involved in the determination of focal length. This enumeration of the number of readings does not count any repeated readings or analogous readings on different rings.

The essential differences in procedure between this method and our own will be made evident in the next section.

V. GENERAL EXPOSITION OF THE PRESENT METHOD¹¹

1. CHARACTERISTIC FEATURES

The present method is characterized by three principal features.

The first characteristic is the use of the *reflection* fringes, the interferometer consisting of two parallel mirrors, one of which is

¹⁰ Ann. d. Physik, 80, p. 836.

¹¹ This section should be considered in connection with the fundamental principles outlined under III above, the present method being a particular case of the general method there outlined.

heavily silvered, while the other is silvered to a transmission of about 10 per cent, the light entering and leaving the interferometer through this partial mirror and in direction perpendicular to it.¹² The silver mirrors are supported on plane glass plates. Light leaving the interferometer falls upon a very short focus converging lens ($F = 20$ mm), and the interference pattern of bright concentric rings is formed in the principal focal plane of this lens. The wide-open slit of a spectrometer is also placed in this focal plane and along a diameter of the rings. Observations are made at the spectrometer telescope. For each wave-length the observer sees an image of the spectrometer slit like a bright ribbon and in each ribbon the system of fringes due to that wave-length. As the difference of path is decreased the rings contract to bright spots and disappear one after the other at the center of the ring system, the difference of path being decreased by one wave-length between two successive disappearances. As will presently be shown, the measurement of the order of interference in this method depends essentially upon locating the relative positions of one interferometer mirror for the respective disappearances of certain bright spots in the fringe systems for the standard and the unknown wave-lengths. The precision with which these positions can be located limits the precision attainable in the measured wave-length. Hence the importance of being able to locate these positions precisely. The two essentials for attaining this end in the present method are the use of the *reflection fringes* and the *short focus lens* to form their image on the spectrometer slit. With this arrangement the observer can adjust the position of the mirror to the disappearance of the bright spot with great satisfaction and nicety. Regarding the quantitative measure of the precision of this adjustment I shall speak further on.

The second characteristic of the method is the use of the double increment in the distance between the two mirrors instead of the double distance as the difference of path.

The third characteristic is the use of the method of flexure to measure the fractional part of the order of interference. Use of the flexure of elastic material is, of course, not new in interferometry. It has been widely used in making fine adjustments,

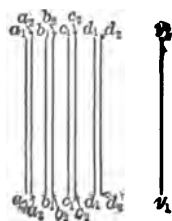
¹² Hamy, Jour. de Phy. (4) 5, pp. 789-809; 1906. Priest, This Bulletin, 5, pp. 483-484; 1909.

and Michelson used torsional flexure to measure the "fractions" by means of his compensator. Fabry and Perot made use of minute flexures to adjust the mirrors of their interferometer, but, so far as I know, did not make measurements in that way. The general principle of the method as used in our apparatus is this: A lever, a weak spring, and a strong spring are so arranged that a displacement of 7500 of one end of the lever produces a displacement of about 1 of the interferometer mirror. We obtain by trial the displacement of the lever end necessary to produce a change of one wave-length in the difference of path. Assuming¹³ the apparatus to be obeying Hooke's law we can then calculate the increment of the difference of path produced by any measured displacement of the lever end. In this way fractional parts of the order of interference can be measured.¹⁴

2. ABSTRACT OUTLINE OF PROCEDURE

ESSENTIAL ARRANGEMENTS AND CONDITIONS

In Fig. 1 let y_1 be the trace of the heavily silvered mirror, its plane being perpendicular to the plane of the figure. Let a_1 , b_1 , c_1 , d_1 be traces of the partial mirror in positions respectively as follows:



NOTE ON FIG. 1.—Not to scale. Relative positions of a_1 and a_2 , b_1 and b_2 , c_1 and c_2 , d_1 and d_2 as to right and left must not be understood as determined by the figure, the purpose of which is merely to illustrate the indicated order a , b , c , d , y_1 , y_2 .

Fig. 1

a_1 is the position for disappearance of a bright spot in the fringe system for λ_r .

d_1 is the position for disappearance of the bright spot of *next lower order* in the same fringe system.

¹³ The assumption that the large displacement of the lever end is proportional to the small displacement of the mirror can easily be tested experimentally by measuring the large displacements corresponding to successive increases of $\pm 1\lambda$ in the difference of path, and I have often done this.

¹⁴ For the actual procedure followed, see VII, 2 and 3; and VIII below, where it is shown also that the effect of small departures from Hooke's law can be eliminated.

b_1 is the position for first disappearance of a bright spot in system for λ_z after leaving position a_1 .

c_1 is the position for first disappearance of a bright spot in the system for λ_r , after leaving position a_1 .

Let y_2 be trace of the heavily silvered mirror in another position, parallel to its first position and such that

$$\overline{a_1 y_2} > \overline{a_1 y_1}$$

For this position of the heavily silvered mirror, let a_2 , b_2 , c_2 , d_2 have significance analogous to a_1 , b_1 , c_1 , d_1 , for the first position.

The equation we use for evaluating λ_z in terms of λ_r is

$$(\Delta p_z)\lambda_z = 2(\overline{a_2 y_2} - \overline{a_1 y_1}) = 2\Delta t = (\Delta p_r)\lambda_r \quad (19)$$

We require the numerical values of Δp_r and Δp_z

DETERMINATION OF Δp_r

Δp_r is determined by the following conditions.

$$\Delta p_r = \text{an integer (by conditions of experiment)}. \quad (20)$$

$$\Delta p_r = \frac{2\Delta t}{\lambda_r} \quad (21)$$

$$(\Delta p_r) \frac{\lambda_z}{\lambda_r} = \text{some integer} + \Delta \phi_r \quad (22)$$

For a given measured value of Δt , Δt_a , conditions (20) and (21) determine that Δp_r must be an integer within the group of minor limit L and major limit $L+l$, where L is determined from λ_r and Δt_a while l is determined by the accuracy of Δt_a . For a given measured value of $\Delta \phi_r$, say $\Delta \phi_{rm}$, conditions (20) and (22) determine that Δp_r must be an integer in an arithmetic series of which the common difference, j , is determined by the ratio $\frac{\lambda_z}{\lambda_r}$. To determine Δp_{re} we require then measured values of Δt and $\Delta \phi_r$, conditions being arranged to give l and j such values that only one member of the arithmetic series can fall within the group L to $L+l$. Δp_r is then determined uniquely as the integer within the group L to $L+l$ satisfying (22) for $\Delta \phi_r = \Delta \phi_{rm}$.

DETERMINATION OF Δp_x

Δp_x is determined by the following conditions.

$$\Delta p_x = \frac{2\Delta t_e}{\lambda_x} = \frac{\Delta p_{se}\lambda_s}{\lambda_x} \quad (23)$$

$$\Delta p_x = \text{an integer} + \Delta\phi_x \quad (24)$$

Substituting the known values of Δp_{se} and λ_s , and λ_{xa} the previously known approximate value of λ_x , condition (23) determines Δp_x to be within a range of values the limits of which are fixed by the uncertainty in λ_{xa} . Substituting for $\Delta\phi_x$ its measured value, conditions (23) and (24) then determine Δp_{se} , the accurate value of Δp_x , uniquely, provided that the uncertainty in λ_{xa} was such as to make the uncertainty in Δp_{xa} [from solution of (23)] numerically less than 0.5. [In symbols, $+0.5 > \frac{(\Delta p_x) \delta\lambda_x}{\lambda_x} > -0.5$] (25)

To determine Δp_{se} , we require then the values of Δp_{se} and λ_s , the measured value of $\Delta\phi_x$, and λ_{xa} , an approximate value of λ_x , with maximum uncertainty $\delta\lambda_x$, such that

$$+0.5 \frac{\lambda_x}{\Delta p_x} > \delta\lambda_x > -0.5 \frac{\lambda_x}{\Delta p_x} \quad (26)$$

The magnitudes to be obtained experimentally are then Δt_a , $\Delta\phi_r$, and $\Delta\phi_x$.

REDUCTION OF REQUIRED QUANTITIES TO DIRECTLY MEASURED QUANTITIES

The measurement of Δt_a is a *direct* measurement obtained as the difference of two scale readings. A scale carried by the carriage carrying the movable mirror is read by a fixed micrometer-microscope before and after the displacement.

$\Delta\phi_r$ and $\Delta\phi_x$ are evaluated by means of the following equations,

$$\Delta\phi_r = \frac{\overline{a_2c_2}}{a_2d_2} \cdot \frac{\lambda_s}{\lambda_r} - \frac{\overline{a_1c_1}}{a_1d_1} \cdot \frac{\lambda_s}{\lambda_r} \quad (27)$$

and

$$\Delta\phi_x = \frac{\overline{a_2b_2}}{a_2d_2} \cdot \frac{\lambda_s}{\lambda_x} - \frac{\overline{a_1b_1}}{a_1d_1} \cdot \frac{\lambda_s}{\lambda_x} \quad (28)$$

the truth of which is evident from the construction given above.

Inasmuch as conditions of measurement make the uncertainty in $\Delta\phi_r$ and $\Delta\phi_x$ greater than one part in 1000, there will always be at hand in advance sufficiently accurate values of λ_s , λ_r , and λ_x for giving the ratios $\frac{\lambda_s}{\lambda_r}$ and $\frac{\lambda_s}{\lambda_x}$. The ratios $\frac{\overline{a_1c_1}}{\overline{a_1d_1}}$, $\frac{\overline{a_1b_1}}{\overline{a_1d_1}}$, $\frac{\overline{a_2c_2}}{\overline{a_2d_2}}$, $\frac{\overline{a_2b_2}}{\overline{a_2d_2}}$, are obtained by the direct measurement of six lengths $\overline{A_1C_1}$, $\overline{A_1B_1}$, $\overline{A_1D_1}$, $\overline{A_2C_2}$, $\overline{A_2B_2}$, $\overline{A_2D_2}$, directly proportional respectively to $\overline{a_1c_1}$, $\overline{a_1b_1}$, $\overline{a_1d_1}$, $\overline{a_2c_2}$, $\overline{a_2b_2}$, $\overline{a_2d_2}$, so that

$$\frac{\overline{a_1c_1}}{\overline{a_1d_1}} = \frac{\overline{A_1C_1}}{\overline{A_1D_1}}, \quad \frac{\overline{a_1b_1}}{\overline{a_1d_1}} = \frac{\overline{A_1B_1}}{\overline{A_1D_1}} \quad (29)$$

$$\frac{\overline{a_2c_2}}{\overline{a_2d_2}} = \frac{\overline{A_2C_2}}{\overline{A_2D_2}} \quad \text{and} \quad \frac{\overline{a_2b_2}}{\overline{a_2d_2}} = \frac{\overline{A_2B_2}}{\overline{A_2D_2}} \quad (30)$$

The lengths $\overline{A_1C_1}$, $\overline{A_1B_1}$, $\overline{A_1D_1}$, $\overline{A_2C_2}$, $\overline{A_2B_2}$, $\overline{A_2D_2}$, are measured in terms of the revolutions of a micrometer screw, the point of which bears on one end of a lever, the other end of which is attached by means of a weak spiral spring to a strong flat spring carrying the partial mirror, the strengths of the springs being adjusted so as to give the proportionality factor, 7500, mentioned above.

3. SUMMARY OF REQUISITE DATA

The following is a list of the data requisite for the determination of a wave-length.

A. Reference standards and previously known values.

The standard reference wave-length, λ_s , and the auxiliary reference wave-length, λ_r .

An approximate value of λ_x , λ_{xa} , such that

$$+0.5 \frac{\lambda_x}{\Delta p_x} > \delta \lambda_x > -0.5 \frac{\lambda_x}{\Delta p_x}$$

Correction (K) of the scale used on the mirror carriage.

B. Experimental data.

The displacement of the heavily silvered mirror in terms of scale divisions of the scale on the carriage.

The six lengths $\overline{A_1C_1}$, $\overline{A_1B_1}$, $\overline{A_1D_1}$, $\overline{A_2C_2}$, $\overline{A_2B_2}$, $\overline{A_2D_2}$, in terms of revolutions of the micrometer screw.

Temperature of the air and the scale on the mirror carriage.

Atmospheric pressure. (*Atmospheric temperature and pressure do not ordinarily enter the computation, and are only recorded to supply complete record in case of future question.*)

The experimental data reduce to twelve scale readings, viz:

(1) M_1 , M_2 , micrometer-microscope readings on the scale before and after displacement of the carriage; (2) A_1 , B_1 , C_1 , D_1 , A_2 , B_2 , C_2 , D_2 , micrometer head readings; (3) thermometer reading; (4) barometer reading.

As to the way these several scale readings enter the calculation the following facts should be noticed.

Under ordinary laboratory conditions the temperature and pressure of the air influence the relative values so slightly as not to come into consideration at all.

M_1 , M_2 , C_1 , and C_2 , and the temperature of the scale are used only in determining the integral part of the order of interference, and as checks upon the reliability of the determination.

The precision of the measured wave-length is a function only of the accidental errors of A_1 , B_1 , D_1 , A_2 , B_2 , D_2 , and can be made to be sensibly a function of only the errors of A_1 , B_1 , A_2 , B_2 .

VI. THE APPARATUS AND ITS ARRANGEMENT

The apparatus requisite for practicing the method consists of—

An interferometer, the essential elements and accessory parts of which are shown in Figs. 2 and 3 and enumerated in the key accompanying those figures; and

Accessory apparatus for illumination and observation arranged as shown in Fig. 4 and enumerated in the accompanying key.

The following is a list of the important special parts of this apparatus actually used in this laboratory.

Glass plates (1, 2, Fig. 2), by Pettidier, Chicago.

Interferometer bed, mirror mountings, and flexure apparatus, designed by Dr. S. W. Stratton and constructed by Mr. O. G. Lange in the Bureau of Standards shop.

Micrometer screw (11, Fig. 2), by Brown and Sharpe, 0.5 mm pitch, reading millimeters with smallest divisions on head indicating 0.01 mm.

Scale (17, Fig. 2), 5 cm long, on silver, by Zeiss. Smallest division = 0.1 mm.

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11. Micrometer

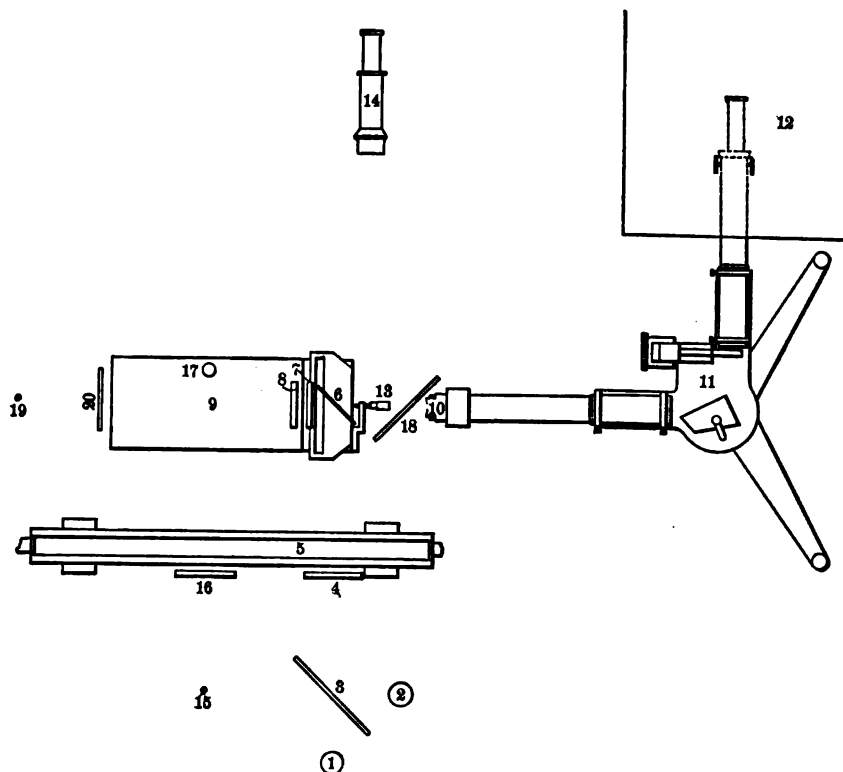


Fig. 4.—General plan of apparatus

KEY.

- | | |
|---|---|
| 1. Source. | 12. Dark observing hood over spectrometer ocular. |
| 2. Source. | 13. Micrometer. |
| 3. Transparent mirror for superposing two beams. | 14. Telescope used by assistant observer in reading micrometer. |
| 4. Condensing lens. | 15. Nernst lamp for illuminating scale on carriage. |
| 5. Water screen to protect interferometer from heat of lamps and heater. | 16. Condensing lens. |
| 6. Transparent mirror for reflecting light into the interferometer. | 17. Microscope. |
| 7. Partial interferometer mirror. | 18. Removable mirror for observing the fringes, looking from the side. |
| 8. Full interferometer mirror. | 19. Nernst lamp used in the preliminary adjustment of the parallelism of 7 and 8. |
| 9. Interferometer bed. | 20. Double convex lens. |
| 10. Short focus (e. g., 20 mm.) converging lens to form real image of interference pattern. | |
| 11. Spectrometer | |

Microscope (18, Fig. 2), with micrometer ocular, by Zeiss. Magnification, about 25. Pitch of micrometer screw, 0.3 mm.

Smallest division on head (0.01) corresponds to 1μ on the scale.

Lens (10, Fig. 4), "Planar," by Zeiss. Focal length = 20 mm.

Spectrometer (11, Fig. 4), Hilger's Constant Deviation type, with dense prism.¹⁵

VII. EXPERIMENTAL DETAILS AND PRECAUTIONS

1. PRELIMINARY ADJUSTMENTS

The apparatus being arranged roughly, as shown in Fig. 4, the sources, 1 and 2, the partial mirrors, 3 and 6, and the lens, 4, must be so adjusted that the center of 7, the image of the center of 4 in 6 and the images of the mid points of 1 and 2 in 6 shall be on a straight line normal to the plane of 7. By sighting and measurements with a millimeter scale the several parts are brought into, roughly, their correct positions. A mirror, 18, is placed between 10 and 6 at a convenient angle, so that the image of 7 can be seen in it on looking from the side toward 14. The observer then moves his head until he sees the image of his observing eye centrally placed in 7. He then turns 6 until the image of the frame of 4 has position concentric with the frame of 7. Small movements of 1 and 2 then serve to bring their distorted images central in the field. To make the first adjustment of the interferometer mirrors to approximate parallelism, a Nernst glower at the principal focus of a converging lens is placed on the opposite side of the interferometer from the spectroscop. The intense light of the glower suffices to penetrate even the heavy silver mirror in perceptible amount, so that in looking into 18 one sees a series of images of the glower due to multiple reflections between 7 and 8. By adjusting the angular position of 8 these images are superposed. The observer, remaining in the same position, now views the interferometer in monochromatic light from 1 or 2, a piece of ground glass having been inserted between the source and the lens 4. The interference fringes are now found and are made to assume the circular form by further angular adjustment of the position of 8. The final adjustment of the parallelism is then made by adjusting the position of 8 until the fringes neither

¹⁵ Adam Hilger, List "A," 1907, Fig. 8, p. 9.

expand nor contract on moving the eye across the field in any direction. (There will usually be a small error in the plates that can not be corrected by altering their position.) The mirror, 18, is now removed. The lens, 10, having been previously focused on the slit, and the telescope ocular on the image of the slit, the observer now goes to the spectroscope. By slightly turning the whole spectroscope the center of the image of the interference rings is brought upon the slit. By moving the whole spectroscope in direction perpendicular to the collimator axis the position of best illumination is found. The rings and the slit are finally sharply focused by proper adjustments of the lens, 10, and the spectroscope ocular.

2. COURSE OF A SINGLE DETERMINATION

The preliminary adjustments being completed, the data for one determination are obtained and recorded as follows.

A. General data relative to the determination are recorded as shown in Form 1. The numbers referring to apparatus are merely Bureau of Standards' inventory numbers, except the numbers of mirrors, which are serial numbers, as prepared. "Exper'l Value" is the arbitrary weight assigned the determination by the observer on account of experimental conditions. Only three numbers, 1, 2, and 3, are used, 3 denoting best conditions and 1 the poorest conditions under which the determination would be given any weight at all.

B. Data for computing ϕ_{z1} and ϕ_{r1} are recorded as shown in Form 2. Columns headed A_1, B_1, C_1, D_1 are the micrometer head readings for positions a_1, b_1, c_1, d_1 of the partial mirror as explained under V. These numbers are read and recorded by an assistant observer as the observer at the spectroscope makes the corresponding settings by turning the micrometer screw. The eight numbers in each set (the sets being numbered I, II, III, etc.) are recorded in order across the page from left to right in the upper row of four and then back again from right to left in the lower row of four. The ten sets are obtained one after the other with as little delay as possible between; and any marked interruption is noted on the margin.

C. Ten microscope readings on the scale for this position of the mirror are then obtained and recorded as shown in the left-hand column, Form 6. Temperature of scale is recorded.

D. After the carriage with the full silvered mirror has been displaced through the distance Δt the data for computing ϕ_{x2} and ϕ_{r2} , exactly similar to that for ϕ_{x1} and ϕ_{r1} , are obtained and recorded as shown in Form 4.

E. Ten microscope readings on the scale for this new position of the mirror are then obtained as shown in the right-hand column, Form 6.

3. SPECIAL PRECAUTIONS

Certain conditions are at our command, a proper choice of which will decrease the effect of instrumental defects and render our operations less liable to errors and mistakes. In this connection we take especial precautions as follows.

A. A well known and frequently used interval of the scale (17, Fig. 2) is used to measure Δt . The intervals we use have been independently calibrated by microscopic comparison with the standards of this bureau and by interference methods so as to reduce the chance of a large mistake or blunder in the value of the interval. The interferential calibration was done with the scale in the position in which it is actually used on the carriage.

B. Δt is made very nearly equal to a whole number of scale divisions and the micrometer head readings in M_1 and M_2 are thus made nearly equal. By so doing any error in Δt due to error in screw value is made negligible.

C. All the fractions (ϕ_{x1} , ϕ_{x2} , ϕ_{r1} , ϕ_{r2}) are chosen small. By so doing the effect of any small departure of the behavior of the elastic system from Hooke's law is reduced and the effect of errors in D_1 and D_2 may be made negligible in comparison to similar errors in A_1 , A_2 , B_1 , B_2 , C_1 , and C_2 .

D. ϕ_{x2} is chosen nearly equal to ϕ_{x1} , and ϕ_{r2} nearly equal to ϕ_{r1} . It is apparent how this condition will be effective in eliminating the effects of departures of the elastic system from the conditions of perfect elasticity, for if ϕ_{x1} and ϕ_{r1} are measured a little too large or too small, ϕ_{x2} and ϕ_{r2} will be affected to a similar degree in the same sense, and since we use only the differences ($\phi_{x2} - \phi_{x1}$) and ($\phi_{r2} - \phi_{r1}$) the errors will be eliminated.

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E. $\overline{a_1b_1}$ is chosen nearly equal to but a little less than $\overline{a_1c_1}$, and $\overline{a_2b_2}$ nearly equal to but a little less than $\overline{a_2c_2}$. Since we can calculate, *a priori*, $\Delta\phi_r$ to a better accuracy than the precision attainable in the measured value ($\Delta\phi_{rm} = \phi_{r2} - \phi_{r1}$), the degree of agreement between this calculated value $\Delta\phi_{re}$ and the measured value $\Delta\phi_{rm}$ affords us a very valuable indication of the reliability of the measured fraction $\Delta\phi_x$, for, by the arrangement just mentioned, $\Delta\phi_x$ and $\Delta\phi_r$ are measured under almost identical conditions, such that the instrumental error should be nearly the same in both, but, if anything, greater in $\Delta\phi_r$ than in $\Delta\phi_x$; and we have obtained very closely the true value of the actual error in $\Delta\phi_r$.

Conditions A and B are satisfied by bringing the carriage into position while observing the scale through the microscope. Conditions C, D, and E can then generally be satisfied by very small displacements of the carriage while observing the fringes through the spectroscope and without disturbing conditions A and B to an undesirable extent.

VIII. COMPUTATION

The course of the computation is shown in Forms 2, 3, 4, 5, 6, and 7. Forms 2 and 3 show the computation of ϕ_{r1} and ϕ_{x1} . Forms 4 and 5 show the computation of ϕ_{r2} and ϕ_{x2} . Form 6 shows the computation of Δt_a and ΔP_{sa} . Form 7 shows the determination of ΔP_{se} and Δp_{xe} , and the final computation of the relative wave-length.

IX. TESTS OF PRECISION AND RELIABILITY

1. DEFINITION OF TERMS

By precision we mean the degree of agreement of individual measurements or determinations. As a convenient measure of the precision of the result of a single determination we take its probable error as computed above from the residuals in the 10 measurements of ϕ_1 and 10 measurements of ϕ_2 . When a number of determinations are available, as in the examples below, a further indication of the precision of the method is afforded by the differences between individual results and their mean. By reliability we mean the degree to which the method may be depended upon to give the correct result within the limits of uncertainty fixed by the precision.

2. PRECISION

The magnitude of the irremovable accidental errors is fixed by the acuity of the observer's judgment in estimating the positions for disappearance of the bright centers. In our present apparatus, we probably have superposed upon this the effect of irregularities in the action of the flexure apparatus. The accidental errors due to both of these causes appear in the probable errors of ϕ computed as shown in the forms mentioned above. Table I contains a résumé of probable errors of ϕ computed as above for different conditions indicated in the table, λ , being in all cases cadmium 644.

TABLE I

 E_{ϕ}

		Cd 509		No 585		No 594	
p_a		5300	26900	4600	23400	4500	23000
Determination No.	12			0.0012	0.0019		
	13	0.0012	0.0017	8	16		
	14	18	35	15	19		
	18			14	22	0.0009	0.0014
	19			13	15	10	18
Mean		0.0015	0.0026	0.0012	0.0018	0.0010	0.0016

Noting that the probable error of Δp is given by $\pm \sqrt{E_{\phi_1}^2 + E_{\phi_2}^2}$ and taking Δp as the difference of the two values of p_a under each wave-length; and ϕ_1 and ϕ_2 respectively as the values of ϕ under the small and large values of p_a , it follows from this table that, for these radiations and these values of p , the average probable error of Δp is about one part in 8 000 000. From which we have also: *the average probable error of λ for the best lines is about one part in 8 000 000.*

Further evidence relative to the precision we have attained in the use of this method is given in Table II.

TABLE II

Determination Relative to Cd Standard, of Neon Line, $\lambda_{\text{ca}} = 5852.65$ ¹⁷

Determination No.	Date	Δp	λ as determined	E_{λ}	Residuals
			λ_M	$\text{\AA}$	$\text{\AA}$
11	2/ 5, '10	18811	5852.4888		0.0002
12	2/15	18822	876	± 0.00070	0.0010
13	2/15	18790	884	56	2
14	2/19	18812	873	74	13
16	3/ 1	35919	889	47	3
18	3/ 1	18734	901	81	15
19	3/ 3	18801	882	62	4
Means			5852.4886	0.00065	0.00070

¹⁷ Baly's Table relative to Kayser's Iron Standards. Trans. Roy. Soc., A, 202, p. 183. Proc. Roy. Soc., 72, p. 84; 1903 B. A. Report; 1905, pp. 105-108.

REMARKS ON TABLE

The above table contains every determination I have made of this wave-length relative to Cd 6438.4722 with values of Δp large enough to give this order of accuracy. Preliminary determinations with low orders of interference gave values of from 5852.480 to 5852.497, relative to Cd, 6438.4722 and Hg, 5460.7424. These preliminary approximate values were used as already explained in obtaining the more accurate values given above. No. 11 is based on only 5 measures each of ϕ_1 and ϕ_2 . All the other determinations are based on 10 measures of each of these fractions. In determinations 18 and 19, this wave-length functioned as λ_r instead of λ_x , but as its value is already known within certain limits, the data obtained in these determinations enable us to compute an independent value. No. 12 is subject to slight suspicion.

The following are the important conclusions from the table.

The average probable error of the last six determinations is about one part in 9 000 000.

The average residual of the seven determinations is about one part in 8 400 000.

The maximum residual of the seven determinations is about one part in 3 900 000.

The foregoing data are thought to be sufficient for forming an estimate of the precision attainable in measuring the wave-lengths of the best lines.

3. RELIABILITY

The reliability of a method of measurement depends upon the correctness of the theoretical relations assumed in formulating the method and upon the degree to which the actual experimental procedure satisfies the ideal conditions of the theory. Errors of method are of two classes. It may be that the experimental procedure is such as to always fail, to the same degree and in the same sense, to satisfy the relations on which the method is based. Such a condition will cause a constant error. Or it may be that the experimental result is a function of some variable condition (e. g., temperature or pressure) not recognized in formulating the method. Variation of this condition may cause in one individual determination a positive error and in another a negative error; but unless such errors are small and are, in any set of determinations, likely to be positive as often as negative they can not generally be satisfactorily eliminated by multiplying determinations.

Ordinarily the discussion of the reliability of a method involves a searching investigation of all conceivable sources of error, and we can never be certain that some have not escaped consideration. The obvious sources of errors and mistakes and the procedure for eliminating them have been noted under VII, 3. Fortunately, in the present instance we are relieved of the necessity of making an exhaustive investigation of errors of method because of the possibility of making control determinations. The ratio of the wave-lengths of the green and red radiations of cadmium is known within an uncertainty of only about one part in 20 000 000 from Michelson's work on the meter. The most rigorous test that can be applied to a method for relative wave-lengths is the ability to check this ratio to the accuracy desired in the method under test. One such test would give us an indication of the importance of the first class of errors of method mentioned under "Reliability." It would not, however, relieve us of concern as to the second class. The highest degree of assurance in this regard can only be obtained by running a control determination in connection with every determination of an unknown, and arranging the experiment in such a way that the chance of errors affecting the determination of the unknown and not affecting the control will be minimized and further arranging so that, as far as we can judge

a priori, the error of method in the control will, if anything, tend to be larger than in the determination of the unknown. An examination of our Record Forms 2, 4, and 7 and consideration of the precaution mentioned under VII, 3, E, will show that the determination of $\Delta\phi_{rm}$ affords the data for such a control determination closely interwoven with the determination of the unknown. The identification of ΔP_{ee} , which is the primary use to which $\Delta\phi_{rm}$ is put, does not require a very accurate value of this fraction. However, by obtaining a value of the same order of precision as $\Delta\phi_x$, we have a rigorous check on the reliability of $\Delta\phi_x$. Definitely, we have

$$\frac{\Delta\phi_{rm} - \Delta\phi_{rc}}{\Delta\phi_r} = -\frac{\delta\lambda_r}{\lambda_r} \quad (31)$$

where $\delta\lambda_r$ is the error in λ_r due to using $\Delta\phi_{rm}$ for $\Delta\phi_{rc}$. We may safely assume, roughly (say to an approximation of one part in ten), that

$$\frac{\delta\lambda_x}{\lambda_x} = \frac{\delta\lambda_r}{\lambda_r} \quad (32)$$

where $\delta\lambda_x$ is *error of method* in λ_x

Control determinations we have made give actual departures from Michelson's value of the ratio of the red and green cadmium wave-lengths as shown in Table III.

TABLE III

Number of determinations of $\Delta\phi_r$	Departure
10	$\frac{1}{7\ 200\ 000}$
10	$\frac{1}{21\ 600\ 000}$
5	$\frac{1}{3\ 100\ 000}$
10	$\frac{1}{2\ 300\ 000}$

It should be explained that the third and fourth of these values were obtained under unfavorable conditions, which would now be avoided in careful work. Also it should be noted that the third

is based on only five measurements of $\Delta\phi$. *This evidence is sufficient to show that the systematic error can be kept well within the accidental error.* Of course when the wave-lengths of other suitable radiations are well enough known they may be used as controls instead of cadmium green.

The only errors that occur to me as not being detected by the above procedure are those that might be due to peculiarities of the radiation λ_z not shared by the radiation λ_r . Such peculiarities may occur in brightness, hue, and "line structure." An examination of the method will show that a constant difference in brightness between λ_z and λ_r will not introduce an error, but that a change in brightness of either between the measurements of ϕ_1 and ϕ_2 might be expected to do so. However, a simple direct experiment shows that even doubling the current through our Neon lamp does not sensibly change the adjustment for disappearance of the bright center. It will be well to guard against large variations in the intensities of illumination in the field, but no excessive or troublesome precautions need be taken on this account. As to errors due to hue differences, the cadmium green radiation is most favorably situated with respect to red to make this error large. As this is our control we need hardly fear that the errors for other wave-lengths would be greater. As to the errors due to "line structure" it may be said that the presence of satellites may vitiate the measurements. We do not ordinarily obtain correct results in using mercury 5461 as a working standard. As, however, such lines should not be chosen as standards,¹⁸ the difficulties in this connection need not trouble us. Apparently the green cadmium satellite has not troubled us, probably because of its faintness. A change in the width or reversal of a line between the measurement of ϕ_1 and ϕ_2 would no doubt introduce an error. On this account the excitation of the source must be kept constant within limits that may be fixed by previous knowledge of the behavior of the line with changing excitation.

Additional assurance as to the reliability of the method is afforded by the close agreement obtained between the value of Δt measured with the microscope and scale and the more accurate value corresponding to ΔP_{se} . In the seven determinations noted

¹⁸ Fabry and Buisson were able to use this line as a *working* standard, taking the precaution to obtain its relation to the *primary* standard for each pair of mirrors used.

above, these two values always agreed to within a few tenths of a micron, always to within less than one wave-length. For the purpose of determining ΔP_{ee} by the method used, the microscopic determination of Δt need not be of this degree of accuracy, but it is highly satisfactory to have the agreement so good. One feels that he is working with a larger "factor of safety," the chance of blundering in identifying ΔP_{ee} being minimized.

Any conclusions as to the precision and reliability of the method drawn from this paper are of course limited by the extent of the data here presented. It has been shown that in the case of one particular ratio, results of a high degree of precision can be obtained. It has been shown that the method checks, to a high degree of accuracy, the previously known ratio of the red and green cadmium wave-lengths, and a plan for running a rigorous control on each determination has been formulated and presented. Probable trouble with lines of complex structure and variable width has been pointed out. There is no apparent reason why the method should not give, for other lines well suited to serve as standards, results comparable in precision and reliability with those given above, provided proper arrangements for observation can be made.

It is expected that the method will be more extensively tested at this laboratory during the coming year. The author would suggest the desirability of independent tests being undertaken by other observers in other laboratories at the same time.

X. ADVANTAGES OF THE METHOD

This method seems to the author to possess the following advantages over the method of diameters as now universally used in establishing spectroscopic standards.

1. The procedure is more direct, involves fewer assumptions, and permits of a more rigorous control of the errors.
2. There is less chance of apparatus errors, such as are due to imperfections of the mirrors, errors in adjustments, and aberrations of the optical system, the requirements of instrumental perfection being much less severe, thus giving a considerable "factor of safety."
3. The chance of error due to distortion of the photographic image is eliminated. By many this source of error is not consid-

ered serious. However, Albrecht¹⁹ has shown the existence of photographic distortion that must be considered if an accuracy of one part in 4 000 000 is to be obtained in all cases with the method of diameters as now used.

4. The errors due to small temperature changes are eliminated without resorting to a thermostat.

5. The error due to "dispersion of phase" on reflection is automatically eliminated without further discussion or correction. Some idea of the importance and trouble of this correction may be obtained by merely noting that in Eversheim's recent paper on the iron standards 10 pages out of a total of 24 are taken in its discussion.

6. The personal error in the measurement of the order of interference is eliminated to a greater extent than in the method of diameters. It is to be noted that in the method of diameters the wave-length is derived from the measurement of the diameter of a circle. It is obvious that this condition is very different from measuring a length defined by two straight lines and is favorable to the introduction of a personal equation. Data presented by Plund²⁰ shows that his personal error in this measurement whatever its size is not constant from day to day. It is true that this error would be eliminated if the diameters of the rings for the standard and the unknown wave-lengths were equal. But this condition will not in general be satisfied; so that complete compensation will be only accidental.

7. There are fewer chances for accidental error because while in the method of diameters the precision of each determination is sensibly affected by the accidental errors of at least eight scale readings,²¹ the precision of each determination by the present method is affected by the accidental errors of only six scale readings, and by two of these in so slight a degree as to make the precision of the measured wave-length sensibly a function of the errors of only four scale readings.

¹⁹ *Astrop. Jour.* 25, p. 349; 1907.

²⁰ *Astrop. Jour.* 28, p. 210; 1908.

²¹ In all, 14 or 18 scale readings plus the number of readings required for a determination of focal-length are involved in each wave-length determination, but the errors of some of them are small enough to be negligible in comparison with the others.

Data available are not adequate for making a rigorous or very significant comparison of the degrees of precision attained in the two methods. I have tried to estimate the relative precision by comparing the probable errors published by Eversheim²² with my own probable errors given above. Unfortunately, a direct comparison is not significant for the following reasons: (1) Eversheim's method of computing his probable errors is not sufficiently specified; (2) apparently his published probable errors do not take into account *all* the accidental errors to which his method is subject,²³ while mine do include all accidental errors in the present method; (3) the individual wave-length values in his table result from 36 measurements of the order of interference, while the individual values given by me result from only 10 measurements; (4) our data are for different radiations. It is hoped that data sufficient for a significant comparison of the degrees of precision of the two methods will be available in a few months and a definite comparison in this respect is therefore not attempted now.

XI. CONCLUSION

The result of this investigation has been the definite formulation of a method which it is believed will give a closer approximation than has hitherto been attained to the ideal of an uncertainty less than 1 part in 4 000 000. Conclusions as to the precision and reliability of the method have been formulated from experiment and stated above (pp. 599, 602, 603) and the peculiar advantages of the method have been pointed out (X). It is not thought that the results above presented are the best of which the method is capable. Improvements under way or being considered are: (1) Improvement of the flexure apparatus by substituting a cut-steel spring for the wound-wire spring and perhaps the substitution of a crystalline-quartz bar for the steel bar; (2) substitution of a black and white micrometer head that can be more easily and rapidly read than the present one (11, Fig. 2); (3) better achromatism of the lenses in the observing apparatus; (4) a slight

²² Ann. der Phys. 80, pp. 837, 838; November 30, 1909.

²³ Eversheim's probable errors are evidently intended merely to indicate the relative precision of his different measurements and not the precision of the method.

increase in Δp . It should also be noted that in Table II the determinations having the large probable errors actually give the largest residuals. It is hoped in future determinations to keep the probable errors considerably smaller than these largest ones.

We are now using the method to determine about 20 wave-lengths in the red, orange, and yellow of the neon spectrum. These lines are exceedingly well suited to serve as standards on account of their homogeneity, their brightness, and the convenience of maintaining the source. Being in a region of the spectrum where iron lines are unsatisfactory, it is thought their accurate values will be of considerable service to spectroscopists. It is hoped to publish a table of these wave-lengths within a few months.

WASHINGTON, June, 1910.

NOTE.—A photographic method for recording the micrometer-head readings (see VII, 2, B above) is now in use. Besides eliminating the necessity of an assistant observer this modification promises to afford an improvement in the method.

DECEMBER 15, 1910.

INDEX TO VOLUME 6.

A.

- Absolute measurement of electric quantity, Method for the, *Burton McCollum*, 503.
- Acoustic quality, Effect of phase harmonic upon, *M. G. Lloyd* and *P. G. Agnew*, 255.
- Analysis of EMF waves, Approximate experimental method for the, *P. G. Agnew*, 95.
- Artificial illuminants, Daylight efficiency of, *Herbert E. Ives*, 231.
- Agnew, P. G.*, Approximate experimental method for the analysis of EMF waves, 95.
- Determination of the constants of instrument transformers, 281.
- Effect of phase harmonics upon acoustic quality, 255.
- Regulation of potential transformers and magnetizing current, 273.
- Austin, Louis W.*, Comparative sensitiveness of some common detectors of electric oscillations, 527.

B.

- Bars, straight, Determination of the magnetic induction in, *Charles W. Burrows*, 31.
- Buckingham, Edgar*, Definition of the ideal gas, 409.
- Theory of the Hampson liquefier, 125.
- Burgess, George K.*, Estimation of the temperature of copper by means of optical pyrometers, 111.
- Platinum resistance thermometry at high temperatures, 149.
- Burrows, Charles W.*, Determination of the magnetic induction in straight bars, 31.

C.

- Calcium chloride solutions between -35°C and $+20^{\circ}\text{C}$, Specific heat of some, *H. C. Dickinson*, *E. F. Mueller*, and *E. B. George*, 379.
- Capacity, Mica condensers as standards of, *Harvey L. Curtis*, 431.
- Circuits, Coupled, in which the secondary has distributed inductance and capacity, *Louis Cohen*, 247.
- Coblentz, W. W.*, Luminous efficiency of the firefly, 321.
- Note on the thermoelectric properties of tantalum and tungsten, 107.
- Selective radiation from various solids, II., 301.
- Cohen, Louis*, Coupled circuits in which the secondary has distributed inductance and capacity, 247.
- Color, Pure, Method for constructing the natural scale of, *P. G. Nutting*, 89.
- Comparative sensitiveness of some common detectors of electric oscillations, *Louis W. Austin*, 527.
- Condensers, Mica, as standards of capacity, *Harvey L. Curtis*, 431.
- Copper, Estimation of the temperature of, by means of optical pyrometers, *George K. Burgess*, 111.
- Current, magnetizing, Regulation of potential transformers and, *M. G. Lloyd* and *P. G. Agnew*, 273.
- Curtis, Harvey L.*, Mica condensers as standards of capacity, 431.

D.

- Daylight efficiency of artificial illuminants, *Herbert E. Ives*, 231.
- Definition of the ideal gas, *Edgar Buckingham*, 409.

Detectors of electric oscillations, Comparative sensitiveness of, *Louis W. Austin*, 527.

Determination of the constants of instrument transformers, *P. G. Agnew* and *T. T. Fitch*, 281.

Determination of the magnetic induction in straight bars, *Charles W. Burrows*, 31.

Determination of the ratio of transformation and of the phase relations in transformers, *E. B. Rosa* and *M. G. Lloyd*, 1.

Dickinson, H. C., Specific heat of some calcium chloride solutions between -35°C and $+20^{\circ}\text{C}$, 379.

E.

Effect of phase harmonics upon acoustic quality, *M. G. Lloyd* and *P. G. Agnew*, 255.

Efficiency, Luminous, of the firefly, *Herbert E. Ives* and *W. W. Coblentz*, 321.

Electric oscillations, Comparative sensitiveness of some common detectors of, *Louis W. Austin*, 527.

Electric quantity, Method for the absolute measurement of, *Burton McCollum*, 503.

Estimation of the temperature of copper by means of optical pyrometers, *George K. Burgess*, 111.

F.

Firefly, Luminous efficiency of the, *Herbert E. Ives* and *W. W. Coblentz*, 321.

Fitch, T. T., Determination of the constants of instrument transformers, 281.

G.

Galvanometer, vibration, Theoretical and experimental study of the, *Frank Wenner*, 347.

Gas, ideal, definition of the, *Edgar Buckingham*, 409.

George, E. B., Specific heat of some calcium chloride solutions between -35°C and $+20^{\circ}\text{C}$, 379.

Grover, Frederick W., Mutual inductance of two parallel coaxial circles in terms of hypergeometrical series, 489.

H.

Hampson liquefier, Theory of the, *Edgar Buckingham*, 125.

Harmonics, phase, Effect of upon acoustic quality, *M. J. Lloyd* and *P. G. Agnew*, 255.

Heat, Specific, of some calcium chloride solutions between -35°C and $+20^{\circ}\text{C}$, *H. C. Dickinson*, *E. F. Mueller* and *E. B. George*, 379.

High temperatures, Platinum resistance thermometry at, *C. W. Waidner* and *G. K. Burgess*, 149.

Hypergeometrical series, Mutual inductance of two parallel coaxial circles in terms of, *Frederick W. Grover*, 489.

I.

Ideal gas, Definition of the, *Edgar Buckingham*, 409.

Illuminants, artificial, Daylight efficiency of, *Herbert E. Ives*, 231.

Inductance, Mutual, of two parallel coaxial circles in terms of hypergeometrical series, *Frederick W. Grover*, 489.

Induction, magnetic, in straight bars, Determination of the, *Charles W. Burrows*, 31.

Instrument transformers, Determination of the constants of, *P. G. Agnew* and *T. T. Fitch*, 281.

Ives, Herbert E., Daylight efficiency of artificial illuminants, 231.

Luminous efficiency of the firefly, 321.

White light from the mercury arc and its complementary, 265.

L.

Light, White, from the mercury arc and its complementary, *Herbert E. Ives*, 265.

Liquefier, Hampson, Theory of the, *Edgar Buckingham*, 125.

- Lloyd, M. G.*, Determination of the ratio of transformation and of the phase relations in transformers, 1.
 Effect of phase harmonics upon acoustic quality, 255.
 Regulation of potential transformers and magnetizing current, 273.
 Luminosity and temperature, *P. G. Nutting*, 337.
 Luminous efficiency of the firefly, *Herbert E. Ives*, and *W. W. Coblentz*, 321.

M.

- Magnetizing current, Regulation of potential transformers and, *M. G. Lloyd* and *P. G. Agnew*, 273.
McCollum, Burton, Method for the absolute measurement of electric quantity, 503.
 Mercury arc and its complementary, White light from the, *Herbert E. Ives*, 265.
 Method for the absolute measurement of electric quantity, *Burton McCollum*, 503.
 Method for constructing the natural scale of pure color, *P. G. Nutting*, 89.
 Method for the determination of relative wave lengths, *Irwin G. Priest*, 573.
 Mica condensers as standards of capacity, *Harvey L. Curtis*, 431.
Mueller, E. F., Specific heat of some calcium chloride solutions between -35°C and $+20^{\circ}\text{C}$, 379.
 Mutual inductance of two parallel coaxial circles in terms of hypergeometrical series, *Frederick W. Grover*, 489.

N.

- Natural scale of pure color, Method for constructing the, *P. G. Nutting*, 89.
 Nomenclature, Photometric units, and *E. B. Rosa*, 543.
 Note on the thermoelectric properties of tantalum and tungsten, *W. W. Coblentz*, 107.
Nutting, P. G., Luminosity and temperature, 337.
 Method of constructing the natural scale of pure color, 89.
 Resolving powers of objectives, 121.

O.

- Objectives, Resolving powers of, *P. G. Nutting*, 121.
 Optical pyrometers, Estimation of the temperature of copper by means of, *George K. Burgess*, 111.
 Oscillations, electric, Comparative sensitiveness of some common detectors of, *Louis W. Austin*, 527.

P.

- Photometric units and nomenclature, *E. B. Rosa*, 543.
 Platinum resistance thermometry at high temperatures, *C. W. Waidner* and *G. K. Burgess*, 149.
 Potential transformers and magnetizing current, Regulation of, *M. G. Lloyd* and *P. G. Agnew*, 273.
Priest, Irwin G., Modified method for the determination of relative wave lengths, 573.
 Pure color, Method for constructing the natural scale of, *P. G. Nutting*, 89.
 Pyrometers, optical, Estimation of the temperature of copper by means of, *George K. Burgess*, 111.

Q.

- Quality, acoustic, Effect of phase harmonics upon, *M. G. Lloyd* and *P. G. Agnew*, 255.
 Quantity, electric, Method for the absolute measurement of, *Burton McCollum*, 503.

R.

- Radiation, Selective, from various solids, II, *W. W. Coblentz*, 301.
 Regulation of potential transformers and magnetizing current, *M. G. Lloyd* and *P. G. Agnew*, 273.
 Resistance thermometry, Platinum, at high temperatures, *C. W. Waidner* and *G. K. Burgess*.
 Resolving powers of objectives, *P. G. Nutting*, 121.
Rosa, E. B., Determination of the ratio of transformation and of the phase relations in transformers, 1.
 Photometric units and nomenclature, 543.

S.

- Scale, natural, of pure color, Method for constructing the, *P. G. Nutting*, 89.
- Selective radiation from various solids, II, *W. W. Coblentz*, 301.
- Solids, various, II, Selective radiation from, *W. W. Coblentz*, 301.
- Specific heat of some calcium chloride solutions between -35° C and $+20^{\circ}$ C, *H. C. Dickinson*, *E. F. Mueller* and *E. B. George*.
- Standards of capacity, Mica condensers as, *Harvey L. Curtis*, 431.

T.

- Tantalum, Note on the thermoelectric properties of, *W. W. Coblentz*, 107.
- Temperature, Luminosity and, *P. G. Nutting*, 337.
- Theoretical and experimental study of the vibration galvanometer, *Frank Wenner*, 347.
- Theory of the Hampson liquefier, *Edgar Buckingham*, 125.
- Thermoelectric properties of tantalum and tungsten, Note on the, *W. W. Coblentz*, 107.
- Transformers, Determination of the ratio of transformation and of the phase relations in, *E. B. Rosa* and *M. G. Lloyd*, 1.
- Transformers, instrument, Determination of the constants of, *P. G. Agnew* and *T. T. Fitch*, 281.

- Transformers, potential, and magnetizing current, Regulation of, *M. G. Lloyd* and *P. G. Agnew*, 273.
- Tungsten, Note on the thermoelectric properties of, *W. W. Coblentz*, 107.

U.

- Units, Photometric, and nomenclature, *E. B. Rosa*, 543.

V.

- Various solids, II, Selective radiation from, *W. W. Coblentz*, 301.
- Vibration galvanometer, Theoretical and experimental study of the, *Frank Wenner*, 347.

W.

- Waidner*, *C. W.*, Platinum resistance thermometry at high temperatures, 149.
- Wave lengths, relative, Modified method for the determination of, *Irwin G. Priest*, 573.
- Waves, EMF, Approximate experimental method for the analysis of, *P. G. Agnew*, 95.
- Wenner*, *Frank*, Theoretical and experimental study of the vibration galvanometer, 347.
- White light from the mercury arc and its complementary, *Herbert E. Ives*, 265.



